Fraternal Twins

The Unreasonable Resemblance of the Ricci and Mean Curvature Flows

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6

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Dedicated to Richard S. Hamilton
1943–2024

"Like leaves on trees the race of man is found,
Now green in youth, now withering on the ground;
Another race the following spring supplies;
They fall successive, and successive rise:
So generations in their course decay;
So flourish these, when those are pass'd away."

– *Homer,* The Iliad (6:171), *tr. Alexander Pope.*

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Preface

On June 4, 2019, at a conference hosted by the ETH Zürich on geometric analysis and general relativity in honour of Gerhard Huisken's 60th birthday, Richard Hamilton presented a lecture enigmatically entitled "Fraternal Twins". In this lecture, he presented an overview of the key historical and mathematical developments in the study of the mean curvature and Ricci flows, emphasizing the striking similarities which consistently occur at a superficial level, but also pointing out the imperfection of these similarities, and some of the analytical differences which lie behind them—much like fraternal twins, the two flows appear very alike at first sight, even though they are by no means identical.

Recognition of the likeness of the two flows goes back much further, of course. Indeed, the drawing of parallels between the two flows is now customary amongst experts; it is often exclaimed, for instance, that "Ricci flow is the extrinsic analogue of mean curvature flow", or that "mean curvature flow in n-dimensions behaves like Ricci flow in $2n$-dimensions", or "since P holds for mean curvature flow/Ricci flow, $\tilde{P}$ *must* be true for Ricci flow/mean curvature flow".[1] And the comparison is more than superficial: despite the fact that the two flows continue to be treated independently, often with quite different tools, Hamilton's analogy continues to be vindicated.

[1] Some brave souls even speculate that there is a hidden canonical correspondence between the two; but no such correspondence is yet to be observed.

The aim of this book is to provide an introduction to geometric evolution equations through a study of these twin flows. It contains two parts: the first is dedicated to the mean curvature flow and the second to the Ricci flow, though the order does not matter much: each part may be treated entirely independently of the other. On the other hand, once the reader has gained some familiarity with one twin, they will feel at once an uncanny familiarity with the other.

We do not attempt to provide a comprehensive treatment[2] of our twin subjects but rather offer the reader an enticing *aperitif*, which we hope may whet their appetite for the subject.[3] Each part begins with *The fundamentals*, introducing the reader to each twin, followed by a technical chapter which lays *The groundwork* for further analysis. This second chapter could be skipped on first reading, and referred

[2] This would take up many volumes, and has already been achieved, to a large degree, by others.

[3] Incidentally, we heartily recommend a glass of *Glenlivet (Founder's Reserve)* to accompany this text, not least because "Glenlivet" may be translated as "Valley of the smooth flow".

back to as needed in the later chapters; on the other hand, the patient reader will certainly benefit in the long run from any effort put into the groundwork. The third chapter of each part is concerned with curvature *Pinching and its consequences*, with a focus on the first major milestone in each of our twin subjects—Huisken's theorem on the contraction of convex hypersurfaces to round points under mean curvature flow and Hamilton's theorem on the contraction of three-manifolds of positive Ricci curvature to round points under Ricci flow, respectively. We then study each flow in its smallest nontrivial dimension, where the behaviour is particularly nice. The fifth chapter introduces the reader to a selection of tools and results pertaining to *Singularities and their analysis* for the respective flow (in higher dimensions). We conclude by surveying some of the recent progress *Towards a classification of ancient solutions* to each flow.

Each chapter ends with a selection of exercises, and the book would be well-suited to a one or two semester graduate course in geometry, or even an undergraduate "special topics" course. For a one semester course, one could plausibly cover, e.g., Chapters 1-5, or Chapters 7-11, or selected parts of Chapters 1-4 and 7-10[4].

[4] A great deal of material can be covered by adopting an alternating structure—1,7,2,8,3,9,...—due to much constructive *approximate* redundancy arising from the fraternal resemblance of the two subjects.

The project grew out of notes for a minicourse on the Ricci flow which I presented in a series of lectures at the summer school "Geometric Flows and Relativity" hosted by the Centro de Matemática of the Universidad de la República in Montevideo, Uruguay, in March 2024, which were subsequently used in a special topics course on both the mean curvature and Ricci flows aimed at advanced undergraduate and beginning graduate students at The Australian National University. I am grateful to Theodora Bourni and Martín Reiris for the invitation to speak at the CMAT summer school, and to the outstanding cohort of students who attended my lectures, keeping me on my toes each morning; I am equally grateful to my wife, Kirsty, who—heavily pregnant with our second child—encouraged me to go!

Many individuals have contributed to this book through useful discussions, particularly Ben Andrews, Theodora Bourni, Tim Buttsworth, Bennett Chow, Apostolos Damialis, Ramiro Lafuente, Stephen Lynch, Martín Reiris and Jonathan Zhu.

I do not claim priority for any of the mathematical results presented herein, and have endeavoured to provide appropriate bibliographic information throughout. The manuscript was compiled on Overleaf in Tufte-LaTeX and the cover was designed using Adobe Illustrator and Adobe Express. Illustrations were created using GeoGebra and Mathematica. No AI tools were used in any stage of the preparation.

Mat Langford
Canberra, August 16, 2025

Part I

Mean curvature flow

Make a soap bubble and observe it;
you could spend a whole life studying it.

– Sir William Thomson, Lord Kelvin

The next time you stare into a beer,
contemplate the bubbles.

– Kenneth Chang, *In Bubbles and Metal, the Art of Shape-Shifting*

Preamble to Part I

In 1952, in a short discussion appearing in an appendix to a paper by Cyril S. Smith,[5] John von Neumann arrived at a mean curvature driven motion for the dynamics of bubbles in a foam as a result of surface tension and the diffusion of gas between neighbouring bubbles. Four years later, William W. Mullins derived the same curvature driven motion for the dynamics of grain boundaries in annealing metals, a process which also seems to be governed by surface tension and interfacial diffusion.[6] These appear to be the earliest appearances of the mean curvature flow in the scientific literature. This is a remarkable fact given that the mean curvature flow has a very natural interpretation as "the heat equation for submanifolds", and its steady state equation—the minimal surface equation—had been introduced and studied by Lagrange already in 1762![7]

[5] Smith, "Shape of metal grains".

[6] Mullins, "Two-dimensional motion of idealized grain boundaries".

[7] By comparison, the heat equation had been introduced by Fourier, in 1822, a mere 40 years after Laplace had developed his eponymous equation.

A systematic mathematical analysis of the mean curvature flow had to wait even longer. It wasn't until 1984 that Gerhard Huisken (inspired in no small part by Hamilton's 1982 introduction of the Ricci flow of Riemannian metrics) brought the full arsenal of differential geometry and partial differential equations to bear on the problem, proving the well-known theorem now carrying his name.[8]

[8] Although it must be noted that, in another fascinating historical peculiarity, geometric measure theoretic *weak solutions* to the mean curvature flow were actually studied a little earlier, in 1979, by Brakke, *The motion of a surface by its mean curvature*. A good introduction to these "Brakke flows" can be found in Ecker, *Regularity theory for mean curvature flow* and Tonegawa, *Brakke's mean curvature flow*; we shall not study them here.

Since that time, our understanding of this beautiful equation has developed rapidly, and several new applications have emerged (for instance in image processing, geometry and topology, and general relativity). Applications aside, the mean curvature flow gives rise to many remarkable and beautiful geometric structures (e.g. solitons, ancient solutions) and analytic features (e.g. differential Harnack inequalities, pseudolocality, gradient structures) and as such is a fascinating area of study for topologists, geometers, and analysts alike.

We shall present here an introduction to the mean curvature flow leading up to the foundations of some modern developments.[9] We assume the reader has some basic familiarity with partial differential equations and the geometry of Euclidean submanifolds. For background, the reader may refer, for instance, to the books of Olver[10] and Kühnel.[11]

[9] There are now a number of texts on the subject, including the excellent lectures of Mantegazza, *Lecture notes on mean curvature flow*, the beautiful book of Ecker, *Regularity theory for mean curvature flow* and the more recent tome of Andrews, Chow, et al., *Extrinsic geometric flows*, each of which this part has drawn upon to some degree.

[10] Olver, *Introduction to partial differential equations*.

[11] Kühnel, *Differential geometry*.

1

The fundamentals

A smooth one-parameter family $\{X_t\}_{t\in I}$ of smooth immersions $X_t : M^n \to \mathbb{R}^{n+k}$ of a smooth[1] n-manifold M^n into Euclidean space[2] $\mathbb{R}^{n+k}$ EVOLVES BY/SATISFIES/IS A MEAN CURVATURE FLOW[3] if

$$\frac{dX_t}{dt} = \vec{\mathrm{H}}_{X_t}, \tag{1.1}$$

where (upon identifying tangent spaces to $\mathbb{R}^n$ with $\mathbb{R}^n$ in the canonical way) $\vec{\mathrm{H}}_{X_t}$ is the MEAN CURVATURE VECTOR associated to X_t and the time derivative is understood in the usual sense: for any $x \in M^n$,

$$\frac{dX_t}{dt}(x) \doteqdot \lim_{h\to 0} \frac{X_{t+h}(x) - X_t(x)}{h}.$$

If we represent $X_t(x)$ and $\vec{\mathrm{H}}_{X_t}(x)$ with respect to the canonical basis $\{e_\alpha\}_{\alpha=1}^{n+1}$ for $\mathbb{R}^{n+1}$ as

$$X_t(x) = X^\alpha(x,t)e_\alpha \quad \text{and} \quad \vec{\mathrm{H}}_{X_t}(x) = \vec{\mathrm{H}}^\alpha(x,t)e_\alpha$$

and introduce local coordinates $\{x^i : U \to \mathbb{R}\}_{i=1}^n$ in some region $U \subset M^n$, then for each $x \in U$ we see that[4]

$$\begin{aligned}\frac{\partial X^\alpha}{\partial t} &= \vec{\mathrm{H}}^\alpha \\ &= g^{ij}\vec{\mathrm{II}}_{ij}^\alpha \qquad (1.2)\\ &= g^{ij}\left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} - g^{k\ell}\delta_{\beta\gamma}\frac{\partial^2 X^\beta}{\partial x^i\partial x^j}\frac{\partial X^\gamma}{\partial x^\ell}\frac{\partial X^\alpha}{\partial x^k}\right), \qquad (1.3)\end{aligned}$$

where $g^{ij}(\cdot,t)$ are the dual components (matrix inverse) of the induced metric g_{X_t}, whose components are given by

$$g_{ij} = \delta_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^i}\frac{\partial X^\beta}{\partial x^j}.$$

The equation (1.2) is thus a system of nonlinear second order partial differential equations. Unappealing, certainly, but it does have the redeeming feature that it is weakly parabolic.

[1] Henceforth, we shall stop using the qualifier "smooth" so irritatingly often, leaving it for the most part to the reader to decide how regular they wish a given object to be in order to make sense of a given statement.

[2] The interested reader may wish to consider the question of how to proceed when $\mathbb{R}^{n+k}$ is replaced by a general Riemannian manifold (N^{n+k}, h). We will be mostly concerned here with the case $k = 1$.

[3] In fact, we shall soon replace this by a more abstract definition, which may appear more complicated at first but has many advantages. The two definitions are equivalent in the sense that there is a canonical bijection between their solutions.

[4] Note that we follow the convention $D_{dXU}(dXV) = dX\nabla_U V + \vec{\mathrm{II}}(U,V)$ for the SECOND FUNDAMENTAL FORM $\vec{\mathrm{II}}$, where $dX : TM^n \to T\mathbb{R}^{n+k}$ denotes the differential of $X : M^n \to \mathbb{R}^{n+k}$.

Observe that, with respect to g_{X_t}-normal coordinates for M^n about a point x, the mean curvature flow system takes the form

$$\frac{\partial X^\alpha}{\partial t} = \sum_{k=1}^{n} \frac{\partial^2 X^\alpha}{\partial x^k \partial x^k} \tag{1.4}$$

at (x,t). Even though this equation only holds at the point x and the time t, it suggests that **we should view the mean curvature flow as a kind of geometric heat equation for immersions**[5]. We shall soon see that it is quite right to do so, but before pursuing this further, let us first establish some additional useful intuition, this time more geometric.

[5] In fact, there is a very natural way to view the mean curvature flow as a literal (albeit abstract) heat equation: identifying the differential dX of the position vector X with a section of the bundle $T^*M \otimes X^*T\mathbb{R}^{n+k}$, which we equip with the metric and connection canonically induced by those on T^*M and $X^*T\mathbb{R}^{n+k}$, the mean curvature vector may be recognized as the divergence of dX—the "Laplacian" of X.

1.1 *Invariance properties*

The mean curvature flow is invariant under certain canonical operations[6], in the sense that these operations take one solution and produce another.

[6] The following list is not intended to be exhaustive.

1.1.1 *Pullback by diffeomorphisms*

If $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow and $\phi : N^n \to M^n$ is a diffeomorphism, then (since the mean curvature vector is invariant under diffeomorphisms)

$$\frac{d(X_t \circ \phi)}{dt}(x) = \frac{dX_t}{dt}(\phi(x)) = \vec{\mathrm{H}}_{X_t}(\phi(x)) = \vec{\mathrm{H}}_{X_t\circ\phi}(x).$$

That is, $\{X_t \circ \phi : N^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow on N^n. This is not at all surprising.

On the other hand, if we allow the diffeomorphism to change with time[7], then we pick up an extra term due to the chain rule:

[7] We shall always assume the group property $\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1+t_2}$ for one-parameter families of diffeomorphisms ϕ_t.

$$\frac{d(X_t \circ \phi_t)}{dt} = \vec{\mathrm{H}}_{X_t\circ\phi_t} + (dX_t)_{X_t\circ\phi_t} V_{\phi_t},$$

where V is the vector field on M^n defined by

$$V(\phi_t(x)) \doteqdot \frac{d}{dt}(t \mapsto \phi_t(x)).$$

The converse of this statement is that if X_t satisfies the equation

$$\frac{dX_t}{dt} = \vec{\mathrm{H}}_{X_t} + dX_t\, V$$

for some vector field V, then the family of immersions $X_t \circ \phi_{-t} : M^n \to \mathbb{R}^{n+k}$ satisfies mean curvature flow, where ϕ_t is the flow of V.

1.1.2 *Time translations*

If $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow and $\tau \in \mathbb{R}$, then clearly $\{X_{t+\tau} : M^n \to \mathbb{R}^{n+k}\}_{t\in I-\tau}$ is a mean curvature flow.

1.1.3 Ambient isometries

If $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow and $v \in \mathbb{R}^{n+1}$, $O \in \mathrm{O}(n+k)$, then $\{x \in M^n \mapsto X_t(x) + v \in \mathbb{R}^{n+k}\}_{t\in I}$ and $\{x \in M^n \mapsto OX_t(x) \in \mathbb{R}^{n+k}\}_{t\in I}$ are mean curvature flows.

1.1.4 Parabolic rescaling

If $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow and $\lambda > 0$, then

$$\frac{d(\lambda X_{\lambda^{-2}t})}{dt}(x) = \lambda^{-1}\vec{\mathrm{H}}_{X_{\lambda^{-2}t}} = \vec{\mathrm{H}}_{\lambda X_{\lambda^{-2}t}},$$

so $\{\lambda X_{\lambda^{-2}t} : M^n \to \mathbb{R}^{n+k}\}_{t\in\lambda^2 I}$ is a mean curvature flow.

1.1.5 Orthogonal sums with flat factors

If $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow on M^n and $\ell \in \mathbb{N}$, then $\{(x,y) \in M^n \times \mathbb{R}^\ell \mapsto (X_t(x), y) \in \mathbb{R}^{n+k} \times \mathbb{R}^\ell\}_{t\in I}$ is a mean curvature flow.

1.1.6 Quotients and lifts

Let $q : N^n \to M^n = N^n/G$ be a quotient map (induced by a proper and free action of a Lie group G on N^n). If $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ evolves by mean curvature, then so does its lift to N^n. Conversely, if $\{X_t : N^n \to \mathbb{R}^{n+k}\}_{t\in I}$ is a mean curvature flow which is constant on the fibres of the quotient, then it descends to M^n and evolves by mean curvature.

$$\begin{array}{ccc} N^n & \xrightarrow{\;q\;} & N^n/G \\ & \tilde{X}_t \searrow & \downarrow X_t \\ & & \mathbb{R}^{n+k} \end{array}$$

1.2 Invariant solutions (a.k.a. self-similar solutions/solitons)

The continuous symmetries of mean curvature flow (domain diffeomorphism, time translation, scaling and orientation preserving ambient isometry) give rise to special types of solutions: those that evolve purely by some combination of these symmetries. There are four primary types (but more generally one might consider combinations of these motions).

1.2.1 Translating self-similar solutions

A solution $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in\mathbb{R}}$ to mean curvature flow is called a TRANSLATING SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ of M^n such that

$$X_{t-\varepsilon}\circ\phi_\varepsilon+\varepsilon v=X_t$$

for all ε and t. Differentiating with respect to ε at $\varepsilon=0$, we find that such a solution must satisfy the equation

$$-\vec{\mathrm{H}}_{X_t}+dX_t\,V+v=0$$

for all t, where V is the vector field tangent to $t\mapsto\phi_t$. Resolving tangential and normal components yields

$$\vec{\mathrm{H}}_{X_t}=v^\perp\ \text{ and }\ dX_t\,V=-v^\top.$$

An immersion $X:M^n\to\mathbb{R}^{n+k}$ satisfying

$$\vec{\mathrm{H}}_X=v^\perp$$

for some $v\in\mathbb{R}^{n+k}$ is called a TRANSLATOR.

Conversely, if $X:M^n\to\mathbb{R}^{n+k}$ is a translator, then the family of immersions $X_t\doteqdot X\circ\phi_t+tv$, $t\in\mathbb{R}$, where ϕ_t is the flow of $-(dX)^{-1}v^\top$, satisfies

$$\frac{dX_t}{dt}=dX\,V+v=-v^\top+v=v^\perp=\vec{\mathrm{H}}_{X_t}.$$

1.2.2 *Rotating self-similar solutions*

A solution $\{X_t:M^n\to\mathbb{R}^{n+k}\}_{t\in\mathbb{R}}$ to mean curvature flow is called a ROTATING SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ of M^n such that

$$\mathrm{e}^{\varepsilon A}X_{t-\varepsilon}\circ\phi_\varepsilon=X_t$$

for all ε and t for some $A\in\mathfrak{so}(n+k)$. Differentiating with respect to ε at $\varepsilon=0$, we find that such a solution must satisfy the equation

$$-\vec{\mathrm{H}}_{X_t}+dX_t\,V+AX=0$$

for all t, where V is the vector field tangent to $t\mapsto\phi_t$. Resolving tangential and normal components yields

$$\vec{\mathrm{H}}_{X_t}=(AX)^\perp\ \text{ and }\ dX_t\,V=-(AX)^\top.$$

An immersion $X:M^n\to\mathbb{R}^{n+k}$ satisfying

$$\vec{\mathrm{H}}_X=(AX)^\perp$$

for some $A\in\mathfrak{so}(n+k)$ is called a ROTATOR.

Conversely, if $X:M^n\to\mathbb{R}^{n+k}$ is a rotator, then the family of immersions $X_t\doteqdot\mathrm{e}^{tA}X\circ\phi_t$, $t\in\mathbb{R}$, where ϕ_t is the flow of $V\doteqdot-(dX)^{-1}(AX)^\top$, satisfies

$$\frac{dX_t}{dt}=\mathrm{e}^{tA}(dX\,V+AX)=\mathrm{e}^{tA}(AX)^\perp=\mathrm{e}^{tA}\vec{\mathrm{H}}_X=\vec{\mathrm{H}}_{X_t}.$$

1.2.3 *Shrinking/expanding self-similar solutions*

A solution $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in I}$ to mean curvature flow is called a HOMOTHETIC SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in I}$ of M^n such that

$$\mathrm{e}^{\varepsilon}\phi_{\varepsilon}^{*}X_{\mathrm{e}^{-2\varepsilon}t} = X_t$$

for all $t \in I$ and ε such that $\mathrm{e}^{-2\varepsilon}t \in I$. Differentiating with respect to ε at $\varepsilon = 0$, we find that such a solution must satisfy the equation

$$X_t + dX_t V - 2t\vec{\mathrm{H}}_{X_t} = 0$$

for all $t \in I$. There are two cases: if $I = (-\infty, 0)$, then $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in(-\infty,0)}$ is called a SHRINKING SELF-SIMILAR SOLUTION. If $I = (0,\infty)$, then $\{X_t : M^n \to \mathbb{R}^{n+k}\}_{t\in(0,\infty)}$ is called an EXPANDING SELF-SIMILAR SOLUTION.

Resolving tangential and normal components, we find that

$$X_t^{\perp} = 2t\vec{\mathrm{H}}_{X_t} \;\text{ and }\; X_t^{\top} = -dX_t V\,.$$

An immersion $X : M^n \to \mathbb{R}^{n+k}$ satisfying

$$\vec{\mathrm{H}}_X = -\frac{1}{2}X^{\perp} \;\text{ resp. }\; \vec{\mathrm{H}}_X = \frac{1}{2}X^{\perp}\,.$$

is called a SHRINKER resp. EXPANDER.

Conversely, if $X : M^n \to \mathbb{R}^{n+k}$ is a shrinker resp. an expander, then the family $\{X_t \doteqdot \sqrt{-t}\phi^*_{\log\sqrt{-t}}X\}_{t\in\mathbb{R}}$ resp. $\{X_t \doteqdot \sqrt{t}\phi^*_{\log\sqrt{t}}X\}_{t\in\mathbb{R}}$ satisfies

$$\frac{dX_t}{dt} = \vec{\mathrm{H}}_X\,.$$

1.2.4 *Examples generated by minimal immersions*

Recall that an immersion $X : M^n \to \mathbb{R}^{n+k}$ is MINIMAL if

$$\vec{\mathrm{H}} = 0\,.$$

Minimal immersions provide (rather dull) examples of mean curvature flows via $\{x \in M^n \mapsto X(x) \in \mathbb{R}^{n+k}\}_{t\in(-\infty,\infty)}$. These may also be viewed as (trivial) translating mean curvature flows (by taking $v = 0$).

Minimal immersions also generate slightly less trivial translating mean curvature flows: given any $v \in \{0\} \times \mathbb{R}^{\ell}$, the product $\{(x,y) \in M^n \times \mathbb{R}^{\ell} \mapsto (X(x), y) \in \mathbb{R}^{n+k} \times \mathbb{R}^{\ell}\}_{t\in(-\infty,\infty)}$ is still minimal (and hence static under mean curvature flow), but also invariant under translation in the v direction, and may thus be viewed as a translating mean curvature flow with nontrivial bulk velocity v. In particular, we may view static affine subsbaces as translating mean curvature flows with respect to parallel translation vectors.

Analogously, minimal *cones* (including linear subspaces) may be viewed as nontrivial shrinking or expanding mean curvature flows.

1.3 *Explicit solutions*

Certain "explicit" solutions can be constructed "by hand" by imposing suitable symmetry or other algebraic ansätze. We present three examples here, but there are many more examples which have been discovered by analogous methods.

By imposing a large enough symmetry group, the mean curvature flow equation may be reduced to a (possibly complicated) system of ordinary differential equations.

Example 1 (The shrinking sphere)**.** We seek a solution to mean curvature flow starting from a round sphere, $X_0(M^n) = S^n_{r_0}$ in $\mathbb{R}^{n+1}$. Since we *expect* roundness to be preserved, we suppose *a priori* that the timeslices are always round,

Vanishing like a
Sigh and slowly
Disappear
Disappear
Vanish
Vanish
Into the air
Slowly disappear
Never really here.
– A Perfect Circle, "Vanishing"

$$X_t(x) = r(t)\frac{X_0(x)}{r_0}.$$

The mean curvature vector of X_t is then

$$\vec{\mathrm{H}}_{X_t} = -\frac{n}{r(t)}\frac{X_0}{r_0}$$

while the time derivative is

$$\frac{dX_t}{dt} = r'\frac{X_0}{r_0}.$$

Equating the two yields $rr' = -n$, and hence

$$r^2(t) = r_0^2 - 2nt, \quad t \in (-\infty, \tfrac{r_0^2}{2n}).$$

When $r_0 = 0$, this solution is called the (standard) SHRINKING SPHERE.

■

We can play a similar game with self-similar solutions, though in this case—since the time evolution is already trivial—we may relax the symmetry by one degree of freedom. In particular, in one space dimension, no symmetry conditions are required.

Example 2 (The Grim Reaper)**.** We seek a nontrivial one-dimensional translator in the plane. I.e. a curve $\gamma : M^1 \to \mathbb{R}^2$ which satisfies

$$\kappa = -\langle \mathsf{N}, v\rangle$$

for some nonzero[8] vector $v \in \mathbb{R}^2$, where N is a choice of unit normal field and κ is the corresponding choice of curvature function. To this end, assume that γ is clockwise oriented and parametrized by arclength s (so that $\mathsf{T} \doteqdot \gamma_s = \mathsf{J}\mathsf{N}$, where J denotes counterclockwise rotation through angle $\frac{\pi}{2}$) and define

[8] The solutions corresponding to $v = 0$ are not so interesting.

$$x(s) \doteqdot \langle \mathsf{N}, e\rangle \quad \text{and} \quad y(s) \doteqdot \langle \mathsf{T}, e\rangle,$$

where $e \doteqdot \frac{v}{|v|}$. By the Frenet–Serret equations,

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \kappa \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

we are led to consider the system

$$\begin{cases} \begin{bmatrix} x \\ y \end{bmatrix}_s = \begin{bmatrix} -xy \\ x^2 \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \end{cases}$$

subject to the geometric constraint $x_0^2 + y_0^2 = 1$. If $x_0 = 0$, then the unique solution is $(x, y) = (0, 1)$, which integrates to one of the (parallel) straight lines

$$\gamma(s) = \gamma(0) + se, \quad p \in \mathbb{R}^2.$$

In particular, if $x_0 \neq 0$, then it must remain nonzero. Consider, then, the curve parameter

$$\theta(s) \doteqdot -\int_0^s x(\sigma)\, d\sigma.$$

Observe that

$$\begin{bmatrix} x \\ y \end{bmatrix}_\theta = \begin{bmatrix} -y \\ x \end{bmatrix}$$

and hence

$$(x, y) = (\cos(\theta + \theta_0), \sin(\theta + \theta_0)),$$

where $(x_0, y_0) = (\cos\theta_0, \sin\theta_0)$. Setting $e \doteqdot \frac{v}{|v|}$, we have

$$\mathsf{T} = x\mathrm{J}e + ye$$

and we conclude that

$$\begin{aligned} \gamma(s) - \gamma(0) &= \mathrm{J}e \int_0^s \cos(\theta + \theta_0)\, ds + e \int_0^s \sin(\theta + \theta_0)\, ds \\ &= \mathrm{J}e \int_0^{\theta(s)} d\theta + e \int_0^{\theta(s)} \tan(\theta + \theta_0)\, d\theta \\ &= (\theta(s) - \theta_0)\mathrm{J}e - \log\cos(\theta(s) - \theta_0)e. \end{aligned}$$

When $v = e_2$, $\gamma(0) = 0$ and $\theta_0 = 0$, this solution is called the (standard) GRIM REAPER. For general initial parameter values, the solution is obtained from the standard Grim Reaper by translation, rotation and scaling. ■

Observe that the computations of Example 2 actually yield the following elementary but important theorem.

Theorem 1.1. *The straight lines and the Grim Reapers are the only one-dimensional translators.*

In higher dimensions, we have the following examples.

Example 3 (The radio-dish soliton)**.** For each $n \geq 2$, there exists a (unique modulo rigid motions and scaling) convex, axially symmetric translator in $\mathbb{R}^{n+1}$. It is the graph of a (convex) function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying $u(x) = \phi(|x|)$ with

$$\phi(r) = \frac{r^2}{2(n-1)} - \log r + O(r^{-1}) \text{ as } r \to \infty.$$

This is proved by seeking axially symmetric solutions $u(x) = \phi(|x|)$ to the graphical[9] translator equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}. \tag{1.5}$$

The axial symmetry ensures that ϕ satisfies the equation

$$\frac{\phi_{rr}}{1+\phi_r^2} + (n-1)\frac{\phi_r}{r} = 1$$

and hence $\Phi \doteqdot \phi_r$ satisfies the equation

$$\frac{\Phi_r}{1+\Phi^2} + (n-1)\frac{\Phi}{r} = 1.$$

Local existence and uniqueness of a solution $\Phi : [r_0, \infty) \to \mathbb{R}$ upon prescribing $\Phi(r_0)$ is a consequence of the Picard–Lindelöf theorem and a straightforward barrier construction, so long as $r_0 > 0$. The asymptotics are established in[10] by barrier and bootstrapping arguments. By a barrier argument and the Arzelà–Ascoli theorem, the desired solution can then be obtained by taking a limit of solutions $\Phi_k : [\frac{1}{k}, \infty)$ arising from initial conditions $\Phi_k(\frac{1}{k}) = \frac{1}{nk}$ (see, for instance[11]). Alternatively, one may apply standard elliptic PDE methods to solve the graphical translator equation (1.5) in a ball about the origin with zero Dirichlet data on the boundary (producing an axially symmetric solution due to the maximum principle), and then extend to all of $\mathbb{R}^n$ using the above ODE arguments.[12] ■

Example 4 (Shrinking doughnuts[13])**.** For each $n \geq 2$, there exists an axially symmetric shrinker in $\mathbb{R}^{n+1}$ which is obtained by rotating about the x-axis an embedding of S^1 which lies in the upper half-plane $\Sigma \doteqdot \{(x, r) : r > 0\}$. (The resulting surface is therefore topologically a handle $S^1 \times S^{n-1}$.)

Indeed, a curve in Σ generates an axially symmetric shrinker in $\mathbb{R}^{n+1}$ if and only if it is a geodesic in the metric[14]

[9] (Mean) convexity guarantees graphicality via the translator equation.

[10] Clutterbuck, Schnürer, and Schulze, "Stability of translating solutions to mean curvature flow".

[11] Rengaswami, "Classification of bowl-type translators to fully nonlinear curvature flows".

[12] See, for example, Andrews, Chow, et al., *Extrinsic geometric flows*, §13.8.

[13] Sigurd B. Angenent, "Shrinking doughnuts"

[14] See Exercise 5.2.

$$\sigma \doteqdot r^{2(n-1)} e^{-\frac{x^2+r^2}{4}} (dx^2 + dr^2).$$

The geodesic equation for (Σ, σ) can always be solved locally for prescribed initial data $(p, v) \in T\Sigma$. However, most of these geodesics will neither remain embedded nor close up. By a "shooting" argument, it is possible to pinpoint a geodesic arc which meets the r-axis orthogonally at both endpoints, hence defining the desired closed geodesic in Σ upon reflection about the r-axis. ∎

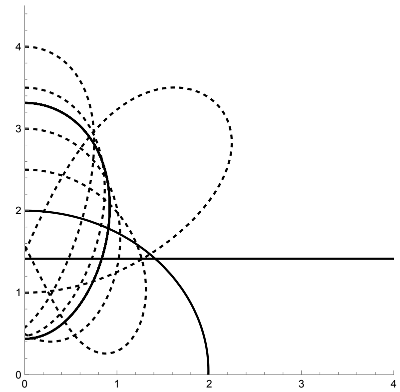


Figure 1.1: Some geodesics in (Σ, σ) (profiles of the shrinking sphere, shrinking cylinder and shrinking doughnut among them). Cf. ibid., Figure 3a.

1.4 *Uniqueness and (short-time) existence of solutions*

We would like to exhibit the mean curvature flow equation as an equation or system of equations for which known methods from the theory of partial differential equations may be applied. There is indeed a general short-time existence theory which applies to strictly parabolic second order partial differential equations for maps between manifolds (with compact domain). Unfortunately, this theory cannot be directly applied to the mean curvature flow due to the lack of strict parabolicity.

For nonlinear equations, parabolicity is determined by the *linearization*.

Lemma 1.2 (Linearization of the mean curvature flow). *Suppose that the two parameter family of immersions $X_t^\varepsilon : M^n \to \mathbb{R}^{n+1}$, $t \in I$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, forms a one-parameter family of mean curvature flows $\{X_t^\varepsilon : M^n \to \mathbb{R}^{n+1}\}_{t\in I}$ about $X_t \doteqdot X_t^0$. The variation field $Y_t \doteqdot \frac{d}{d\varepsilon}\big|_{\varepsilon=0} X_t^\varepsilon$ satisfies, in any local coordinate chart,*

$$\frac{\partial Y^\alpha}{\partial t} = g^{ij}\left(\frac{\partial^2 Y^\alpha}{\partial x^i \partial x^j} - \frac{\partial^2 Y^\beta}{\partial x^i \partial x^j}\frac{\partial X^\beta}{\partial x^k}\frac{\partial X^\alpha}{\partial x^\ell}\right) + \textit{lower order terms}, \qquad (1.6)$$

where $g^{ij}(\cdot, t)$ are the dual components of the metric g_t induced by X_t.

Proof. We leave the proof as an exercise. □

The equation (1.6) is weakly but not strictly parabolic. The lack of strict parabolicity is due to the identity $\pi_{dX_t(TM)}\vec{H}_{X_t} = 0$, which guarantees that $\pi_{dX_t(TM)}\frac{dY_t}{dt} = 0$. Treating this equation as a constraint, Gage and Hamilton[15] are able to prove short-time existence using direct methods (in particular, the Nash—Moser implicit function theorem) following Hamilton's earlier work on the Ricci flow.[16] Soon after, de Turck found a way to relate the mean curvature flow to a strictly parabolic equation, to which the standard theory may be more readily applied. We will present yet another proof, due to Huisken and Polden[17] (see also[18]).

[15] Gage and R. S. Hamilton, "The heat equation shrinking convex plane curves".

[16] Richard S. Hamilton, "Three-manifolds with positive Ricci curvature".

[17] Huisken and Polden, "Geometric evolution equations for hypersurfaces".

[18] Mantegazza, *Lecture notes on mean curvature flow*.

Theorem 1.3 (Short time existence and uniqueness)**.** *Let M^n be a compact manifold. Given any immersion $X_0 : M^n \to \mathbb{R}^{n+1}$, there exists $\delta > 0$ and a mean curvature flow $\{X_t : M^n \to \mathbb{R}^{n+1}\}_{t\in(0,\delta)}$ such that X_t converges uniformly to X_0 as $t \to 0$ (in the smooth sense if X_0 is smooth). Moreover, any other mean curvature flow starting from X_0 agrees with X_t on their common interval of existence. Finally, the mean curvature flow $\{X_t : M^n \to \mathbb{R}^{n+1}\}_{t\in(0,\delta)}$ depends continuously on X_0 (in the smooth sense if X_0 is smooth).*

Sketch of the Huisken–Polden argument[19]. Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be an immersion of a compact manifold M^n. Given a smooth function $u : M^n \times [0, T) \to \mathbb{R}^{n+1}$, we can form a family of smooth maps $X_t : M^n \to \mathbb{R}^{n+1}$ by setting $X_t(x) \doteqdot X_0(x) + u(x,t)\,\mathsf{N}_0(x)$. If $u(x,0) \equiv 0$, then $X_t|_{t=0}$ agrees with our initial immersion X_0. Moreover, by smoothness of u and compactness of M^n, X_t is an immersion for sufficiently small times. By a somewhat involved calculation, this family of immersions will satisfy

$$\left\langle \frac{dX_t}{dt}, \mathsf{N}_{X_t} \right\rangle = -\,\mathrm{H}_{X_t}$$

if and only if u satisfies an equation of the form

$$u_t(x,t) = \Delta_{X_t} u(x,t) + P(\nabla u(x,t), u(x,t), x),$$

where Δ_{X_t} is the Laplacian induced by X_t. But since this equation is strictly parabolic (and M^n is compact), it admits a unique solution with zero initial condition. Existence of our mean curvature flow now follows by composing the corresponding family of normal graphs with a (unique) time-dependent diffeomorphism.

Uniqueness and continuous dependence also follow from this argument since, due to compactness of M^n, any small perturbation of X_0 may be represented as a normal graph over X_0. □

[19] We only present the argument in the setting of hypersurfaces, but it can actually be generalized to any codimension, due to suitable existence theory for strictly parabolic *systems*.

1.5 *The time-dependent geometric formalism*

A one-parameter family $\{X_t\}_{t\in I}$ of immersions $X_t : M^n \to \mathbb{R}^{n+k}$ may (perhaps more properly) be viewed as a map $X : M^n \times I \to \mathbb{R}^{n+k}$ via $(x,t) \mapsto X(x,t) \doteqdot X_t(x)$; we call such a map a TIME-DEPENDENT IMMERSION. We may exhibit the time derivative $\frac{dX_t}{dt}$ as a section $(x,t) \mapsto \frac{dX_t}{dt}(x)$ of the PULLBACK BUNDLE[20] $X^*T\mathbb{R}^{n+k} \to M^n \times I$. Indeed,

$$\frac{dX_t}{dt}(x) = dX_{(x,t)}\,\partial_t|_{(x,t)},$$

where the CANONICAL VECTOR FIELD $\partial_t \in \Gamma(T(M^n \times I))$ is defined by

$$\partial_t f(x,t) = \left.\frac{d}{dh}\right|_{h=0} f(x,t+h) \ \text{ for } \ f \in C^\infty(M^n \times I).$$

[20] This is the "obvious" vector bundle over $M^n \times I$ whose fibre at (x,t) is $T_{X(x,t)}\mathbb{R}^{n+k}$.

Denoting $\partial_t X \doteqdot dX\partial_t$ and identifying $\vec{\mathrm{H}}_{X_t}$ with a section $(x,t) \mapsto \vec{\mathrm{H}}(x,t) \doteqdot \vec{\mathrm{H}}_{X_t}(x)$ of $X^*T\mathbb{R}^{n+k}$, the mean curvature flow may be recast as the equation

$$\partial_t X = \vec{\mathrm{H}}. \tag{1.7}$$

Now, this may seem like abstract nonsense (and it is), but it does have a more pragmatic purpose: any time-dependent immersion $X : M^n \times I \to \mathbb{R}^{n+k}$ induces a natural (and computationally convenient) "time-dependent" extrinsic geometric formalism on M^n, which is entirely analogous to the extrinsic geometric formalism induced by a (time-independent) immersion. This geometry is exhibited on the SPATIAL TANGENT BUNDLE[21]

$$\mathfrak{S} \doteqdot \{\xi \in T(M \times I) : dt(\xi) = 0\}$$

of $M \times I$ through the assignment of a (canonical) metric and connection. The TIME-DEPENDENT METRIC $g \in \Gamma(\mathfrak{S}^* \odot \mathfrak{S})$ is defined by

$$g_{(x,t)}(u,v) \doteqdot \left\langle dX_{(x,t)}u\,,\, dX_{(x,t)}v \right\rangle \text{ for } u,v \in \mathfrak{S}_{(x,t)}$$

and the TIME-DEPENDENT CONNECTION $\nabla : T(M \times I) \times \Gamma(\mathfrak{S}) \to \mathfrak{S}$ is defined by

$$dX\nabla_\xi U \doteqdot (D_{dX\xi}(dXU))^\top \text{ for } \xi \in T(M \times I), U \in \Gamma(\mathfrak{S}).$$

The time-dependent metric induces the orthogonal splitting

$$X^*T\mathbb{R}^{n+k} = dX(\mathfrak{S}) \oplus^\perp \mathfrak{N}.$$

The vector bundle $\mathfrak{N} \to M^n \times I$ defined by this splitting is called the NORMAL BUNDLE of X. Since the time-dependent metric coincides at each time t with the metric induced by X_t, the fibre of $\mathfrak{N}$ at (x,t) is the fibre of the normal bundle to X_t at x. Projecting the ambient connection onto $\mathfrak{N}$ produces the TIME-DEPENDENT SECOND FUNDAMENTAL FORM $\vec{\mathrm{II}} \in \Gamma(T^*(M \times I) \otimes \mathfrak{S}^* \otimes \mathfrak{N})$:

$$\vec{\mathrm{II}}(\xi, U) \doteqdot D_{dX\xi}(dXU) - dX(\nabla_\xi U) \text{ for } \xi \in T(M \times I),\ U \in \Gamma(\mathfrak{S}).$$

Note that $\mathfrak{S}$ induces a canonical splitting

$$T(M \times I) = \mathfrak{S} \oplus \mathbb{R}\partial_t.$$

With respect to this splitting, the restriction of the time-dependent connection and second fundamental form to $\mathfrak{S}$ may at any fixed time t be identified with the (Levi–Civita) connection and second fundamental form induced by X_t. In particular, $\vec{\mathrm{H}} = \mathrm{tr}_g(\vec{\mathrm{II}}|_{\mathfrak{S}})$.

The time-dependent connection provides a natural notion of differentiation in the time direction of TIME-DEPENDENT VECTOR FIELDS

[21] Here, $t : M \times I \to \mathbb{R}$ denotes the projection onto the second factor.
Note that the fibres of $\mathfrak{S}$ are canonically identified with those of TM; however, $\mathfrak{S}$ is a bundle over $M^n \times I$ (not M^n), which means that its sections are "time-dependent".

(sections of $\mathfrak{S}$). The upshot is that this notion is computationally very convenient, as it is compatible with the time-dependent metric:

$$\begin{aligned}\partial_t(g(U,V)) &= \partial_t \langle dXU\,,\,dXV\rangle \\ &= \left\langle D_{dX\partial_t}(dXU)\,,\,dXV\right\rangle + \left\langle dXU\,,\,D_{dX\partial_t}(dXV)\right\rangle \\ &= g(\nabla_t U, V) + g(U, \nabla_t V)\end{aligned}$$

for $U, V \in \Gamma(\mathfrak{S})$, where we are denoting $\nabla_t \doteqdot \nabla_{\partial_t}$.

Observe that

$$\begin{aligned}g(\nabla_t U, V) &= \left\langle D_{dX\partial_t}(dXU)\,,\,dXV\right\rangle \\ &= \langle D_{dXU}(dX\partial_t) + [dX\partial_t, dXU]\,,\,dXV\rangle \\ &= \left\langle D_{dXU}\big((dX\partial_t)^\top + (dX\partial_t)^\perp\big) + [dX\partial_t, dXU]\,,\,dXV\right\rangle \\ &= \left\langle dX\nabla_U(\partial_t X)^\top + dX\mathrm{W}^{(\partial_t X)^\perp}(U) + dX[\partial_t, U]\,,\,dXV\right\rangle \\ &= g\left([\partial_t, U] + \nabla_U(\partial_t X)^\top + \mathrm{W}^{(\partial_t X)^\perp}(U), V\right), \qquad (1.8)\end{aligned}$$

where the TIME-DEPENDENT WEINGARTEN MAP $\mathrm{W}^N \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S})$ associated with the normal field $N \in \Gamma(\mathfrak{N})$ is defined for any $\xi \in \mathfrak{S}$ by[22]

$$dX\mathrm{W}^N(\xi) \doteqdot (D_{dX\xi}N)^\top.$$

[22] Note that the normal projection

$$\nabla_\xi N = (D_{dX\xi}N)^\perp \qquad (1.9)$$

defines a (TIME-DEPENDENT) CONNECTION $\nabla : T(M^n \times I) \times \Gamma(\mathfrak{N}) \to \mathfrak{N}$ on the normal bundle $\mathfrak{N}$.

Returning to (1.8), we have established that

$$\nabla_t U = [\partial_t, U] + \nabla_U(\partial_t X)^\top + \mathrm{W}^{(\partial_t X)^\perp}(U). \qquad (1.10)$$

In particular, for mean curvature flow,

$$\nabla_t U = [\partial_t, U] + \mathrm{W}^{\vec{\mathrm{H}}}(U). \qquad (1.11)$$

Note that Equation (1.10) is sufficient to determine ∇, since $\nabla_\xi U$ is given by the Levi-Civita formula when $\xi \in \mathfrak{S}$.

In the sequel, when it is clear that we are working in the "time-dependent" setting, we shall conflate $\mathfrak{S}$ with TM^n and write NM^n for $\mathfrak{N}$.

1.6 Exercises

It takes two, it's up to me and you, to prove it.
– Gossip, "Heavy Cross"

Exercise 1.1. Suppose that the family of graphs $M^n_t \doteqdot \operatorname{graph} u(\cdot, t)$ induced by a function $u : \Omega \times I \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, evolve by mean curvature flow, in the sense that the parametrizations $X_t(x) \doteqdot (x, u(x,t))$ satisfy

$$\left(\frac{dX_t}{dt}\right)^\perp = \vec{\mathrm{H}}_{X_t}.$$

Show that u satisfies

$$\partial_t u = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Exercise 1.2. Suppose that the family of level sets $M_t^n \doteqdot \{X \in \Omega : u(X) = t\}$ of a nondegenerate function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^{n+1}$, evolve by mean curvature flow, in the sense that there exist local parametrizations $X_t : U \to \mathbb{R}^{n+1}$ satisfying

$$\frac{dX_t}{dt} = \vec{\mathrm{H}}_{X_t}.$$

Show that u satisfies

$$-1 = |Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right). \tag{1.12}$$

Exercise 1.3. (a) Show that the induced connections, ∇^ε, of a one-parameter family of immersions $X_\varepsilon : M^n \to \mathbb{R}^{n+1}$ satisfy

$$\begin{aligned} 2\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\nabla^\varepsilon\right)(U,V,W) \qquad\qquad\qquad\qquad (1.13)\\ = \nabla^\varepsilon_U \mathrm{S}\,\mathrm{II}_X(V,W) + \nabla^\varepsilon_V \mathrm{S}\,\mathrm{II}_X(U,W) - \nabla^\varepsilon_W \mathrm{S}\,\mathrm{II}_X(U,V)\\ + \mathrm{S}\left(\nabla^\varepsilon_U \mathrm{II}_X(V,W) + \nabla^\varepsilon_V \mathrm{II}_X(U,W) - \nabla^\varepsilon_W \mathrm{II}_X(U,V)\right). \end{aligned}$$

where $X \doteqdot X_0$, $\mathrm{S} \doteqdot -\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X_\varepsilon\right)\cdot \mathsf{N}_X$, and

$$\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\nabla^\varepsilon\right)(U,V,W) \doteqdot dX\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\nabla^\varepsilon\right)(U,V)\cdot dXW.$$

(b) Deduce that, under mean curvature flow,

$$\begin{aligned} \frac{d\nabla^{X_t}}{dt}(U,V,W) \qquad\qquad\qquad\qquad (1.14)\\ = \nabla^{X_t}_U \mathrm{H}_{X_t}\,\mathrm{II}_{X_t}(V,W) + \nabla^{X_t}_V \mathrm{H}_{X_t}\,\mathrm{II}_{X_t}(U,W) - \nabla^{X_t}_W \mathrm{H}_{X_t}\,\mathrm{II}_{X_t}(U,V)\\ + \mathrm{H}_{X_t}\left(\nabla^{X_t}_U \mathrm{II}_{X_t}(V,W) + \nabla^{X_t}_V \mathrm{II}_{X_t}(U,W) - \nabla^{X_t}_W \mathrm{II}_{X_t}(U,V)\right), \end{aligned}$$

Exercise 1.4. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a time-dependent immersion. Suppose that $\partial_t X \in \Gamma(\mathfrak{N})$. Choosing a (local) unit normal vector field N, write $\partial_t X = -F\,\mathsf{N}$. Show that

$$\mathrm{W}^{\mathsf{N}}(\partial_t) = \nabla F.$$

Hint: This an extension of the first Weingarten identity (for time-independent immersions) to the extra time direction.

Exercise 1.5. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a time-dependent immersion. Suppose that $\partial_t X \in \Gamma(\mathfrak{N})$. Choosing a (local) unit normal vector field N, write $\partial_t X = -F\,\mathsf{N}$. Show that

$$\nabla_t \mathrm{II} = \nabla^2 F + F\,\mathrm{II}^2.$$

Hint: This is an extension of the Codazzi identity (for time-independent immersions) to the extra time direction.

Exercise 1.6. Let $X : M^n \to \mathbb{R}^{n+1}$ be a hypersurface. Use the Gauss, Codazzi and Weingarten equations to obtain Simons' identity

$$\nabla_{(u}\nabla_{v)}\mathrm{II}(w,z) - \nabla_{(w}\nabla_{z)}\mathrm{II}(u,v) = \mathrm{II}(u,v)\mathrm{II}^2(w,z) - \mathrm{II}(w,z)\mathrm{II}^2(u,v), \quad (1.15)$$

where the brackets indicate symmetrization.

2
The groundwork

2.1 The maximum principle

The maximum principle is a fundamental tool in the analysis of partial differential equations of parabolic type, and the mean curvature flow is no exception.

2.1.1 Maximum principle for scalars

Proposition 2.1. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a mean curvature flow on a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies*

$$(\partial_t - \Delta - \nabla_b - c)u \leq 0$$

for some time-dependent vector field b and some locally bounded function $c : M^n \times [0,T) \to \mathbb{R}$, where the Laplacian Δ is taken with respect to the induced time-dependent metric. If $\max_{M^n\times\{0\}} u \leq 0$, then

$$\max_{M^n\times\{t\}} u \leq 0 \text{ for all } t \in [0,T]. \tag{2.1}$$

If $c \equiv 0$, then

$$\max_{M^n\times[0,T]} u = \max_{M^n\times\{0\}} u. \tag{2.2}$$

Proof. Given $\sigma \in (0,T)$ and $\varepsilon > 0$, consider the modification $u_{\sigma,\varepsilon}(x,t) \doteqdot u(x,t) - \varepsilon e^{(C+1)t}$, where $C \doteqdot \max_{M^n\times[0,\sigma]} c$. We claim that $u_{\sigma,\varepsilon} < 0$ in $M^n \times [0,\sigma]$. Suppose, to the contrary, that $u_{\sigma,\varepsilon}(x_0,t_0) \geq 0$ for some point $(x_0,t_0) \in M^n \times [0,\sigma]$. Since $u_{\sigma,\varepsilon}(\cdot,0) < 0$, there exists a positive earliest such time, which we take to be t_0, in which case $u_{\sigma,\varepsilon}(x_0,t_0) = 0$. At the point (x_0,t_0),

$$\begin{aligned} 0 \leq (\partial_t - \Delta - \nabla_b)u_{\sigma,\varepsilon} &\leq cu - \varepsilon(C+1)e^{(C+1)t} \\ &= \varepsilon e^{(C+1)t}c - \varepsilon(C+1)e^{(C+1)t} \\ &\leq -\varepsilon e^{(C+1)t} \\ &< 0, \end{aligned}$$

which is absurd. We conclude that $u_{\sigma,\varepsilon} < 0$ in $M^n \times [0,\sigma]$. But $\varepsilon > 0$ and $\sigma \in (0,T)$ were arbitrary. Taking $\varepsilon \to 0$ and then $\sigma \to T$ yields the claim. □

Of course, the same argument applies with the inequalities reversed, leading to a *minimum principle*.

The following ODE COMPARISON PRINCIPLE is an immediate consequence of the maximum principle.

Proposition 2.2 (ODE comparison principle)**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a mean curvature flow on a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies*

$$(\partial_t - \Delta - \nabla_b)u \le F(u) \,, \tag{2.3}$$

for some time-dependent vector field b and some locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$, where the Laplacian Δ and covariant derivative ∇ are taken with respect to the induced time-dependent metric. If $u \le \phi_0$ at $t = 0$ for some $\phi_0 \in \mathbb{R}$, then $u(x,t) \le \phi(t)$ for all $x \in M^n$ and $0 \le t < T$, where ϕ is the solution to the ODE

$$\begin{cases} \dfrac{d\phi}{dt} = F(\phi) \quad in \;\; (0,T), \\ \phi(0) = \phi_0 . \end{cases} \tag{2.4}$$

Proof. Fix $s \in (0,T)$. Since F is locally Lipschitz, there exists some $L < \infty$ such that

$$\begin{aligned}(\partial_t - \Delta - \nabla_b)(u-\phi) &\le F(u) - F(\phi) \\ &\le L|u-\phi| = L\,\mathrm{sign}(u-\phi)(u-\phi)\end{aligned}$$

in $M^n \times (0,s]$, where $\mathrm{sign}(u-\phi)$ is the sign of the expression $u - \phi$. The claim now follows, within $M^n \times [0,s]$, from Theorem 9.1. Taking $s \to T$ completes the proof. □

Again, one can reverse the inequalities to obtain the corresponding ODE comparison from below.

The strong maximum principle also passes to the geometric setting.

Proposition 2.3. *Let $X : M^n \times (0,T) \to \mathbb{R}^{n+1}$ be a mean curvature flow on a connected manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T))$ is nonpositive and satisfies*

$$(\partial_t - \Delta - \nabla_b - c)u \le 0 \tag{2.5}$$

for some time-dependent vector field b and some function $c : M^n \times (0,T) \to \mathbb{R}$, where the Laplacian Δ and covariant derivative ∇ are taken with respect to the induced time-dependent metric. If $u(x_0,t_0) = 0$ for some $(x_0,t_0) \in M^n \times (0,T)$, then $u(x,t) = 0$ for all $(x,t) \in M^n \times (0,t_0]$.

Proof. In local coordinates $\{x^i\}_{i=1}^n$ for a connected coordinate patch $U \subset M^n$ about x_0, u satisfies

$$\partial_t u \leq g^{ij}u_{ij} + (b^k + g^{ij}\Gamma_{ij}{}^k)u_k + cu\,.$$

The classical strong maximum principle then implies that $u \equiv 0$ in $U \times (0, t_0]$. Since M^n is connected, the claim follows from a standard 'open-closed' argument. □

2.1.2 *A maximum principle for symmetric bilinear forms*

There is also an extremely useful incarnation of the maximum principle which applies to symmetric bilinear forms.

Proposition 2.4 (Tensor maximum principle)**.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be mean curvature flow on a compact manifold M^n. Suppose that $S \in \Gamma(T^*M^n \odot T^*M^n)$ satisfies*

$$(\nabla_t - \Delta - \nabla_b)S_{(x,t)}(v, v) \geq F(x, t, S_{(x,t)})(v, v) \ \text{ for all } \ (x, t, v) \in TM^n$$

for some time-dependent vector field $b \in \Gamma(TM)$ and some TIME-DEPENDENT VERTICAL VECTOR FIELD*—a (time-dependent) section F of $\pi^*(T^*M^n \odot T^*M^n)$—which is Lipschitz in the fibre and satisfies the* NULL EIGENVECTOR CONDITION*:*

$$F(x, t, T_{(x,t)})(v, v) \geq 0 \ \text{ whenever } \ T_{(x,t)} \geq 0 \ \text{ and } \ T_{(x,t)}(v) = 0,$$

where ∇ and Δ are the time-dependent connection and (spatial) Laplacian induced by the time-dependent metric induced by X. If $S_{(x,0)} \geq 0$ for all $x \in M^n$, then $S_{(x,t)} \geq 0$ for all $(x, t) \in M^n \times [0, T)$.

Proof. Fix $\sigma \in (0, T)$ and $\varepsilon > 0$. Setting $C \doteqdot \max_{(x,t)\in M^n \times [0,\sigma]} \operatorname{Lip} F(x, t, \cdot)$, we will show that the tensor

$$S^{\sigma,\varepsilon} \doteqdot S + \varepsilon e^{(C+1)t} g\,,$$

is positive definite in $M^n \times [0, \sigma]$. By hypothesis, $S^{\sigma,\varepsilon}_{(x,0)} > 0$ for all $x \in M^n$. So suppose, contrary to the claim, that there exist $x_0 \in M^n$, $t_0 \in (0, \sigma]$ and $V_0 \in T_{x_0}M^n \setminus \{0\}$ such that $S^{\sigma,\varepsilon}_{(x,t)} > 0$ for each $(x, t) \in M^n \times [0, t_0)$ but $S^{\sigma,\varepsilon}_{(x_0,t_0)}(V_0, V_0) = 0$. Extend V_0 locally in space by solving

$$\nabla_{\gamma'} V \equiv 0$$

along radial g_{t_0}-geodesics γ emanating from x_0 and then extend the resulting local vector field in the time direction by solving

$$\nabla_t V \equiv 0\,.$$

Then $\nabla V(x_0, t_0) = 0$ and $\nabla_t V(x_0, t_0) = 0$. We claim that we also have $\Delta V(x_0, t_0) = 0$. To see this, let $\{e_i\}_{i=1}^n$ be an orthonormal frame at

x_0 and parallel translate it along geodesics emanating from x_0, all of this respect to g_{t_0}. We then may compute using $e_i = \gamma_i'$ along γ_i with $\gamma_i'(0) = e_i$ that

$$\Delta V(x_0, t_0) = \sum_{i=1}^{n} \left(\nabla_{e_i}(\nabla_{e_i} V) - \nabla_{\nabla_{e_i} e_i} V \right) (x_0, t_0) = 0.$$

Now set

$$s_{\sigma,\varepsilon}(x,t) \doteqdot S^{\sigma,\varepsilon}_{(x,t)}(V_{(x,t)}, V_{(x,t)})$$

for (x,t) near (x_0,t_0). Then $s_{\sigma,\varepsilon}(x,t) \geq 0$ for (x,t) in a small parabolic neighborhood $B_r(x_0,t_0) \times (t_0 - r^2, t_0]$ of (x_0,t_0) and $s_{\sigma,\varepsilon}(x_0,t_0) = 0$, and hence

$$\begin{aligned}
0 &\geq (\partial_t - \Delta - \nabla_b) s_{\sigma,\varepsilon}|_{(x_0,t_0)} \\
&= (\nabla_t - \Delta - \nabla_b) S^{\sigma,\varepsilon}|_{(x_0,t_0)}(V_0, V_0) \\
&\geq F(x_0, t_0, S_{(x_0,t_0)})(V_0,V_0) + \varepsilon(C+1)\mathrm{e}^{(C+1)t} g_{(x_0,t_0)}(V_0,V_0) \\
&\geq F(x_0, t_0, S^{\sigma,\varepsilon}_{(x_0,t_0)})(V_0,V_0) - C\big(S^{\sigma,\varepsilon}_{(x_0,t_0)} - S_{(x_0,t_0)}\big)(V_0,V_0) \\
&\quad + \varepsilon(C+1)\mathrm{e}^{(C+1)t} g_{(x_0,t_0)}(V_0,V_0) \\
&\geq \varepsilon \mathrm{e}^{(C+1)t_0} g_{(x_0,t_0)}(V_0,V_0) \\
&> 0,
\end{aligned}$$

which is absurd. So $S^{\sigma,\varepsilon}$ indeed remains positive definite in $[0,\sigma]$. The claim follows since $\sigma \in (0,T)$ and $\varepsilon > 0$ are arbitrary. □

2.2 *Evolution of geometry under mean curvature flow*

The mean curvature flow equation induces diffusion equations of various types for the various geometric attributes of the evolving interface.

2.2.1 *Displacement estimates*

If $X : M^n \times I \to \mathbb{R}$ evolves by mean curvature flow, then, for any basepoint $(p_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$, a short calculation reveals that the squared parabolic distance function $|X - p_0|^2 + 2n(t - t_0)$ satisfies the heat equation induced by X:

$$(\partial_t - \Delta)\big(|X - p_0|^2 + 2n(t - t_0)\big) = 0.$$

When M^n is compact, the maximum principle then ensures that upper and lower bounds for $|X - p_0|^2 + 2n(t - t_0)$ are preserved. In particular, we obtain the following extrinsic distance estimates.

Proposition 2.5. *Let $X : M^n \times I \to \mathbb{R}$ be a mean curvature flow. If $t_0 \in I$ and $X(M^n, t_0)$ is disjoint from the sphere $\partial B_r(p_0)$, then $X(M^n, t)$ is disjoint from the sphere $\partial B_{\sqrt{r^2 - 2n(t-t_0)}}(p_0)$ for all $t \in I \cap [t_0, \infty)$.*

In particular, if the initial hypersurface lies inside the ball $B_r(p_0)$, then[1] the hypersurface $X(M^n, t)$ must lie inside the ball $B_{\sqrt{r^2-2nt}}(p_0)$. What happens as $t \to \frac{r^2}{2n}$? We can only conclude that the solution simply does not exist long enough to find out.

[1] By the Jordan–Schoenflies theorem, for example.

Corollary 2.6. *If M^n is compact and $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ evolves by mean curvature, then*

$$T \leq \frac{r_0^2}{2n},$$

where r_0 is the circumradius of $X(M^n, 0)$.

Note that[2] the boundary of the ball $B_{\sqrt{r^2-2n(t-t_0)}}(p_0)$ evolves by mean curvature; so Proposition 2.5 says that the mean curvature evolution of a compact hypersurface and a sphere which is disjoint from it must "avoid" each other in their common interval of existence. This turns out to be true for any pair of proper mean curvature flows, so long as at least one of them is compact.

[2] Recall Example 1.

Proposition 2.7 (The avoidance principle)**.** *Let $X_j : M_j^n \times [0, T_j) \to \mathbb{R}^{n+1}$, $j = 1, 2$, be a pair of proper mean curvature flows, with at least one of the domains M_j^n compact. If $X_1(M_1^n, 0) \cap X_2(M_2^n, 0) = \emptyset$, then $X_1(M_1^n, t) \cap X_2(M_2^n, t) = \emptyset$ for all $t \in [0, T_1) \cap [0, T_2)$. In fact, the distance $\min_{(x,y)\in M_1^n \times M_2^n} d(x, y, t)$ between the two solutions is nondecreasing.*

Proof. Note first that the infimum $\inf_{(x,y)\in M_1^n \times M_2^n} d(x, y, t)$ is indeed attained at each time t (since one of the two spatial domains is compact and the other flow is proper). Given $\varepsilon > 0$, consider the function $d_\varepsilon(x, y, t) \doteqdot \mathrm{e}^{\varepsilon(1+t)} d(x, y, t)$. Note that $d_\varepsilon(x, y, 0) > \min_{M_1^n \times M_2^n} d(\cdot, \cdot, 0)$. Suppose then that, for some $\varepsilon > 0$, $d_\varepsilon(x, y, t)$ does not remain strictly greater than $\min_{M_1^n \times M_2^n} d(\cdot, \cdot, 0)$. Then there must be some positive time t_0 and some pair of points $(x_0, y_0) \in M_1^n \times M_2^n$ such that $d_\varepsilon(\cdot, \cdot, t) > d_0$ for $t < t_0$ but $d_\varepsilon(x_0, y_0, t_0) = d_0$. At this point,

$$\partial_t d_\varepsilon \leq 0, \quad \nabla d_\varepsilon = 0, \quad \text{and} \quad \nabla^2 d_\varepsilon \geq 0, \tag{2.6}$$

where ∇ is the product connection on $M_1^n \times M_2^n$ induced by those on M_1^n and M_2^n

Given $U, V \in T_x M_1^n \cong (dX_1)_{(x,t)}(T_x M_1^n)$ and $W, Y \in T_y M_2^n \cong (dX_2)_{(x,t)}(T_y M_2^n)$, we have

$$U(d) = -\langle U, w \rangle \tag{2.7}$$

and

$$W(d) = \langle W, w \rangle\,, \tag{2.8}$$

where

$$w\,(x, y, t) \doteqdot \frac{X_2\,(y, t) - X_1\,(x, t)}{d\,(x, y, t)}\,. \tag{2.9}$$

Observe that

$$D_V w = -\frac{1}{d}\left(V - \langle V, w\rangle w\right) \text{ and } D_Y w = \frac{1}{d}\left(Y - \langle Y, w\rangle w\right).$$

Extending U, V to vector fields in a neighborhood of x and extending W, Y to vector fields in a neighborhood of y, we compute

$$\begin{aligned} V\left(U(d)\right) &= -\left\langle D_V U, w\right\rangle - \left\langle U, D_V w\right\rangle \\ &= -\left\langle \nabla_V U, w\right\rangle + \mathrm{II}_1\left(V, U\right)\left\langle \mathsf{N}_1, w\right\rangle + \left\langle U, \frac{1}{d}\left(V - \langle V, w\rangle w\right)\right\rangle \end{aligned}$$

since $D_V U = \nabla_V U - \mathrm{II}_1\left(V, U\right)\mathsf{N}_1$ for any vector field U tangent to M_1^n near x. Thus, for $U, V \in T_x M_1$,

$$\left(\nabla^2 d\right)(V, U) = \mathrm{II}_1\left(V, U\right)\left\langle \mathsf{N}_1, w\right\rangle + \frac{1}{d}\left(\langle U, V\rangle - \langle U, w\rangle\langle V, w\rangle\right).$$

Taking the trace then yields

$$\Delta_1 d = \mathrm{H}_1\left\langle \mathsf{N}_1, w\right\rangle + \frac{1}{d}\left(n - \left|w^{\top_1}\right|^2\right), \tag{2.10}$$

where $w^{\top_1}$ denotes the projection of w onto $(dX_1)_{(x,t)} T_x M_1$.

Similarly, for $W, Y \in T_y M_2$,

$$\left(\nabla^2 d\right)(Y, W) = -\mathrm{II}_2\left(Y, W\right)\left\langle \mathsf{N}_2, w\right\rangle + \frac{1}{d}\left(\langle W, Y\rangle - \langle W, w\rangle\langle Y, w\rangle\right),$$

and

$$\Delta_2 d = -\mathrm{H}_2\left\langle \mathsf{N}_2, w\right\rangle + \frac{1}{d}\left(n - \left|w^{\top_2}\right|^2\right), \tag{2.11}$$

where $w^{\top_2}$ denotes the projection of w onto $(dX_2)_{(y,t)} T_y M_2$.

By (2.7) and (2.8), $w^{\top_1} = 0$ and $w^{\top_2} = 0$ at (x_0, y_0, t_0). We may choose the orientation of the normals at this point so that $\mathsf{N}_1 = \mathsf{N}_2 = w$, which yields (at$(x_0, y_0, t_0)$)

$$\Delta_1 d = \mathrm{H}_1 + \frac{n}{d} \text{ and } \Delta_2 d = -\mathrm{H}_2 + \frac{n}{d}.$$

Next we compute, for $U \in T_x M_1$ and $W \in T_y M_2$, the "cross-term"

$$\begin{aligned} \left(\nabla^2 d\right)(U, W) &= U\left(W(d)\right) \\ &= \left\langle W, D_U w\right\rangle \\ &= -\frac{1}{d}\left\langle W, U - \langle U, w\rangle w\right\rangle \\ &= -\frac{1}{d}\left(\langle W, U\rangle - \langle U, w\rangle\langle W, w\rangle\right). \end{aligned} \tag{2.12}$$

Since the tangent planes at $X(x_0, t_0)$ and $Y(y_0, t_0)$ are parallel, and orthogonal to $w(x_0, y_0, t_0)$, we can trace this with respect to an orthonormal basis $\{e_i\}_{i=1}^n$ for the subspace $(dX_1)_{(x_0,t_0)} T_{x_0} M_1 = w^{\perp}(x_0, y_0, t_0) = (dX_2)_{(y_0,t_0)} T_{y_0} M_2$ to obtain

$$\sum_{i=1}^n e_i(e_i d) = -\frac{n}{d}.$$

We thus find, at the point (x_0, y_0, t_0),

$$\sum_{i=1}^{n}(e_i^1 + e_i^2)\left((e_i^1 + e_i^2)d\right) = \Delta_1 d + \Delta_2 d + 2\sum_{i=1}^{n} e_i^1\left(e_i^2 d\right) \tag{2.13}$$
$$= \mathrm{H}_1 - \mathrm{H}_2\ .$$

On the other hand,

$$\partial_t d\,(x, y, t) = \frac{1}{d}\left\langle X_1(x,t) - X_2(y,t), -\mathrm{H}_1\,\mathsf{N}_1 + \mathrm{H}_2\,\mathsf{N}_2\right\rangle \tag{2.14}$$
$$= \left\langle w, -\mathrm{H}_1\,\mathsf{N}_1 + \mathrm{H}_2\,\mathsf{N}_2\right\rangle .$$
$$= \mathrm{H}_2 - \mathrm{H}_1 \tag{2.15}$$

with the final inequality holding at the point (x_0, y_0, t_0). We conclude that

$$0 \geq \partial_t\left(\mathrm{e}^{\varepsilon(1+t)}d\right)|_{(x_0,y_0,t_0)} \tag{2.16}$$
$$= \varepsilon \mathrm{e}^{\varepsilon(1+t_0)}d + \mathrm{e}^{\varepsilon(1+t_0)}\left\langle w, -\mathrm{H}_1\,\mathsf{N}_1 + \mathrm{H}_2\,\mathsf{N}_2\right\rangle$$
$$= \varepsilon \mathrm{e}^{\varepsilon(1+t_0)}d + \mathrm{e}^{\varepsilon(1+t_0)}\left(\mathrm{H}_1 - \mathrm{H}_2\right)$$
$$\geq \varepsilon d_0$$

at the point (x_0, y_0, t_0), which is absurd. □

This argument can also be used to prevent a solitary mean curvature flow from forming self-intersections through the coming together of points at a large intrinsic distance, though an additional argument is required to prevent cusps from appearing out of a single point.

Lemma 2.8. *If $X : M^n \to \mathbb{R}^{n+1}$ is an immersion of a compact manifold M^n, and $\max_{M^n}|\mathrm{II}| \leq K$, then*

$$|X(y) - X(x)| \geq \frac{2}{K}\sin\left(\frac{K\ell(x,y)}{2}\right) \tag{2.17}$$

for all x, y with $\ell(x,y) \leq \frac{\pi}{K}$, where $\ell(x,y)$ is the intrinsic distance on M^n from x to y.

Proof. Let x and y be two points of M^n, joined by a unit speed, minimizing geodesic segment $\gamma : [-\ell/2, \ell/2] \to M^n$ of intrinsic length $\ell \leq \frac{\pi}{K}$, and let $\theta(s)$ denote the angle between $(X \circ \gamma)'(s)$ and $\mathsf{T}_0 \doteqdot (X \circ \gamma)'(0)$. Observe that

$$\left|\frac{d}{ds}(X \circ \gamma)'(s) \cdot \mathsf{T}_0\right| = \left|\mathrm{II}_{\gamma(s)}(\gamma'(s), \gamma'(s))\,\mathsf{N}(\gamma(s)) \cdot \mathsf{T}_0\right|$$
$$\leq K$$

and hence,

$$|\theta(s)| \leq K|s| \leq \frac{K\ell}{2} \leq \frac{K\pi}{2}.$$

Thus,

$$\begin{aligned}
|X(x)-X(y)| &\geq (X(x)-X(y))\cdot \mathsf{T}_0\\
&= \int_{-\ell/2}^{\ell/2}(X\circ\gamma)'(s)\cdot \mathsf{T}_0 \, ds\\
&= \int_{-\ell/2}^{\ell/2}\cos(\theta(s))\, ds\\
&\geq \int_{-\ell/2}^{\ell/2}\cos(Ks)\, ds\\
&= \frac{2}{K}\sin\left(\frac{K\ell}{2}\right). \qquad \square
\end{aligned}$$

Applying the argument of Proposition 2.7 outside neighbourhood of the diagonal in which Lemma 2.8 applies, yields the following.

Proposition 2.9 (Embeddedness is preserved). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a mean curvature flow on a compact domain M^n. If $X(\cdot,0)$ is an embedding, then so is $X(\cdot,t)$ for each $t\in[0,T)$.*

2.2.2 *The first variation of area*

Recall that, on any submanifold $X : M^n \to \mathbb{R}^{n+1}$, the area of any compact subset $K \subset M^n$ is defined by

$$\operatorname{area}(K,X) \doteqdot \int_K d\mu \doteqdot \sum_\alpha \int_{x_\alpha(U_\alpha)} (x_\alpha^{-1})^*\left(\rho_\alpha\sqrt{\det g_\alpha}\right) dx\,,$$

where $\{(U_\alpha, x_\alpha)\}_\alpha$ is any locally finite covering of K, $\{\rho_\alpha\}_\alpha$ is any subordinate partition of unity, dx is the Lebesgue measure on $\mathbb{R}^n$, and g_α is the component matrix of the induced metric g with respect to the α-th coordinate chart. If $\{X_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a one-parameter family of immersions of M^n with $X_0 = X$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} X_\varepsilon = \vec{F} \in \Gamma(NM^n)$, then, with respect to any coordinate chart,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^\varepsilon_{ij} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left\langle \partial_i X_\varepsilon\,,\, \partial_j X_\varepsilon\right\rangle = 2\left\langle \nabla^\perp_i \vec{F}\,,\, \partial_j X\right\rangle = -2\left\langle \vec{F}\,,\, \vec{\mathrm{II}}_{ij}\right\rangle,$$

and hence

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\sqrt{\det g_\varepsilon} = \frac{1}{2}\sqrt{\det g}\,\mathrm{tr}_g\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} g^\varepsilon\right) = -\sqrt{\det g}\left\langle \vec{F}\,,\, \vec{\mathrm{H}}\right\rangle.$$

We thus obtain the FIRST VARIATION FORMULA (FOR AREA):

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\operatorname{area}(K,X_\varepsilon) = -\int_K \left\langle \vec{F}\,,\, \vec{\mathrm{H}}\right\rangle d\mu\,.$$

We conclude that,

Proposition 2.10. *along a mean curvature flow*[3] $X : M^n \times I \to \mathbb{R}^{n+1}$,

$$\frac{d}{dt}\,\mathrm{area}(K,\cdot) = -\int_K |\vec{\mathrm{H}}|^2\, d\mu \qquad (2.18)$$

for any compact $K \subset M^n$, *where* $\mathrm{area}(K,t) \doteqdot \mathrm{area}(K, X_t)$.

[3] We invite the reader to verify that this formula holds in any codimension and in any ambient space.

In particular, the area of a compact hypersurface is nonincreasing under mean curvature flow.

2.2.3 *Evolution of the normal*

By Exercise 1.4,

Proposition 2.11. *if* $X : M^n \times I \to \mathbb{R}^{n+1}$ *evolves by mean curvature, then, for any (local) choice of unit normal* N,

$$D_t\,\mathsf{N} = \nabla \mathrm{H}.$$

2.2.4 *The first variation of volume*

If our hypersurface is the boundary $\partial\Omega$ of a bounded open subset $\Omega \subset \mathbb{R}^{n+1}$, then the divergence theorem ensures that

$$\mathrm{volume}(\Omega) = \int_\Omega d\mathcal{L} = \frac{1}{n+1}\int_\Omega \mathrm{div}\, X d\mathcal{L} = \frac{1}{n+1}\int_{\partial\Omega} \langle X\,,\,\mathsf{N}\rangle\, d\mu\,,$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^{n+1}$, X is the position vector-field on $\mathbb{R}^{n+1}$, and N is the outward unit normal field to $\partial\Omega$. Thus, if $\{\partial\Omega_\varepsilon\}_{\varepsilon\in I}$ are a one-parameter family of (bounded) boundaries with $\Omega_0 = \Omega$ and normal variation field $\vec{F} = -F\,\mathsf{N}$, then (expressing the integral with respect to a covering by coordinate charts dominating a partition of unity) we find that

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{volume}(\Omega_\varepsilon) &= \frac{1}{n+1}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_{\partial\Omega_\varepsilon}\langle X\,,\,\mathsf{N}\rangle\, d\mu\\ &= -\frac{1}{n+1}\int_{\partial\Omega}\left(F - \langle X\,,\,\nabla F\rangle + \langle X\,,\,\mathsf{N}\rangle\,\mathrm{H}\,F\right)\, d\mu\,.\end{aligned}$$

With the divergence theorem in mind, we rewrite

$$\begin{aligned}\langle X\,,\,\nabla F\rangle &= \left\langle X^\top\,,\,\nabla F\right\rangle\\ &= \mathrm{div}(FX^\top) - F\,\mathrm{div}\,X^\top\,.\end{aligned}$$

Since

$$\begin{aligned}F\,\mathrm{div}\,X^\top &= F\,\mathrm{tr}_{T\partial\Omega}\, D(X - \langle X\,,\,\mathsf{N}\rangle\,\mathsf{N})\\ &= F(n - \langle X\,,\,\mathsf{N}\rangle\,\mathrm{H})\,,\end{aligned}$$

we arrive at the FIRST VARIATION FORMULA (FOR ENCLOSED VOLUME):

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \text{volume}(\Omega_\varepsilon) = -\int_{\partial\Omega} F\, d\mu\,.$$

In particular,

Proposition 2.12. *if $K \subset \mathbb{R}^{n+1}$ is compact and $\{\Omega_t\}_{t\in I}$ are a family of open sets for which the hypersurfaces $\{\partial\Omega_t \cap K\}_{t\in I}$ evolve by mean curvature, then*

$$\frac{d}{dt}\,\text{volume}(\Omega_t \cap K) = -\int_{\partial\Omega_t\cap K} \mathrm{H}\, d\mu\,. \tag{2.19}$$

2.2.5 *The Jacobi equation*

An important equation related to the mean curvature flow is the linear equation

$$(\partial_t - \Delta)u = |\mathrm{II}|^2 u\,. \tag{2.20}$$

We will refer to (2.20) as the JACOBI EQUATION (for mean curvature flow). It arises as the equation satisfied by the normal variation of a smooth family of solutions to mean curvature flow.

Lemma 2.13. *Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a mean curvature flow and $\{X_\varepsilon : M \times I \to \mathbb{R}^{n+1}\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ a one-parameter family of mean curvature flows with $X_0 = X$. The normal component,*

$$v \doteqdot \left\langle \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X_\varepsilon\,,\, \mathsf{N}\right\rangle,$$

of the variation field $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} X_\varepsilon$ is a solution to the Jacobi equation (2.20).

2.2.6 *Evolution of the mean curvature*

Given a mean curvature flow $X : M^n \times I \to \mathbb{R}^{n+1}$, consider the one-parameter family of mean curvature flows $\{X_\varepsilon : M^n \times I \to \mathbb{R}^{n+1}\}_{\varepsilon\in I_\varepsilon}$ defined by $X_\varepsilon(x,t) \doteqdot X(x, t-\varepsilon)$. The normal component of the variation field is

$$\left\langle \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X_\varepsilon\,,\, \mathsf{N}\right\rangle = \mathrm{H}\,.$$

We thus obtain:

Proposition 2.14. *If $X : M^n \times I \to \mathbb{R}^{n+1}$ evolves by mean curvature, then*

$$(\partial_t - \Delta)\,\mathrm{H} = |\mathrm{II}|^2\,\mathrm{H}\,. \tag{2.21}$$

The maximum principle then immediately implies the following.

Corollary 2.15. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a mean curvature with M^n compact. If $\mathrm{H}(\cdot,0) \geq 0$, then $\mathrm{H} \geq 0$. In fact, $\mathrm{H}(\cdot,t) > 0$ for all $t > 0$.*[4]

[4] Indeed, any connected (possibly noncompact) mean curvature flow satisfying $\mathrm{H} \geq 0$, must satisfy either $\mathrm{H} > 0$ at interior times, or $\mathrm{H} \equiv 0$.

2.2.7 Evolution of the second fundamental form

By Exercise 1.5,

$$\nabla_t \mathrm{II} = \nabla^2 \mathrm{H} + \mathrm{H}\,\mathrm{II}^2.$$

Tracing Simons' commutation formula (1.15) yields

$$\nabla^2 \mathrm{H} = \Delta \mathrm{II} + |\mathrm{II}|^2 \mathrm{II} - \mathrm{II}^2\,\mathrm{H}\,.$$

We thus obtain the following evolution equation for the second fundamental form.

Proposition 2.16. *If $X : M^n \times I \to \mathbb{R}^{n+1}$ evolves by mean curvature, then*

$$(\nabla_t - \Delta)\mathrm{II} = |\mathrm{II}|^2 \mathrm{II}. \tag{2.22}$$

2.3 Global-in-space Bernstein estimates and long-time existence

The evolution equation for II immediately yields[5] an evolution equation for $|\mathrm{II}|^2$:

$$\begin{aligned}(\partial_t - \Delta)|\mathrm{II}|^2 &= 2g((\nabla_t - \Delta)\mathrm{II}, \mathrm{II}) - 2|\nabla \mathrm{II}|^2 \\ &= 2|\mathrm{II}|^4 - 2|\nabla \mathrm{II}|^2.\end{aligned}$$

[5] The ease with which this equation is established is a return on our investment in constructing the time-dependent connection. If you are skeptical, try Exercise 2.1.

In order to establish evolution equations for derivatives of II, we shall need to commute the heat operator with the covariant differential. Since only "rough" evolution equations will be required, the following lemma will be sufficient.

Let us denote by $S * T$ any tensor which is a linear combination of metric contractions of the tensor product of S and T (of the same type).

Lemma 2.17. *Along a mean curvature flow $X : M^n \times I \to \mathbb{R}^{n+1}$,*

$$[\nabla_t - \Delta, \nabla]T = \mathrm{II} * \mathrm{II} * \nabla T + \mathrm{II} * \nabla \mathrm{II} * T.$$

From this, we find that

$$\begin{aligned}(\partial_t - \Delta)|\nabla \mathrm{II}|^2 &= 2g((\nabla_t - \Delta)\nabla \mathrm{II}, \nabla \mathrm{II}) - 2|\nabla^2 \mathrm{II}|^2 \\ &= 2g(\nabla(\nabla_t - \Delta)\mathrm{II} + \mathrm{II} * \mathrm{II} * \nabla \mathrm{II} + \mathrm{II} * \nabla \mathrm{II}, \nabla \mathrm{II}) \\ &\quad - 2|\nabla^2 \mathrm{II}|^2 \\ &= \mathrm{II} * \mathrm{II} * \nabla \mathrm{II} * \nabla \mathrm{II} - 2|\nabla^2 \mathrm{II}|^2.\end{aligned}$$

If $|\mathrm{II}|$ remains bounded on the time interval $[0, T]$, then we can estimate

$$(\partial_t - \Delta)|\nabla \mathrm{II}|^2 \le C|\nabla \mathrm{II}|^2,$$

where C depends only on n and the bound for $|\mathrm{II}|$. The ODE comparison principle then implies that $|\nabla \mathrm{II}|^2$ grows at most exponentially on $[0, T]$:

$$|\nabla \mathrm{II}|^2 \le \max_{t=0} |\nabla \mathrm{II}|^2 \mathrm{e}^{CT}.$$

This estimate takes a more natural form if we exploit its scale invariance: since $|\mathrm{II}|$ scales (under parabolic rescaling of our mean curvature flow) like the inverse square of distance, whereas t scales as distance squared, the constant CT will be scale invariant. If we introduce the scale parameter $r = \sqrt{T}$ and assume that $|\mathrm{II}| \leq Kr^{-1}$ for $t \in [0, r^2]$ (a scale-invarant assumption), then the estimate becomes

$$|\nabla \mathrm{II}|^2 \leq C_1 \max_{t=0} |\nabla \mathrm{II}|^2,$$

where C_1 depends only on K and n.

We can also obtain a time-interior version of this estimate: consider, for some to-be-determined constant a, the combination

$$Q \doteqdot 2t|\nabla \mathrm{II}|^2 + a|\mathrm{II}|^2.$$

Observe that

$$\begin{aligned}(\partial_t - \Delta)Q &= 2|\nabla \mathrm{II}|^2 + 2t(\partial_t - \Delta)|\nabla \mathrm{II}|^2 + a(\partial_t - \Delta)|\mathrm{II}|^2 \\ &\leq 2|\nabla \mathrm{II}|^2 + 2tC_1|\mathrm{II}|^2|\nabla \mathrm{II}|^2 + a(C_0|\mathrm{II}|^3 - 2|\nabla \mathrm{II}|^2) \\ &= 2(1 + C_1 t|\mathrm{II}|^2 - a)|\nabla \mathrm{II}|^2 + aC_0|\mathrm{II}|^4.\end{aligned}$$

If we know that $|\mathrm{II}|$ is bounded by Kr^{-2} on $M \times [0, r^2]$, then

$$(\partial_t - \Delta)Q \leq 2(1 + C_1 K - a)|\nabla \mathrm{II}|^2 + aC_0 K^3 r^{-4}$$

Thus, if we choose $a = 1 + C_1 K$, then the ODE comparison principle yields

$$t|\nabla \mathrm{II}|^2 \leq Q \leq \max_{t=0} Q + aC_0 K^3 r^{-4} t \leq aK^2(1 + C_0 K) r^{-2}.$$

That is,

$$|\nabla \mathrm{II}| \leq \frac{Dr^{-1}}{\sqrt{t}},$$

where $D^2 \doteqdot aK^2(1 + C_0 K)$. This is another manifestation of the diffusive nature of the mean curvature flow: even if the curvature is arbitrarily rough at the initial time, it becomes much more regular only a short-time later.

An inductive extension of this argument yields the following estimates.

Proposition 2.18 ((Global-in-space) Bernstein estimates[6]). *For every $n \in \mathbb{N}$, $K < \infty$ and $m \in \mathbb{N}$, there exists $C_m < \infty$ with the following property. Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a mean curvature flow on a compact manifold M^n. If*

$$|\mathrm{II}_{(x,t)}| \leq Kr^{-1} \text{ for all } (x, t) \in M^n \times [0, r^2],$$

then

$$|\nabla^m \mathrm{II}_{(x,t)}| \leq C_m \max_{M \times \{0\}} |\nabla^m \mathrm{II}| \text{ for all } (x, t) \in M^n \times [0, r^2]$$

[6] Ecker and Huisken, "Mean curvature evolution of entire graphs".

and

$$|\nabla^m \mathrm{II}_{(x,t)}| \le \frac{C_m r^{-1}}{t^{\frac{m}{2}}} \quad \textit{for all} \quad (x,t) \in M^n \times [0, r^2]\,.$$

A fundamental application of the global-in-space Bernstein estimates is the following characterization of finite time singularities.

Theorem 2.19 (Long-time existence)**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a* MAXIMAL[7] *mean curvature flow on a compact manifold M^n. If $T < \infty$, then*

$$\limsup_{t\to T} \max_{M\times\{t\}} |\mathrm{II}| = \infty\,.$$

[7] I.e. there is no mean curvature flow $X' : M^n \times [0,T') \to \mathbb{R}^{n+1}$ with $T' > T$ such that $X'(x,t) = X(x,t)$ for all $t < T$.

Sketch of the proof. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ with $T < \infty$ be a maximal mean curvature flow on a compact manifold M^n and suppose, contrary to the claim, that

$$|\mathrm{II}| \le K \;\text{ on }\; M^n \times [0,T)\,.$$

By the Bernstein estimates, we also have bounds on $M^n \times [0,T)$ for $\nabla^m \mathrm{II}$ for all m. These geometric estimates can be converted, by an inductive argument, to estimates in C^k for the immersions X_t in any local coordinate chart. The only subtlety is the $k = 0$ and $k = 1$ cases; to control these terms, we observe that, for any $x \in M^n$ and any $v \in T_x M^n$,

$$\left|\frac{d}{dt}\log\left(g_{(x,t)}(v,v)\right)\right| = \left|\frac{2\mathrm{H}\mathrm{II}_{(x,t)}(v,v)}{g_{(x,t)}(v,v)}\right| \le C\,.$$

Integrating, we find that $g_{(x,t)}$ remains uniformly equivalent to $g_{(x,0)}$ under the evolution. This gives a uniform estimate in C^1 and also a uniform lower bound for $|dX|$. A displacement estimate is obtained by integrating the flow equation:

$$\begin{aligned} |X(x,t) - X(x,0)| &= \left|\int_0^t \partial_t X(x,\tau)\,d\tau\right| \\ &= \left|\int_0^t \vec{\mathrm{H}}(x,\tau)\,d\tau\right| \\ &\le \int_0^t |\mathrm{H}(x,\tau)|\,d\tau \\ &\le Kt \le KT\,. \end{aligned}$$

Now cover M^n by finitely many compact sets K_α which each lie to the interior of some coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. Due to the uniform estimates in C^k for each k, the Arzelá–Ascoli theorem implies that, for any sequence of times $t_j \to T$, we can find, for each compact set K_α, a subsequence of times such that the restriction of X to the domain of the chart ϕ_α converges uniformly on K_α in the smooth topology to some limit. Taking appropriate subsequences, we can find limits along the same sequence of times which agree on overlaps. Due to the uniform lower bound for dX, these limits thus define a global smooth immersion of M^n into $\mathbb{R}^{n+1}$, which we now evolve by the mean curvature flow using our short-time existence theorem. The so extended family of immersions is smooth at each time and it is also smooth in time across the jump time T since time derivatives of X are related to spatial derivatives via the mean curvature flow equation. But this is impossible since our original mean curvature flow was assumed to be maximal. □

Proposition 2.20. *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be the maximal mean curvature flow on a compact manifold M^n. If $T < \infty$, then*

$$\max_{M^n \times \{t\}} |\mathrm{II}| \geq \frac{C}{\sqrt{T-t}},$$

where C depends only on n.

Proof. Since $\limsup_{t \nearrow T} \max_{M^n \times \{t\}} |\mathrm{II}| = \infty$ and

$$(\partial_t - \Delta)|\mathrm{II}|^2 \leq c(n)|\mathrm{II}|^4,$$

the claim follows from the ODE comparison principle. □

2.4 *Local-in-space Bernstein estimates and the compactness theorem*

By introducing spatial cutoff functions into the above argument, one may derive the following local-in-space estimates.

Proposition 2.21 (Fully local Bernstein estimates[8]). *For every $n \in \mathbb{N}$, $K < \infty$ and $m \in \mathbb{N}$, there exists $C_m < \infty$ with the following property. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a mean curvature flow on a manifold M^n. If $[t - r^2, t] \subset I$, X_s is proper with respect to the ambient ball $B_r(p)$ for each $s \in [t - r^2, t]$, and $\sup_{B_r(p) \times [t-r^2, t]} |\mathrm{II}| \leq K r^{-2}$, then*

$$|\nabla^m \mathrm{II}_{(x,t)}| \leq C_m r^{-m-1}.$$

[8] Ecker and Huisken, "Interior estimates for hypersurfaces moving by mean curvature".

Combining these estimates with the Cheeger–Gromov compactness theorem for Riemannian manifolds with bounded geometry yields the following compactness theorem for mean curvature flows under modest geometric assumptions.

Theorem 2.22 (Compactness of the space of mean curvature flows with bounded geometry). *Let $\{(X_k : M_k \times I_k \to \mathbb{R}^{n+1}, x_k)\}_{k \in \mathbb{N}}$ be a sequence of pointed mean curvature flows and (o, t_0) a point in spacetime $\mathbb{R}^{n+1} \times \mathbb{R}$. Suppose that the following conditions hold for every k:*

1. $X_k(x_k, t_0) = o$, $t_0 \in I \doteqdot [\alpha, \omega] \subset I_k$ *and* $X_k(\cdot, t)$ *is proper with respect to the ambient ball* $B_r(o)$.

2. $\max_{\overline{B}_r(o_k)\times I} |\mathrm{II}_{X_k}| \le C < \infty$.

There exists a pointed mean curvature flow $(X : M^n \times I \to \mathbb{R}^{n+1}, x_0)$ *with* $X(x, t_0) = 0$ *such that, after passing to a subsequence, the mean curvature flows flows* $\big(X_k|_{\overline{B_{\frac{r}{2}}(o_k,\alpha)}\times I_k} : \overline{B_{\frac{r}{2}}(o_k,\alpha)} \times I_k \to \mathbb{R}^{n+1}, o_k\big)$ *converge uniformly in the smooth sense to the mean curvature flow* $\big(X|_{\overline{B_{\frac{r}{2}}(o,0)}\times I} : \overline{B_{\frac{r}{2}}(o,0)} \times I \to \mathbb{R}^{n+1}, o\big)$. *That is, there exists a sequence of diffeomorphisms* $\phi_k : \overline{B_{\frac{r}{2}}(o,0)} \to M_k$ *with* $\phi_k(o) = o_k$ *such that* $\phi_k^* X_k \to X$ *uniformly in the smooth topology.*[9]

By taking limits along diagonal subsequences, one can obtain a complete limit under global bounds on the curvature. Note though that the limit can lose or gain topology, and different subsequences can take different limits. Compact limits are better behaved, however (as in this case the convergence is necessarily uniform).

2.5 *Estimates for the curvature*

Theorems 9.15 and 9.18 provide estimates to all orders for as long as the curvature remains bounded. In some situations, the curvature itself can be bounded in terms of lower order data. The following clever argument, due to Chou,[10] provides bounds for the mean curvature under a uniform "starshapedness" condition.[11]

Proposition 2.23. *If a compact mean curvature flow* $X : M^n \times [0, T] \to \mathbb{R}^{n+1}$ *satisfies*

$$\vartheta \le (X - p)\cdot \mathsf{N} \le \Theta,$$

then it also satisfies

$$|\mathrm{H}| \le \max\left\{2\vartheta^{-1}\Theta \max_{M^n\times\{0\}} |\mathrm{H}|, 8n\vartheta^{-2}\Theta\right\}$$

and

$$|\mathrm{H}| \le \max\left\{\frac{\Theta}{2t}, 8n\vartheta^{-1}\right\}.$$

Proof. The key observation is that the reaction term in the evolution equation

$$(\partial_t - \Delta)(X-p)\cdot\mathsf{N} = |\mathrm{II}|^2 (X-p)\cdot \mathsf{N} - 2\,\mathrm{H}$$

can be used to absorb that in

$$(\partial_t - \Delta)\,\mathrm{H} = |\mathrm{II}|^2\,\mathrm{H},$$

[9] This is an "intrinsic" notion of convergence, as we only consider neighbourhoods of points in the *domain* space. It has the advantage of providing information even when the flows are only immersed, or lose embeddedness in the limit, but the disadvantage that it loses extrinsic information (e.g. discarding extrinsically nearby components). Under additional conditions (anything sufficient to ensure that the evolving hypersurfaces along the sequence bound regions which do not degenerate in the limit), a similar compactness statement can be made with convergence holding in the "extrinsic sense" (where neighbourhoods of points in the *target* space are considered).

[10] Tso (Chou), "Deforming a hypersurface by its Gauss–Kronecker curvature".

[11] Cf. Ecker and Huisken, "Mean curvature evolution of entire graphs".

with a fortuitously good term, $-2\mathrm{H}$, to spare. Indeed, wherever $v \doteqdot (X-p)\cdot \mathsf{N} \geq \vartheta > 0$, we have

$$
\begin{aligned}
(\partial_t - \Delta)\frac{\mathrm{H}}{v-\frac{\vartheta}{2}} &= \frac{1}{v-\frac{\vartheta}{2}}\left((\partial_t-\Delta)\,\mathrm{H} - \frac{\mathrm{H}}{v-\frac{\vartheta}{2}}(\partial_t-\Delta)v\right)\\
&\quad + 2\nabla\frac{\mathrm{H}}{v-\frac{\vartheta}{2}}\cdot\frac{\nabla v}{v-\frac{\vartheta}{2}}\\
&= \frac{\mathrm{H}}{v-\frac{\vartheta}{2}}\left(2\frac{\mathrm{H}}{v-\frac{\vartheta}{2}} - \frac{\vartheta}{2}\frac{|\mathrm{II}|^2}{v-\frac{\vartheta}{2}}\right) + 2\nabla\frac{\mathrm{H}}{v-\frac{\vartheta}{2}}\cdot\frac{\nabla v}{v-\frac{\vartheta}{2}}.
\end{aligned}
$$

Estimating $|\mathrm{II}|^2 \geq \frac{1}{n}\mathrm{H}^2$, we find, at any positive interior maximum of $\frac{\mathrm{H}}{v-\frac{\vartheta}{2}}$, that

$$
\frac{\vartheta}{2n}\frac{\mathrm{H}^2}{v-\frac{\vartheta}{2}} \leq 2\frac{\mathrm{H}}{v-\frac{\vartheta}{2}} \tag{2.23}
$$

and hence, at such a point,

$$
\frac{\mathrm{H}}{v-\frac{\vartheta}{2}} \leq 8n\vartheta^{-2}.
$$

At a nonpositive interior minimum, we still obtain (2.23), which now implies $\mathrm{H} \geq 0$.

On the other hand, if no new minima or maxima of $\frac{|\mathrm{H}|}{v-\frac{\vartheta}{2}}$ form, then

$$
\Theta^{-1}|\mathrm{H}| \leq \frac{|\mathrm{H}|}{v-\frac{\vartheta}{2}} \leq \max_{M^n\times\{0\}}\frac{|\mathrm{H}|}{v-\frac{\vartheta}{2}} \leq 2\vartheta^{-1}\max_{M^n\times\{0\}}|\mathrm{H}|.
$$

The first claim is proved.

To establish the second claim, we instead consider instead the function $\frac{t\,\mathrm{H}}{v-\frac{\vartheta}{2}}$, which at a positive interior maximum satisfies

$$
\begin{aligned}
0 \leq (\partial_t-\Delta)\frac{t\,\mathrm{H}}{v-\frac{\vartheta}{2}} &= t(\partial_t-\Delta)\frac{\mathrm{H}}{v-\frac{\vartheta}{2}} + \frac{\mathrm{H}}{v-\frac{\vartheta}{2}}\\
&= \frac{\mathrm{H}}{v-\frac{\vartheta}{2}}\left(2\frac{t\,\mathrm{H}}{v-\frac{\vartheta}{2}} - \frac{\vartheta}{2}\frac{t|\mathrm{II}|^2}{v-\frac{\vartheta}{2}} + 1\right)
\end{aligned}
$$

and hence

$$
\frac{\vartheta}{2n}\frac{t\,\mathrm{H}^2}{v-\frac{\vartheta}{2}} \leq 2\frac{t\,\mathrm{H}}{v-\frac{\vartheta}{2}} + 1. \tag{2.24}
$$

If $\mathrm{H} \geq 8n\vartheta^{-1}$, then we may conclude that

$$
\frac{t\,\mathrm{H}}{v-\frac{\vartheta}{2}} \leq \frac{1}{2}.
$$

At a negative interior minimum, we still obtain the inequality (2.24), which now yields

$$
\frac{t\,\mathrm{H}}{v-\frac{\vartheta}{2}} \geq -\frac{1}{2}.
$$

Since $\max_{M^n\times\{0\}}\frac{t|\mathrm{H}|}{v-\frac{\vartheta}{2}} = 0$, the second claim is proved. $\square$

Introducing a cut-off function yields the following local-in-space estimates, which we state in a streamlined and scale invariant form.[12]

Proposition 2.24. *Suppose that the mean curvature flow* $X : M^n \times I \to \mathbb{R}^{n+1}$ *is properly defined*[13] *in* $B_{\Lambda r}(p) \times (\omega - \Lambda^2 K^2 r^2, \omega]$. *If*

$$(X - p) \cdot \mathsf{N} \geq \vartheta r > 0 \ \ in \ \ B_{\Lambda r}(p) \times (\omega - \Lambda^2 K^2 r^2, \omega],$$

then

$$\sup_{B_r(p)\times(\omega - K^2r^2, \omega]} |\mathrm{H}| \leq C(n, \vartheta, \Lambda, L) r^{-1}, \tag{2.25}$$

where $L \doteqdot \sup_{B_{2r}(p)\times\{\omega - 4K^2r^2\}} |\mathrm{H}|$, *and*

$$\sup_{B_r(p)\times(\omega - K^2r^2, \omega]} |\mathrm{H}| \leq C(n, \vartheta, \Lambda, K) r^{-1}. \tag{2.26}$$

Proof. Without loss of generality, we may assume that $r = 1$, $p = 0$ and $\omega = 0$. We introduce the cut-off function

$$\eta \doteqdot 1 - |X|^2$$

and consider, in B_1,

$$\begin{aligned}(\partial_t - \Delta)\frac{\eta\,\mathrm{H}}{v - \frac{\vartheta}{2}} = {}& \frac{1}{v - \frac{\vartheta}{2}}\left(\eta(\partial_t - \Delta)\,\mathrm{H} + \mathrm{H}(\partial_t - \Delta)\eta - \frac{\eta\,\mathrm{H}}{v - \frac{\vartheta}{2}}(\partial_t - \Delta)v\right)\\ & - 2\frac{\nabla\eta\cdot\nabla\mathrm{H}}{v - \frac{\vartheta}{2}} + 2\nabla\frac{\eta\mathrm{H}}{v - \frac{\vartheta}{2}}\cdot\frac{\nabla v}{v - \frac{\vartheta}{2}}.\end{aligned}$$

At a positive interior maximum of $\frac{\eta\,\mathrm{H}}{v - \frac{\vartheta}{2}}$, we find that

$$\begin{aligned}-\frac{\nabla\eta\cdot\nabla\mathrm{H}}{v - \frac{\vartheta}{2}} &= \frac{\eta\,\mathrm{H}}{v - \frac{\vartheta}{2}}\left(\frac{\nabla\eta}{\eta} - \frac{\nabla v}{v - \frac{\vartheta}{2}}\right)\frac{\nabla\eta}{\eta}\\ &= \frac{\mathrm{H}}{v - \frac{\vartheta}{2}}\sum_{i=1}^n\left(2 + \frac{\eta\kappa_i}{v - \frac{\vartheta}{2}}\right)\frac{X_i^2}{\eta}\\ &\leq \frac{\mathrm{H}}{v - \frac{\vartheta}{2}}\left(2 + \frac{\eta|\mathrm{II}|}{v - \frac{\vartheta}{2}}\right)\frac{|X|^2}{\eta}.\end{aligned}$$

Since

$$(\partial_t - \Delta)\eta = 2n,$$

we deduce that

$$\begin{aligned}\tfrac{\vartheta^2}{4}\frac{\eta^2|\mathrm{II}|^2}{(v - \frac{\vartheta}{2})^2} &\leq \tfrac{\vartheta}{2}\frac{\eta^2|\mathrm{II}|^2}{v - \frac{\vartheta}{2}}\\ &\leq 2\frac{\eta^2\,\mathrm{H}}{v - \frac{\vartheta}{2}} + 2n\eta + 2\left(2 + \frac{\eta|\mathrm{II}|}{v - \frac{\vartheta}{2}}\right)|X|^2\\ &\leq 2\frac{\eta\,\mathrm{H}}{v - \frac{\vartheta}{2}} + 2\frac{\eta|\mathrm{II}|}{v - \frac{\vartheta}{2}} + 2(n+2)\end{aligned}$$

[12] Estimates which are stronger at large scales were established by Lynch, "Convexity and gradient estimates for fully nonlinear curvature flows", Theorem 4.3.

[13] I.e. the map $(X, t) : (x, t) \mapsto (X(x, t), t) \in \mathbb{R}^{n+1} \times \mathbb{R}$ is PROPER with respect to $B_{2r}(p) \times (\omega - 4K^2r^2, \omega]$, in the sense that $(X, t)^{-1}(K) \Subset M^n \times I$ for every compact subset $K \subset B_{2r}(p) \times (\omega - 4K^2r^2, \omega]$.

at such a point. Rearranging, we obtain

$$\frac{\eta|\mathrm{II}|}{v-\frac{\vartheta}{2}}\left(\frac{\vartheta^2}{4}\frac{\eta|\mathrm{II}|}{v-\frac{\vartheta}{2}}-2\right)\le 2\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}+2(n+2)\,. \tag{2.27}$$

If, at this point, $\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}\ge 8\sqrt{n}(1+2\sqrt{n})\vartheta^{-2}$, then

$$\begin{aligned}4\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}&\le\frac{\eta|\mathrm{II}|}{v-\frac{\vartheta}{2}}\left(\frac{\vartheta^2}{4}\frac{\eta|\mathrm{II}|}{v-\frac{\vartheta}{2}}-2\right)\\&\le 2\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}+2(n+2)\end{aligned}$$

and hence

$$\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}\le n+2\,.$$

At a nonpositive interior minimum, we still obtain (2.27). If, at this point, $\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}\le -8\sqrt{n}(1+2\sqrt{n})\vartheta^{-2}$, then (2.27) implies that

$$\frac{\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}\ge -(n+2)\,.$$

The first claim is proved. To establish the second, apply the above argument to the function $\frac{t\eta\,\mathrm{H}}{v-\frac{\vartheta}{2}}$. □

These estimates show that, along a mean curvature flow, uniform starshapedness in some parabolic region implies a uniform bound for its curvature in any smaller region. This is particularly useful when the evolving hypersurfaces are convex[14], since the conditions

$$\vartheta\le(X-p)\cdot\mathsf{N}\le\Theta$$

are automatically satisfied (at all times) whenever the ball $B_\vartheta(p)$ is enclosed by the final timeslice and the ball $B_\Theta(p)$ encloses the initial timeslice, respectively. We thus obtain, in particular, the following rather remarkable estimate.

Proposition 2.25. *Suppose that the boundaries $\{\partial\Omega_t\}_{t\in I}$ of the convex regions $\Omega_t\subset\mathbb{R}^{n+1}$ are properly defined and evolve by mean curvature flow in $B_{\Lambda r}(p)\times(\omega-\Lambda^2K^2r^2,\omega]$. If $B_{\vartheta r}(p)\subset\Omega_\omega$, then*

$$\sup_{B_r(p)\times(\omega-K^2r^2,\omega]}\mathrm{H}\le C(n,\vartheta,\Lambda,K)r^{-1}\,. \tag{2.28}$$

Note that this estimate immediately yields regularity estimates to all orders, via the Bernstein estimates, since the inequality $|\mathrm{II}|^2\le\mathrm{H}^2$ holds whenever $\mathrm{II}\ge 0$. As a consequence, we find that *any sequence of convex mean curvature flows which are properly defined in $B_{2r}\times(-4r^2,0]$ and whose final timeslices pass through the origin and enclose a ball of uniform size has a subsequence which converges, in $B_r\times(-r^2,0]$, uniformly in the smooth topology.*

In fact, an estimate of the form (2.28) also holds when the evolving hypersurfaces are only *almost convex*.[15]

[14] We shall see in Chapter 3 that convexity is preserved under the flow.

[15] This estimate is not particularly useful, however, as (nonzero) lower bounds for the principal curvatures are not preserved by the flow. We will substantially improve it in §5.5.1.

Proposition 2.26. *Suppose that the boundaries $\{\partial\Omega_t\}_{t\in I}$ of the regions $\Omega_t \subset \mathbb{R}^{n+1}$ are properly defined and evolve by mean curvature flow in $B_{\Lambda r}(p)\times(\omega-4\Lambda^2K^2r^2,\omega]$, and that the regions $\Omega_t\cap B_{\Lambda r}(p)$ are connected for all $t\in(\omega-4\Lambda^2K^2r^2,\omega]$. If*

$$\kappa_1 \geq -\delta r^{-1} \text{ in } B_{\Lambda r}(p)\times(\omega-\Lambda^2K^2r^2,\omega] \text{ and } B_{\vartheta r}(p)\subset\Omega_\omega$$

for some $\vartheta<1<\Lambda$ and $\delta<\frac{2\vartheta}{\vartheta^2+\Lambda^2}$, then

$$\sup_{B_r(p)\times(\omega-K^2r^2,\omega]}|\mathrm{H}| \leq C(n,\vartheta,\Lambda,K)r^{-1}. \tag{2.29}$$

Proof. We claim that the flow is uniformly starshaped in $B_{\Lambda r}(p)\times(\omega-\Lambda^2K^2r^2,\omega]$, with a (scale normalized) constant depending only on ϑ, Λ and δ (in which case the claim follows from Proposition (2.24)). To do this, we introduce the (standard) ε-TRUMPET[16] $T_\varepsilon(p,x,r)$, which is the locus of points reached by a rigid circular arc that joins the point x to the circle in $p+(x-p)^\perp$ of radius r about p, and whose tangent line at x intersects the radius of this circle at a distance εr from p.

[16] The MOUTHPIECE is located at the point x and the BELL bounds a disk of radius r centred at p. The aperture of the mouthpiece (equivalently, the outward curvature) is determined by the parameter ε.

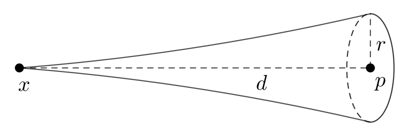


Figure 2.1: An ε-trumpet, $T_\varepsilon(p,x,r)$.

Figure 2.2: Buisine player and religious figure. Unknown Illustrator, *Manuscript of Saint-Esprit*.

Note that the portion of the boundary $\partial\Omega_t$ lying in $B_{\Lambda r}(p)$ is starsphaped about the point p precisely when the segments $T_\varepsilon(p,x,0)$ are contained in Ω_t for all $x\in\partial\Omega_t\cap B_{\Lambda r}(p)$. Now, if the *trumpet* $T_\varepsilon(p,x,\vartheta r)$ happens to be contained in Ω_t for some $x\in\partial\Omega_t\cap B_{\Lambda r}(p)$, then, setting $d \doteqdot |x-p|$, we may estimate

$$\begin{aligned}
\mathsf{N}(x,t)\cdot\frac{x-p}{|x-p|} &\geq \mathsf{N}_{T_\varepsilon(p,x,\vartheta r)}(x)\cdot\frac{x-p}{|x-p|}\\
&= \frac{\varepsilon\vartheta r}{\sqrt{d^2+\varepsilon^2\vartheta^2r^2}}\\
&\geq \frac{\varepsilon}{\sqrt{\vartheta^{-2}\Lambda^2+\varepsilon^2}}.
\end{aligned}$$

So such points are *uniformly* starshaped about p (relative to the scale r) with constant $\frac{\varepsilon\vartheta}{\sqrt{\vartheta^{-2}\Lambda^2+\varepsilon^2}}$.

On the other hand, if some trumpet $T_\varepsilon(p,x,\vartheta r)$, $x\in\partial\Omega_t\cap B_{\Lambda r}(p)$, is *not* contained in Ω_t, then we can find[17] some other trumpet $T_\varepsilon(p,y,\vartheta r)$, $y\in\partial\Omega_t\cap B_{\Lambda r}(p)$, which makes first order contact with $\partial\Omega_t\cap B_{\Lambda r}(p)$ at some interior point x', at which

[17] See Lynch, "Convexity and gradient estimates for fully nonlinear curvature flows", Lemma 4.6.

$$\begin{aligned}
\kappa_1(x',t) &\leq (\kappa_1)_{T_\varepsilon(p,y,\vartheta r)}(x',t)\\
&= -\frac{2\vartheta r}{\vartheta^2r^2+d^2}\frac{(1-\varepsilon)d}{\sqrt{\varepsilon^2\vartheta^2r^2+d^2}}\\
&\leq -\frac{1}{r}\frac{2\vartheta}{\vartheta^2+\Lambda^2}\frac{1-\varepsilon}{\sqrt{1+\varepsilon^2\vartheta^2}}.
\end{aligned}$$

This is in contradiction with our hypothesis if $\varepsilon\leq\varepsilon_0=\varepsilon_0(\vartheta,\Lambda,\delta)$. We conclude that $T_{\varepsilon_0}(x,p,\vartheta r)\subset\Omega_t$ for *all* $x\in\partial\Omega_t$, as desired. □

2.6 Exercises

Exercise 2.1. Suppose that $\{X_t : M^n \to \mathbb{R}^{n+1}\}_{t\in I}$ evolves by mean curvature. Given some coordinate chart $\{x^i : U \to \mathbb{R}\}_{i=1}^n$ for M^n, consider the components

$$g_{ij}(\cdot,t) = \langle \partial_i X_t\,,\, \partial_j X_t\rangle \quad\text{and}\quad \mathrm{II}_{ij}(\cdot,t) = -\left\langle \frac{\partial^2 X_t}{\partial x^i \partial x^j}\,,\, \mathsf{N}_{X_t}\right\rangle$$

of the metric g_{X_t} and second fundamental form II_{X_t} of X_t, where N_{X_t} is a local choice of unit normal field, and the components $g^{ij}(\cdot,t)$ of the dual metric.

(a) Show that

$$0 = \left\langle \mathsf{N}_{X_t}\,,\, \frac{d}{dt}\mathsf{N}_{X_t}\right\rangle \quad\text{and}\quad 0 = \left\langle \partial_i X_t\,,\, \frac{d}{dt}\mathsf{N}_{X_t}\right\rangle + \left\langle D_{\partial_i X_t}\vec{\mathrm{H}}_{X_t}\,,\, \mathsf{N}_{X_t}\right\rangle .$$

(b) Deduce that

$$\frac{d}{dt}\mathsf{N}_{X_t} = \nabla\,\mathrm{H}(\cdot,t),$$

where H is defined by $\vec{\mathrm{H}}_{X_t} = -\,\mathrm{H}(\cdot,t)\,\mathsf{N}_{X_t}$.

(c) Show that

$$\partial_t\,\mathrm{II}_{ij} = \nabla_i\nabla_j\,\mathrm{H} - \mathrm{H}\,g^{k\ell}\,\mathrm{II}_{ik}\,\mathrm{II}_{\ell j}\,.$$

(d) Deduce from Simons' identity that

$$(\partial_t - \Delta)\,\mathrm{II}_{ij} = |\mathrm{II}|^2\,\mathrm{II}_{ij} - 2\,\mathrm{H}\,g^{k\ell}\,\mathrm{II}_{ik}\,\mathrm{II}_{\ell j}\,.$$

(e) Show that

$$\partial_t g_{ij} = -2\,\mathrm{H}\,\mathrm{II}_{ij}\,.$$

(f) Show that

$$\partial_t g^{ij} = 2\,\mathrm{H}\,g^{ik}g^{j\ell}\,\mathrm{II}_{k\ell}\,.$$

(g) Writing $|\mathrm{II}|^2 = g^{ik}g^{j\ell}\,\mathrm{II}_{ij}\,\mathrm{II}_{k\ell}$, deduce that

$$(\partial_t - \Delta)|\mathrm{II}|^2 = 2|\mathrm{II}|^4 - 2|\nabla\mathrm{II}|^2.$$

Exercise 2.2. Let $X : M^n \times I \to \mathbb{R}^{n+1}$ be a mean curvature flow on a manifold M^n. Suppose that $B_r(x_0)\times(t_0-r^2,t_0] \Subset M^n\times I$ and that $S \in \Gamma(T^*M^n \odot T^*M^n)$ satisfies

$$(\nabla_t - \Delta - \nabla_b)S_{(x,t)}(v,v) \geq F(x,t,S_{(x,t)})(v,v) \quad\text{and}\quad S_{(x,t)}(v,v) \geq 0$$

for all $(x,t) \in B_r(x_0)\times(t_0-r^2,t_0]$ and $v \in T^n_x B_r(x_0)$ for some time-dependent vector field $b \in \Gamma(TM^n)$ and some time-dependent vertical section F of $\pi^*(T^*M^n \odot T^*M^n)$ which is Lipschitz in the fibre and satisfies the null eigenvector condition. Prove that $\min_{|v|=1} S_{(x,t)}(v,v) = 0$ for all $(x,t) \in B_r(x_0)\times(t_0-r^2,t_0]$ if $S_{(x_0,t_0)}(v_0,v_0) = 0$ for some nonzero $v_0 \in T_{x_0}M^n$.

Exercise 2.3. Show that the ODE comparison principle does indeed prove Proposition 2.20.

Exercise 2.4. Given any $r > 0$, $p \neq x \in \mathbb{R}^{n+1}$ and $\varepsilon > 0$, consider the trumpet $T_\varepsilon(p, x, r)$.

(a) Show that the aperture of $T_\varepsilon(p, x, r)$ is given by

$$\mathsf{N}_{T_\varepsilon(p,x,r)}(x) \cdot \frac{x-p}{|x-p|} = \frac{\varepsilon r}{\sqrt{|x-p|^2 + \varepsilon^2 r^2}}.$$

(b) Show that the smallest principal curvature of $T_\varepsilon(p, x, r)$ is constant and equal to

$$\kappa_1 = -\frac{2r}{|x-p|^2 + r^2} \frac{(1-\varepsilon)|x-p|}{\sqrt{\varepsilon^2 r^2 + |x-p|^2}}.$$

Exercise 2.5. Show that the evolution $\{\partial\Omega_t\}_{t\in[0,T)}$ by mean curvature of a bounded, convex, locally uniformly convex hypersurface, $\partial\Omega$, may be continued (smoothly) until either

1. the smallest principal curvature of $\partial\Omega_t$ tends to zero[18], or
2. the inradius of Ω_t tends to zero.

Hint: Use Proposition 2.25.

[18] We shall see in Chapter 3 that this will never occur.

3
Pinching and its consequences

We have seen that positivity of the mean curvature is preserved under the mean curvature flow, by applying the (scalar) maximum principle to the reaction-diffusion equation for the mean curvature. The reaction terms in the evolution equation for the second fundamental form enjoy a richer algebraic structure. Understanding this structure (in relation to the tensor maximum principle) is a crucial step in understanding the long term behaviour of the mean curvature flow. We will explore this paradigm in this chapter.

3.1 Contraction of convex hypersurfaces to round points

The tensor maximum principle guarantees that positivity of the second fundamental form is preserved.

Proposition 3.1. *Let* $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ *be a mean curvature flow on a compact manifold* M^n*. If, for some* $\alpha \geq 0$, $\mathrm{II}|_{t=0} \geq \alpha g$*, then* $\mathrm{II} \geq \alpha g$ *for all* $t > 0$

Proof. Recalling (2.22), we see that the tensor $S \doteqdot \mathrm{II} - \alpha g$ satisfies

$$(\nabla_t - \Delta) S = |S + \alpha g|^2 (S + \alpha g).$$

Since $\alpha \geq 0$, the reaction term $F(\cdot, T) \doteqdot |T + \alpha g|^2 (T + \alpha g)$ is clearly nonnegative in the direction of any null eigenvector v of T. The tensor maximum principle implies that $\mathrm{II} \geq 0$ at all points and times. Since M^n is compact, no connected component can split off a line, and we conclude from Proposition 3.2 and a straightforward covering argument that $\mathrm{II} > 0$ at positive times. □

The strong maximum principle provides a useful rigidity statement for nonnegative curvature.

Proposition 3.2. *Let* $X : M^n \times I \to \mathbb{R}^{n+1}$ *be a mean curvature flow satisfying* $\mathrm{II} \geq 0$*. If* $\mathrm{II}_{(x_0,t_0)}(v_0, v_0) = 0$ *at some* $(x_0, t_0, v_0) \in TM^n$*, then*

X SPLITS OFF A FLAT FACTOR: *there exist* $r > 0$*,* $m \in \{1, \dots, n\}$*, an* m*-dimensional linear subspace* $L \subset \mathbb{R}^{n+1}$*, and a family of submanifolds* $\Sigma_t^{n-m} \subset L^\perp$*,* $t \in (t_0 - r^2, t_0]$*, such that*

- $dX(\ker \mathrm{II}_{(x,t)}) = L^m$ *for all* $(x,t) \in B_r(x_0, t_0) \times (t_0 - r^2, t_0]$
- X_t *embeds* $(B_r(x_0, t_0), g_t)$ *isometrically into* $L^m \times^\perp \Sigma_t^{n-m}$ *for each* $t \in (t_0 - r^2, t_0]$
- *the family* $\{\Sigma_t^{n-m}\}_{t \in (t_0 - r^2, t_0]}$ *evolves by mean curvature flow.*

Proof. The strong maximum principle implies[1] that the smallest principal curvature $\kappa_1(x,t) \doteqdot \min_{|v|=1} \mathrm{II}_{(x,t)}(v,v)$ vanishes identically in $B_r(x_0) \times (t_0 - r^2, t_0]$ whenever $B_r(x_0) \times (t_0 - r^2, t_0] \Subset M^n \times I$. In what follows, we restrict attention to such a parabolic cylinder.

[1] See Exercise 2.2.

Consider the (time-dependent) subbundles $\ker \mathrm{II}$ and $(\ker \mathrm{II})^\perp$ of TM^n. Given any $V \in \Gamma(\ker \mathrm{II})$ and $W \in \Gamma(TM^n)$,

$$0 \equiv \nabla_W(\mathrm{II}(V)) = \nabla_W \mathrm{II}(V) + \mathrm{II}(\nabla_W V) \tag{3.1}$$

and hence, for any $U \in \Gamma(\ker \mathrm{II})$,

$$0 \equiv \nabla_W \mathrm{II}(V, U) + \mathrm{II}(\nabla_W V, U) = \nabla_W \mathrm{II}(V, U).$$

By the Codazzi identity and the freedom to choose any W, we find that

$$\nabla_U \mathrm{II}(V) \equiv 0$$

for all $U, V \in \Gamma(\ker \mathrm{II})$. Taking $W = U$ in the gradient identity (3.1) then yields

$$\mathrm{II}(\nabla_U V) \equiv 0.$$

That is, $\nabla_U V \in \Gamma(\ker \mathrm{II})$. Since ∇ is torsion free, we conclude that $[U, V] \in \Gamma(\ker \mathrm{II})$. That is, $\ker \mathrm{II}$ is involutive, and hence, by Frobenius' theorem, integrable.

Next observe that, for any $V \in \Gamma(\ker \mathrm{II})$,

$$0 \equiv \Delta \mathrm{II}(V) + 2\,\mathrm{tr}\,(\nabla_\cdot \mathrm{II}(\nabla_\cdot V)) + \mathrm{II}(\Delta V).$$

Since $\nabla_U V \in \Gamma(\ker \mathrm{II})$ whenever $U \in \Gamma(\ker \mathrm{II})$ and $\nabla_U \mathrm{II}(W) = 0$ whenever $U, W \in \Gamma(\ker \mathrm{II})$, the $\ker \mathrm{II}$ components of the trace term vanish.

So

$$\begin{aligned} 0 \equiv \nabla_t(\mathrm{II}(V)) &= \nabla_t \mathrm{II}(V) - \mathrm{II}(\nabla_t V) \\ &= \Delta \mathrm{II}(V) - \mathrm{II}(\nabla_t V) \\ &= -2\,\mathrm{tr}^\perp\,(\nabla_\cdot \mathrm{II}(\nabla_\cdot V)) + \mathrm{II}((\nabla_t - \Delta)V), \end{aligned}$$

where $\mathrm{tr}^{\perp}$ indicates the trace over $(\ker \mathrm{II})^{\perp}$. Contracting this identity against V and applying the Codazzi identity and the first identity above yields

$$\begin{aligned} 0 \equiv\ & \mathrm{tr}^{\perp} \left(\nabla_{\cdot} \mathrm{II}(\nabla_{\cdot} V, V)\right) \\ =\ & \mathrm{tr}^{\perp} g\left(\nabla_{\cdot} \mathrm{II}(V), \nabla_{\cdot} V\right) \\ =\ & -\mathrm{tr}^{\perp} \mathrm{II}\left(\nabla_{\cdot} V, \nabla_{\cdot} V\right) \end{aligned}$$

for any $V \in \Gamma(\ker \mathrm{II})$. Since $\mathrm{II} > 0$ on $(\ker \mathrm{II})^{\perp}$, we conclude from this that $\nabla_W V \in \Gamma(\ker \mathrm{II})$ whenever $W \in \Gamma((\ker \mathrm{II})^{\perp})$. This means that $\ker \mathrm{II}$ is invariant under parallel translation in space at each time. We also find, for any $U \in \Gamma(\ker \mathrm{II})$, $W \in \Gamma((\ker \mathrm{II})^{\perp})$ and $Y \in \Gamma(TM^n)$, that

$$0 \equiv Yg(U, W) = g(\nabla_Y U, W) + g(U, \nabla_Y W) = g(U, \nabla_Y W)$$

and hence $\nabla_Y W \in (\ker \mathrm{II})^{\perp}$. So $(\ker \mathrm{II})^{\perp}$ is also integrable (at each time) and invariant under spatial parallel transport.

Returning to (3.1), we now conclude, for any $V \in \Gamma(\ker \mathrm{II})$, that

$$\nabla_V \mathrm{II} \equiv 0,$$

and hence

$$\mathrm{tr}^{\perp} \left(\nabla_{\cdot} \mathrm{II}(\nabla_{\cdot} V)\right) = 0.$$

Since $\Delta V \in \Gamma(\ker \mathrm{II})$, we find that

$$0 \equiv \nabla_t(\mathrm{II}(V)) = \mathrm{II}(\nabla_t V).$$

So $\ker \mathrm{II}$ is also parallel in the time direction. But then so is $(\ker \mathrm{II})^{\perp}$, since

$$0 \equiv \partial_t g(U, W) = g(\nabla_t u, W) + g(U, \nabla_t W)$$

for any $U \in \Gamma(\ker \mathrm{II})$ and $W \in \Gamma((\ker \mathrm{II})^{\perp})$.

We now convert this into information about the parametrizations. First observe that $dX(\ker \mathrm{II})$ is parallel in space with respect to the pullback connection ${}^X D$. Indeed, given any $V \in \Gamma(\ker \mathrm{II})$ and any $U \in \Gamma(TM^n)$,

$${}^X D_U(dX(V)) = dX \nabla_U V - \mathrm{II}(U, V)\, \mathsf{N} = dX \nabla_U V \in dX(\ker \mathrm{II}).$$

Since $\nabla_V \mathrm{H} = \mathrm{tr}\, \nabla_V \mathrm{II} = 0$ for any $V \in \Gamma(\ker \mathrm{II})$, we similarly find that $dX(\ker \mathrm{II})$ is parallel in time with respect to the pullback connection ${}^X D$:

$${}^X D_t(dX(V)) = \nabla_V \mathrm{H}\, \mathsf{N} + dX(\nabla_t V) = dX(\nabla_t V) \in dX(\ker \mathrm{II}).$$

We conclude that $dX(\ker \mathrm{II})$ is a constant subspace of $\mathbb{R}^{n+1}$ (after canonically identifying the spaces $T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$). In particular, the nullity $m \in \{1, \ldots, n\}$ of II is constant.

Now consider any geodesic $\gamma(-s_0, s_0) \to M^n$ of the metric at time t, with initial data $(\gamma(0), \gamma'(0)) \in \ker \mathrm{II}_{(x,t)}$. Since $\ker \mathrm{II}$ is invariant under parallel translation, $\gamma'(s) \in \ker \mathrm{II}_{(\gamma(s),t)}$ for all s, and hence

$$^X D_s(dX(\gamma')) = dX(\nabla_s \gamma') - \mathrm{II}(\gamma', \gamma')\,\mathsf{N} = 0.$$

That is, $X \circ \gamma$ is geodesic in $\mathbb{R}^{n+1}$.

By Sard's theorem, we can find $r' > 0$ (arbitrarily close to r) and x_0' (arbitrarily close to x_0) such that the section $\Sigma_t^{n-m} \doteqdot X_t(B_{r'}(x_0, t_0)) \cap \Pi^{n-m+1}$ is a smoothly embedded $(n-m)$-manifold for each $t \in (t_0 - (r')^2, t_0]$, where $\Pi^{n-m+1} \doteqdot X(x_0', t_0) + (dX(\ker \Pi_{(x_0', t_0)}))^{\perp}$. The remaining claims follow. □

Since local convexity guarantees that $|\mathrm{II}|^2 \leq \mathrm{H}^2$, the ODE comparison principle yields the following growth estimates for the mean curvature.

Proposition 3.3. *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a maximal, compact*[2]*, locally uniformly convex*[3] *mean curvature flow.*

$$\min_{M \times \{t\}} \tfrac{1}{\sqrt{n}} \mathrm{H} \leq \frac{1}{\sqrt{2(T-t)}} \leq \max_{M \times \{t\}} \mathrm{H}.$$

[2] I.e. M^n is compact.

[3] I.e. the time-slices $X_t : M^n \to \mathbb{R}^{n+1}$ are locally uniformly convex: $\mathrm{II} > 0$.

Proof. Since $\mathrm{II} > 0$, we may estimate $|\mathrm{II}|^2 \leq \mathrm{H}^2$, and hence

$$\tfrac{1}{n}\mathrm{H}^3 \leq (\partial_t - \Delta)\mathrm{H} \leq \mathrm{H}^3.$$

Since $\limsup_{t \to T} \max_{M^n \times \{t\}} \mathrm{H} = \infty$, the ODE comparison principle yields the claims. □

We next observe that any uniform lower *pinching* is preserved under mean curvature flow.

Proposition 3.4 (Pinching is preserved)**.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a compact, locally uniformly convex mean curvature flow. There exists $\alpha > 0$ such that*

$$\mathrm{II} \geq \alpha \mathrm{H} g > 0$$

at all times.

Proof. Since M^n is compact and $\mathrm{II} > 0$, a constant $\alpha > 0$ may be found such that the inequality holds at the initial time. Given such a constant, consider the tensor $S \doteqdot \mathrm{II} - \alpha \mathrm{H} g$. Recalling (2.22) and (2.21), observe that

$$\begin{aligned}(\nabla_t - \Delta)S &= (\nabla_t - \Delta)\mathrm{II} - \alpha(\partial_t - \Delta)\mathrm{H}g \\ &= |\mathrm{II}|^2 S.\end{aligned}$$

So the claim follows from the tensor maximum principle. □

Consider now the ratio $|\mathring{\mathrm{II}}|^2/\,\mathrm{H}^2$, where $\mathring{\mathrm{II}} = \mathrm{II} - \frac{1}{n}\,\mathrm{H}\,g$ denotes the trace-free part of II. Since $\mathring{\mathrm{II}}$ vanishes precisely at umbilic points, this ratio is a scale invariant pointwise measure of the "roundness" of our hypersurface. We will show that this measure of roundness becomes optimal in regions of very large curvature. Observe first that it does not decay.

Proposition 3.5 (Roundness is preserved)**.** *Along any compact, locally uniformly convex mean curvature flow $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$,*

$$\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} \le \max_{M^n \times \{0\}} \frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}.$$

Proof. Since

$$|\mathring{\mathrm{II}}|^2 = |\mathrm{II}|^2 - \tfrac{1}{n}\mathrm{H}^2,$$

we find that

$$\begin{aligned}
(\partial_t - \Delta)\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} &= (\partial_t - \Delta)\frac{|\mathrm{II}|^2}{\mathrm{H}^2} \\
&= 2g\left((\nabla_t - \Delta)\frac{\mathrm{II}}{\mathrm{H}}, \frac{\mathrm{II}}{\mathrm{H}}\right) - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 \\
&= 2g\left(\frac{(\nabla_t - \Delta)\mathrm{II}}{\mathrm{H}} - (\partial_t - \Delta)\mathrm{H}\frac{\mathrm{II}}{\mathrm{H}^2} + 2\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{\mathrm{II}}{\mathrm{H}}, \frac{\mathrm{II}}{\mathrm{H}}\right) \\
&\quad - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 \\
&= 2\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathrm{II}|^2}{\mathrm{H}^2} - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 \\
&= 2\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 \\
&\le 2\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}.
\end{aligned}$$

So the claim follows from the maximum principle. □

We now show that roundness improves at the onset of a singularity.

Proposition 3.6 (Roundness improves)**.** *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a compact, locally uniformly convex mean curvature flow. For every $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ (which depends only on ε and the initial hypersurface) such that*

$$|\mathring{\mathrm{II}}|^2 \le \varepsilon \mathrm{H}^2 + C_\varepsilon \tag{3.2}$$

at all times.

Proof. Given σ and $\varepsilon > 0$, consider the function $\mathrm{H}^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right)$. We aim to show that this function is bounded uniformly in time for some (very small but positive) σ. The claim then follows from Young's inequality.

We compute

$$
\begin{aligned}
(\partial_t - \Delta)&\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right]\right)\\
&= \mathrm{H}^\sigma(\partial_t-\Delta)\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - 2g\left(\nabla \mathrm{H}^\sigma, \nabla\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}\right) + \left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)(\partial_t-\Delta)\mathrm{H}^\sigma\\
&= \mathrm{H}^\sigma(\partial_t-\Delta)\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - 2\sigma \mathrm{H}^\sigma g\left(\frac{\nabla \mathrm{H}}{\mathrm{H}}, \nabla\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}\right)\\
&\quad + \sigma\mathrm{H}^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)\left[\frac{(\partial_t-\Delta)\mathrm{H}}{\mathrm{H}} + (1-\sigma)\frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2}\right]\\
&= \mathrm{H}^\sigma\left[\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)|\mathrm{II}|^2 + 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2\right.\\
&\qquad\qquad \left. + \sigma(1-\sigma)\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)\frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2}\right].
\end{aligned}
$$

Since

$$
\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right]\right) = \sigma\mathrm{H}^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)\frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2} + \mathrm{H}^\sigma\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2},
$$

we arrive at

$$
\begin{aligned}
(\partial_t - \Delta)&\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right]\right)\\
&= \mathrm{H}^\sigma\left[\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)|\mathrm{II}|^2 - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 - \sigma(1-\sigma)\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)\frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2}\right]\\
&\quad + 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right]\right). \qquad (3.3)
\end{aligned}
$$

Unfortunately, the reaction term, $\sigma H^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2}-\varepsilon\right)|\mathrm{II}|^2$, is an obstruction to applying the maximum principle (the other terms are either negative or gradient terms). Instead, we shall exploit the diffusion term to obtain good integral estimates, which can be bootstrapped to an L^∞ estimate. Before doing this, let us discard some junk that won't be needed. We first estimate

$$
\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 = \mathrm{H}^{-2}\left|\nabla\mathrm{II} - \frac{\mathrm{II}}{\mathrm{H}}\otimes\nabla\mathrm{H}\right| \geq \gamma\frac{|\nabla\mathrm{II}|^2}{\mathrm{H}^2}
$$

for some $\gamma = \gamma(n,\alpha) > 0$, where α is any measure of the initial pinching. This is a consequence of the Codazzi identity and the following purely algebraic statement.

Claim 3.7. *Given $\alpha > 0$ and $n \geq 2$, there exists a constant $\gamma = \gamma(n,\alpha) > 0$ such that*

$$
\left|T - \frac{B}{\operatorname{tr}(B)}\otimes\operatorname{tr}(T)\right| \geq \gamma|T|^2
$$

for any symmetric $B \in \mathbb{R}^n \otimes \mathbb{R}^n$ satisfying $B \geq \alpha\,\mathrm{tr}(B)\mathrm{I} > 0$ and any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof of Claim 3.7. It suffices to establish the inequality when $|B| = |T| = 1$, due to homogeneity. In fact, since $K \doteqdot \{(B,T) : |B| = |T| = 1, B \geq \alpha\,\mathrm{tr}(B)\mathrm{I}\}$ is compact, it suffices to show that

$$\min_{(B,T)\in K} \left| T - \frac{B}{\mathrm{tr}(B)} \otimes \mathrm{tr}(T) \right| > 0.$$

Suppose, then, that this is not the case. Then we can find $(B,T) \in K$ such that

$$T = \frac{B}{\mathrm{tr}(B)} \otimes \mathrm{tr}(T).$$

Tracing both sides appropriately, we find that $\mathrm{tr}(T)$ is an eigenvector of B with eigenvalue $\mathrm{tr}(B)$. But this is impossible when $n \geq 2$ as B is positive definite. ☐

Since $|\nabla \mathrm{H}|^2 \leq n|\nabla \mathrm{II}|^2$ and $|\mathring{\mathrm{II}}|^2 \leq \mathrm{H}^2$, discarding the second negative term in (3.3) (whose coefficient $\sigma(1-\sigma)$ will prove too small to be of use) and applying Young's inequality to the final (gradient) term, we may estimate, wherever $|\mathring{\mathrm{II}}| > 0$,

$$(\partial_t - \Delta)\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right) \leq \mathrm{H}^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right)\left(\sigma|\mathrm{II}|^2 - \gamma\frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2}\right) + \gamma^{-1}\frac{\left|\nabla\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right)\right|^2}{\mathrm{H}^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right)} \tag{3.4}$$

for some (smaller than before) $\gamma = \gamma(n,\alpha) > 0$.

Consider now, for large p (at least 10, say), the function

$$v \doteqdot \left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right)_+^{\frac{p}{2}},$$

where f_+ denotes the positive part, $\max\{0, f\}$, of a function f. If p is large enough, and σ small enough, we shall be able to establish an L^2 estimate for v, which can be bootstrapped to the desired L^∞ estimate by STAMPACCHIA ITERATION. To that end, observe that

$$\frac{|\nabla v|^2}{v^2} = \frac{p^2}{4}\frac{\left|\nabla\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right)\right|^2}{\left(\mathrm{H}^\sigma\left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right)^2}$$

wherever $v > 0$. Applying (3.4), we may thus estimate

$$
\begin{aligned}
(\partial_t - \Delta) v^2 \\
&= p v^2 \frac{(\partial_t - \Delta)\left(\mathrm{H}^\sigma \left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right)}{\mathrm{H}^\sigma \left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right)} - p(p-1)\left|\nabla\left(\mathrm{H}^\sigma \left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right]\right)\right|^2 \\
&\le v^2 \left(\sigma p |\mathrm{II}|^2 - \gamma p \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2}\right) - 2|\nabla v|^2
\end{aligned}
$$

for some $\gamma = \gamma(n, \alpha) > 0$, so long as $p \ge L = L(n, \alpha)$.

The divergence theorem then yields

$$
\begin{aligned}
\frac{d}{dt}\int v^2\, d\mu &= \int (\partial_t v^2 - v^2\, \mathrm{H}^2)\, d\mu \\
&\le \sigma p \int |\mathrm{II}|^2 v^2\, d\mu - \int \left(\gamma p v^2 \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} + 2|\nabla v|^2 + v^2\, \mathrm{H}^2\right) d\mu. \qquad (3.5)
\end{aligned}
$$

Simons' identity allows us to absorb the bad term.

Claim 3.8. *There exists $C = C(n, \alpha) < \infty$ with the following property. If $X : M^n \to \mathbb{R}^{n+1}$ is a compact hypersurface satisfying $\mathrm{II} \ge \alpha\, \mathrm{H}\, g$, then any sufficiently smooth, nonnegative function $u : M^n \to \mathbb{R}$ satisfies*

$$
\int |\mathring{\mathrm{II}}|^2 u^2\, d\mu \le C \int \left(\frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} + \frac{|\nabla \mathrm{II}|}{\mathrm{H}} \frac{|\nabla u|}{u}\right) u^2\, d\mu.
$$

Proof of Claim 3.8. Recall Simons' identity:

$$
\nabla_{(i}\nabla_{j)}\, \mathrm{II}_{k\ell} - \nabla_{(k}\nabla_{\ell)}\, \mathrm{II}_{ij} =: \mathrm{S}_{ijk\ell} = \mathrm{C}_{ijk\ell} \doteqdot \mathrm{II}_{ij}\, \mathrm{II}^2_{k\ell} - \mathrm{II}_{k\ell}\, \mathrm{II}^2_{ij}\,.
$$

Observe that

$$
|\mathrm{C}|^2 = \sum_{i,k=1}^{n} \kappa_i^2 \kappa_k^2 (\kappa_i - \kappa_k)^2 \ge \mathrm{C}^{-1}\, \mathrm{H}^4\, |\mathring{\mathrm{II}}|^2
$$

for some $C = C(n, \alpha) < \infty$. Thus,

$$
\int |\mathring{\mathrm{II}}|^2 u^2\, d\mu \le C \int g(\mathrm{S}, \mathrm{C})\, \mathrm{H}^{-4}\, u^2\, d\mu = C \int \nabla^2\, \mathrm{II} * \frac{\mathrm{II}^3}{\mathrm{H}^4} u^2\, d\mu.
$$

Integrating by parts and estimating crudely then yields the claim. □

Since $|\mathring{\mathrm{II}}|^2 \ge \varepsilon\, \mathrm{H}^2 \ge \varepsilon |\mathrm{II}|^2$ wherever $v > 0$, applying Claim 3.8 to (3.5) and exploiting Young's inequality yields

$$
\frac{d}{dt}\int v^2\, d\mu \le 0
$$

and hence

$$
\int v^2\, d\mu \le \Omega^2 \doteqdot \int v^2\, d\mu \Big|_{t=0} \qquad (3.6)
$$

so long as $p \geq L$ and $\sigma p^{\frac{1}{2}} \leq \ell$ for some constants $L = L(n, \alpha, \varepsilon) < \infty$ and $\ell = \ell(n, \alpha, \varepsilon) > 0$. This is our L^2 estimate.

In order to extract an L^∞ estimate, consider, for each $k \geq k_0 \doteqdot \max_{M^n \times \{0\}} \mathrm{H}^\sigma$, the truncated function

$$v_k^2 \doteqdot \left(\mathrm{H}^\sigma \left[\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon \right] - k \right)_+^p$$

and its support

$$A_k(t) \doteqdot \{(x,t)\, M^n \times [0,T) : v_k(x,t) > 0\}\,.$$

Set also

$$u(k) \doteqdot \iint v_k^2 \, d\mu \, dt \quad \text{and} \quad a(k) \doteqdot \iint_{A_k} d\mu \, dt\,.$$

We want to show that A_k has measure zero for a suitably large value of k (depending only on ε and initial data). Observe that, for any $h \geq k > 0$,

$$(h-k)^p \, a(h) \leq u(k)\,. \tag{3.7}$$

In order to estimate $u(k)$, we observe first that the same arguments which led to (3.5) yield

$$\frac{d}{dt} \int v_k^2 \, d\mu + \int \left(|\nabla v_k|^2 + v_k^2 \, \mathrm{H}^2 \right) d\mu \leq \sigma p \int_{A_k} |\mathrm{II}|^2 v_0^p \, d\mu\,,$$

so long as $p \geq L = L(n, \alpha)$.

We next apply the Sobolev inequality of Michael and Simon[4] in conjunction with Hölder's inequality to estimate

$$\left(\int v_k^{2^*} d\mu \right)^{\frac{1}{2^*}} \leq C \left(\int \left(|\nabla v_k|^2 + v_k^2 \, \mathrm{H}^2 \right) d\mu \right)^{\frac{1}{2}},$$

where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$ and $C = C(n)$ when $n \geq 3$; when $n = 2$, we may take 2^* to be any number greater than one half and $C = C(n, 2^*)|A_k|^{\frac{1}{2^*}} \leq C(n, 2^*)\mu(M^n)^{\frac{1}{2^*}} \leq C(n, 2^*)\mu_0(M^n)^{\frac{1}{2^*}} = C(n, 2^*, X_0)$. We thereby obtain

$$\frac{d}{dt} \int v_k^2 \, d\mu + C^{-1} \left(\int v_k^{2^*} d\mu \right)^{\frac{2}{2^*}} \leq \sigma p \int_{A_k} |\mathrm{II}|^2 v_0^2 \, d\mu\,,$$

where (upon fixing $2^* \geq \frac{1}{2}$) C depends at worst on n and the initial data. Since $|\mathring{\mathrm{II}}|^2 \leq |\mathrm{II}|^2 \leq \mathrm{H}^2$ and, without loss of generality, $C \geq 1$, integrating in time yields

$$\sup_{t \in [0,T)} \int v_k^2 \, d\mu + \int_0^T \left(\int v_k^{2^*} d\mu \right)^{\frac{2}{2^*}} \leq C\sigma p \int_0^T \int_{A_k} \mathrm{H}^2 \, v_0^p \, d\mu \tag{3.8}$$

for $k \geq k_0$.

[4] On any submanifold $M^n \to \mathbb{R}^{n+k}$, every $u \in W^{1,1}(M^n)$ satisfies

$$\left(\int |u|^{1^*} d\mu \right)^{\frac{1}{1^*}} \leq C \int \left(|\nabla u|^2 + |u||\vec{\mathrm{H}}| \right) d\mu,$$

where $\frac{1}{1^*} = 1 - \frac{1}{n}$ and C is a constant which depends only on n, Michael and L. M. Simon, "Sobolev and mean-value inequalities on generalized submanifolds of R^n".

We now use the interpolation inequality for L^p spaces[5] to estimate

$$\int_{A_k} v_k^{2p^*}\,d\mu \le \left(\int_{A_k} v_k^2 d\mu\right)^{p^*-1}\left(\int_{A_k} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}$$

where $p^* = 2 - \frac{2}{2^*}$, which is equal to $\frac{n+2}{2}$ when $n \ge 3$. We then conclude, using Young's inequality, that

$$\begin{aligned}\left(\int_0^T\!\!\int_{A_k} v_k^{2p^*} d\mu dt\right)^{\frac{1}{p^*}} &\le \left(\sup_{t\in[0,T)}\int_{A_k} v_k^{2p^*} d\mu\right)^{1-\frac{1}{p^*}}\left(\int\left(\int_{A_k} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}} dt\right)^{\frac{1}{p^*}}\\ &\le \sup_{t\in[0,T)}\int v_k^2\,d\mu + \int_0^T\left(\int v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}dt. \qquad (3.9)\end{aligned}$$

Returning to (3.8), we use Hölder's inequality and the L^2 estimate (3.6) to estimate (for some $r \ge 1$ to be chosen momentarily)

$$\begin{aligned}\int_0^T\!\!\int_{A_k} \mathrm{H}^2\, v_0^p\,d\mu &\le a(k)^{1-\frac{1}{r}}\left(\int_0^T\!\!\int_{A_k} \mathrm{H}^{2r}\, v_0^{2r}\,d\mu\right)^{\frac{1}{r}}\\ &= a(k)^{1-\frac{1}{r}}\left(\int_0^T\!\!\int_{A_k}\left(\mathrm{H}^{\sigma+\frac{2}{p}}\left[\frac{|\mathring{\mathrm{I\!I}}|^2}{\mathrm{H}^2}-\varepsilon\right]\right)^{pr} d\mu\right)^{\frac{1}{r}}\\ &\le Ca(k)^{1-\frac{1}{r}} \qquad (3.10)\end{aligned}$$

so long as $pr \ge L$ and $(\sigma + \frac{2}{p})p^{\frac{1}{2}} \le \ell$, where C depends on n, the initial data, σ, p and r.

Putting together the estimates (3.7)-(3.10), we arrive at the estimate

$$a(h) \le \frac{Ca(k)^\gamma}{(h-k)^p}$$

for any $h > k > k_0$, where $\gamma = 2 - \frac{1}{p^*} - \frac{1}{r}$, where C depends on n, the initial data, σ, p and r. We may fix r in such a way that $r > \frac{p^*}{p^*-1}$, guaranteeing that $\gamma > 1$. By a simple iteration argument, this rate of decay actually ensures that a reaches zero after a (quantifiably) *finite* number of steps.

Claim 3.9 (Stampacchia's Lemma[6]). *Let $\varphi : [k_0, \infty) \to \mathbb{R}$ be a nonnegative, nonincreasing function. If*

$$\varphi(h) \le \frac{C}{(h-k)^\alpha}\varphi(k)^\beta \qquad (3.11)$$

when $h > k > k_0$ for some constants $C > 0$, $\alpha > 0$ and $\beta > 1$, then

$$\varphi(k_0 + d) = 0,$$

where $d^\alpha = C\,\varphi(k_0)^{\beta-1}\,2^{\alpha\beta/(\beta-1)}$.

[5] I.e.

$$\|u\|_{\frac{\vartheta}{p}+\frac{1-\vartheta}{q}} \le \|u\|_p^\vartheta \|u\|_q^{1-\vartheta}$$

for any $\vartheta \in [0,1]$ and $p, q > 1$.

[6] Kinderlehrer and Stampacchia, *An introduction to variational inequalities and their applications*, Chapter II, Appendix B

Proof of Stampacchia's Lemma. Consider the sequence of numbers k_r prescribed by

$$k_r = k_0 + d - \frac{d}{2^r}, \quad r = 0, 1, 2, \ldots .$$

By assumption,

$$\varphi(k_{r+1}) \le C\frac{2^{(r+1)\alpha}}{d^\alpha}\varphi(k_r)^\beta \quad \text{for all} \quad r = 0, 1, \ldots . \tag{3.12}$$

We will prove by induction that

$$\varphi(k_r) \le \varphi(k_0)2^{-r\mu} \tag{3.13}$$

for all $r \in \mathbb{N}$, where $\mu \doteqdot \frac{\alpha}{\beta-1} > 0$. Clearly (3.13) holds trivially for $r = 0$. Supposing (3.13) holds up to some integer r, we find by (3.12) and the definition of d that

$$\varphi(k_{r+1}) \le C\frac{2^{(r+1)\alpha}}{d^\alpha}\varphi(k_0)^\beta 2^{-r\mu\beta} = \varphi(k_0)2^{-(r+1)\mu}.$$

The claim (3.13) follows. Now, by the monotonicity assumption,

$$0 \le \varphi(k_0 + d) \le \varphi(k_r) \quad \text{for all} \quad r = 0, 1, \ldots .$$

But, by (3.13), $\varphi(k_r) \to 0$ as $r \to \infty$. □

Applying Stampacchia's Lemma, we conclude that

$$\mathrm{H}^\sigma\left(\frac{|\mathring{\mathrm{II}}|^2}{\mathrm{H}^2} - \varepsilon\right) \le C$$

for suitable p and σ (depending only on n, α and ε), which we now fix, where, having fixed such p and σ, C depends only on n, initial data, and ε. The proposition now follows from a simple application of Young's inequality. □

Proposition 3.6 ensures, when $n \ge 2$, that the evolving hypersurface is becoming round at any point where the curvature is becoming large, in the sense that the scale invariant ratio $|\mathring{\mathrm{II}}|/\mathrm{H}$ is becoming small. We already know that $\max \mathrm{H} \ge \frac{1}{\sqrt{2(T-t)}}$ is blowing up at the final time; we thus need to show that $\min \mathrm{H}$ blows up at the same rate. So we should try to control the *gradient* of II. In order to do this, we need to compare $|\nabla \mathrm{II}|^2$ to some function (of curvature) whose evolution equation can overcome the bad reaction terms $\mathrm{II} * \mathrm{II} * \nabla\, \mathrm{II} * \nabla\, \mathrm{II}$ in the evolution equation for $|\nabla \mathrm{II}|^2$. We can exploit the estimate (3.2) in this regard.

Proposition 3.10. *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$, $n \ge 2$, be a compact, locally uniformly convex mean curvature flow. For any $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ (which depends only on ε and the initial data) such that*

$$|\nabla \mathrm{II}|^2 \le \varepsilon \mathrm{H}^4 + C_\varepsilon .$$

Proof. Recall that

$$(\partial_t - \Delta)|\nabla \mathrm{II}|^2 \le c|\mathrm{II}|^2|\nabla \mathrm{II}|^2 - 2|\nabla^2 \mathrm{II}|^2 .$$

Given $\varepsilon > 0$, choose C_ε (as permitted by Proposition 10.5) so that

$$|\mathring{\mathrm{II}}|^2 \le \varepsilon \mathrm{H}^2 + C_\varepsilon$$

and consider, for suitable $C_\varepsilon < \infty$, the function

$$G_\varepsilon \doteqdot 2C_\varepsilon + \varepsilon \mathrm{H}^2 - |\mathring{\mathrm{II}}|^2 \ge C_\varepsilon > 0 .$$

Consider also, for suitable $C_0 < \infty$,

$$G_0 \doteqdot 2C_0 + \tfrac{3}{n+2}\mathrm{H}^2 - |\mathrm{II}|^2 \ge C_0 + \tfrac{n-1}{n(n+2)}\mathrm{H}^2 > 0 .$$

The coefficient of H^2 is chosen with Kato's inequality in mind:

Claim 3.11. *On any hypersurface $M^n \to \mathbb{R}^{n+1}$, $|\nabla \mathrm{II}|^2 \ge \frac{3}{n+2}|\nabla \mathrm{H}|^2$.*

Proof of Claim 3.11. Decomposing $\nabla \mathrm{II} = E + F$ into its trace and trace-free parts

$$E_{ijk} \doteqdot \tfrac{1}{n+2}\left(\nabla_i \mathrm{H}\, g_{jk} + \nabla_j \mathrm{H}\, g_{ki} + \nabla_k \mathrm{H}\, g_{ij}\right)$$

and $F \doteqdot \nabla \mathrm{II} - E$, respectively, we find that

$$|\nabla \mathrm{II}|^2 = |E|^2 + |F|^2 \ge |E|^2 = \tfrac{3}{n+2}|\nabla \mathrm{H}|^2 . \qquad \square$$

Applying Claim 3.11, we find that

$$\begin{aligned}(\partial_t - \Delta)G_0 &= 2(|\nabla \mathrm{II}|^2 - \tfrac{3}{n+2}|\nabla \mathrm{H}|^2) + 2(\tfrac{3}{n+2}\mathrm{H}^2 - |\mathrm{II}|^2)|\mathrm{II}|^2 \\ &\ge 2|\mathrm{II}|^2(G_0 - 2C_0) \\ &\ge -2|\mathrm{II}|^2 G_0 .\end{aligned}$$

Similarly,

$$\begin{aligned}(\partial_t - \Delta)G_\varepsilon &= 2\left(\left(\tfrac{1}{n} + \varepsilon\right)|\mathrm{II}|^2\mathrm{H}^2 - |\mathrm{II}|^4\right) + 2\left(|\nabla \mathrm{II}|^2 - \left(\tfrac{1}{n} + \varepsilon\right)|\nabla \mathrm{H}|^2\right) \\ &\ge 2|\mathrm{II}|^2(G_\varepsilon - 2C_\varepsilon) + 2\left(1 - \tfrac{n+2}{3}(\tfrac{1}{n} + \varepsilon)\right)|\nabla \mathrm{II}|^2 \\ &\ge -2|\mathrm{II}|^2 G_\varepsilon + \tfrac{1}{2}|\nabla \mathrm{II}|^2 ,\end{aligned}$$

so long as $\varepsilon \le \frac{1}{4(n+2)}$, say.

We aim to preserve upper bounds for the function $\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}$. So consider

$$\begin{aligned}(\partial_t - \Delta)\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} &= \frac{(\partial_t - \Delta)|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} - \frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}\left(\frac{(\partial_t - \Delta)G_0}{G_0} - \frac{(\partial_t - \Delta)G_\varepsilon}{G_\varepsilon}\right) \\ &\quad + 2g\left(\nabla \frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}, \frac{\nabla(G_0 G_\varepsilon)}{G_0 G_\varepsilon}\right) + 2\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} g\left(\frac{\nabla G_0}{G_0}, \frac{\nabla G_\varepsilon}{G_\varepsilon}\right) .\end{aligned}$$

We estimate the terms on the first line as above. To control the terms on the second line, observe that, at a new local maximum of $\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}$,

$$0 = \nabla_k \frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} = 2\frac{g(\nabla_k \nabla \mathrm{II}, \nabla \mathrm{II})}{G_0 G_\varepsilon} - \frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}\left(\frac{\nabla_k G_0}{G_0} + \frac{\nabla_k G_\varepsilon}{G_\varepsilon}\right)$$

and hence

$$4\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} g\left(\frac{\nabla G_0}{G_0}, \frac{\nabla G_\varepsilon}{G_\varepsilon}\right) \leq \frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}\left|\frac{\nabla G_0}{G_0} + \frac{\nabla G_\varepsilon}{G_\varepsilon}\right|^2 \leq 4\frac{|\nabla^2 \mathrm{II}|^2}{G_0 G_\varepsilon}.$$

Thus, at such a point,

$$0 \leq (\partial_t - \Delta)\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} \leq \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon}\left((c+4)\frac{|\mathrm{II}|^2}{G_0} - \frac{1}{2}\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}\right)$$

and hence

$$\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} \leq 2(c+4)\frac{|\mathrm{II}|^2}{G_0} \leq 2\,(c+4)\,\tfrac{n(n+4)}{n+1}.$$

We conclude that

$$\frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon} \leq C \doteqdot \max\left\{2\,(c+4)\,\tfrac{n(n+4)}{n+1}, \max_{M^n \times \{0\}} \frac{|\nabla \mathrm{II}|^2}{G_0 G_\varepsilon}\right\}$$

and hence

$$|\nabla \mathrm{II}|^2 \leq C(2C_0 + \tfrac{3}{n+2}\,\mathrm{H}^2)(2C_\varepsilon + \varepsilon\,\mathrm{H}^2),$$

at which point the claim is a straightforward application of Young's inequality. □

We can exploit the gradient estimate in conjunction with Myers' theorem to establish the desired blow-up rate for the mean curvature.

Proposition 3.12. *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$, $n \geq 2$, be the maximal mean curvature flow of a compact, locally uniformly convex initial hypersurface.*

$$\frac{\mathrm{H}_{\max}(t)}{\mathrm{H}_{\min}(t)} \to 1 \text{ and } \operatorname{diam}(M^n, g_{(\cdot,t)}) \to 0 \text{ as } t \to T, \tag{3.14}$$

where $\mathrm{H}_{\max} \doteqdot \max_{M^n} \mathrm{H}$ *and* $\mathrm{H}_{\min} \doteqdot \min_{M^n} \mathrm{H}$.

Proof. By the gradient estimate (Proposition 3.10), for every $\eta > 0$ there is a constant $C_\eta < \infty$ such that

$$|\nabla \mathrm{H}| \leq \tfrac{1}{2}\eta^2 \mathrm{H}^2 + C_\eta.$$

Since $\mathrm{H}_{\max}(t) \to \infty$ as $t \to T$, there is, for every $\eta > 0$, some point $(x_\eta, t_\eta) \in M^n \times [0, T)$ such that

$$\mathrm{H}_\eta^2 \doteqdot \mathrm{H}^2(x_\eta, t_\eta) = \mathrm{H}^2_{\max}(t_\eta) \geq 2C_\eta/\eta^2$$

and hence

$$|\nabla H|(x,t_\eta)\le \eta^2 H^2(x_\eta,t_\eta)$$

for all $x\in M$. Now let γ be a unit speed $g_{(\cdot,t_\eta)}$-geodesic through $\gamma(0)=x_\eta$. For each $s\le L\doteqdot \eta^{-1}H_\eta^{-1}$, the mean value theorem provides some $s_0\in(0,s)$ such that

$$H(\gamma(s),t_\eta)=H_\eta+s\nabla_{\gamma'(s_0)}H(\gamma(s_0),t_\eta)\ge H_\eta(1-\eta). \tag{3.15}$$

Applying the preserved pinching estimate $\mathrm{II}\ge \alpha H g$ and the trace Gauss equation $\mathrm{Rc}=H\,\mathrm{II}-\mathrm{II}^2$, we may estimate

$$\mathrm{Rc}(\gamma',\gamma')\ge (n-1)\alpha^2H^2\ge (1-\eta)(n-1)\alpha^2H_\eta^2$$

for $s\le L$. If $\eta<\frac{1}{2}$, then

$$\mathrm{Rc}(\gamma',\gamma')\ge (n-1)Kg,$$

where $K\doteqdot \frac{\alpha^2}{2}H_\eta^2$. Choosing further $\eta^2\le\frac{\alpha^2}{2\pi}$, we obtain $L\ge \pi K^{-\frac{1}{2}}$. Myers' theorem then implies that every point of M^n is reached by a $g_{(\cdot,t_\eta)}$-geodesic of length at most L and we conclude from (3.15) that

$$H_{\min}(t_\eta)\ge (1-\eta)H_{\max}(t_\eta).$$

Since $H_{\min}$ is nondecreasing, we then have

$$H^2_{\max}(t)\ge(1-\eta)^2H^2_{\max}(t_\eta)\ge\tfrac{1}{4}H_\eta^2\quad\text{for all}\quad t\ge t_\eta,$$

so that the above arguments hold for all $t\ge t_\eta$. We now conclude that, given any $\eta\le\min\{\frac{\alpha}{\sqrt{2\pi}},\frac{1}{2}\}$, there is some time $t_\eta\in[0,T)$ such that

$$\operatorname{diam}(M,g_{(\cdot,t)})\le\frac{1}{\eta H_{\max}(t)}\quad\text{and}\quad H_{\min}(t)\ge(1-\eta)H_{\max}(t)$$

for all $t>t_\eta$. The proposition follows since $H_{\max}(t)\ge\frac{1}{2(T-t)}$. □

It follows that the (extrinsic) diameters of the rescaled hypersurfaces $\frac{1}{\sqrt{2n(T-t)}}X_t$ remain bounded, and their mean curvature converges uniformly to a constant as $t\to T$. Bootstrapping arguments (cf. Theorem 2.19) then yield smooth convergence to a round sphere.

Theorem 3.13 (Huisken's theorem[7]). *Let $X : M^n\times[0,T)\to\mathbb{R}^{n+1}$, $n\ge 2$, be the maximal mean curvature flow starting from a smooth, compact, locally uniformly convex hypersurface $X_0 : M^n\to\mathbb{R}^{n+1}$. There exists $p\in\mathbb{R}^{n+1}$ such that*

$$\frac{X(\cdot,t)-p}{\sqrt{2n(T-t)}}\to\overline{X}$$

uniformly in the smooth topology as $t\to T$, where $\overline{X} : M^n\to\mathbb{R}^{n+1}$ is an embedding whose image is the unit sphere.

[7] Huisken, "Flow by mean curvature of convex surfaces into spheres"

3.2 Pinched hypersurfaces are compact

Huisken's theorem shows that a compact hypersurface of $\mathbb{R}^{n+1}$, $n \geq 2$, which is PINCHED, in the sense that $\mathrm{II} \geq 0$ and

$$\kappa_1 \geq \alpha\kappa_n \ \text{ for some } \ \alpha > 0,$$

contracts to a round point under mean curvature flow. The following theorem of Hamilton[8] shows that the compactness hypothesis is superfluous, unless the hypersurface is a hyperplane.

[8] Richard S. Hamilton, "Convex hypersurfaces with pinched second fundamental form".

Theorem 3.14. *Any pinched, proper hypersurface of $\mathbb{R}^{n+1}$ is either flat, or compact.*

In fact, it is possible to prove Theorem 3.14 using the mean curvature flow, at least under the assmption $\sup|\mathrm{II}| < \infty$. Here is an outline:

Sketch of the proof of Theorem 3.14 (assuming $\sup|\mathrm{II}| < \infty$*).* Suppose, to the contrary, that there exists a non-flat, pinched proper hypersurface (with bounded curvature) which is *not* compact. Evolve it by mean curvature (one can take a limit of flows of compact hypersurfaces which approximate the noncompact initial hypersurface). The curvature bound ensures that the maximum principle may be applied in order to preserve the pinching condition. It is possible to localize Huisken's "roundness improves" estimate by introducing suitable cut-off functions. Thus, if the evolving hypersurface becomes singular in finite time, then we can perform a blow-up procedure to obtain a shrinking sphere solution, violating the noncompactness. Else, the flow exists for all time. In that case, it is possible to perform a blow-up at time infinity to obtain a non-flat, pinched (translating or expanding) soliton solution.[9] But these may be ruled out by direct arguments. □

[9] This step makes use of the *differential Harnack inequality*, which we will discuss in Chapter 5.

3.3 Deforming locally convex hypersurfaces by curvature in Riemannian ambient spaces and the quarter-pinched sphere theorem

One immediate consequence of Huisken's Theorem (Theorem 3.13) is that compact, locally uniformly convex hypersurfaces in $\mathbb{R}^{n+1}$ are smooth spheres and bound smooth balls when $n \geq 2$. But this is quite easy to show without using the mean curvature flow since, by Sacksteder's theorem,[10] such hypersurfaces are globally convex. On the other hand, the relationship between local and global convexity for hypersurfaces in general Riemannian ambient spaces is far more subtle. Now, the mean curvature flow may also be employed to deform hypersurfaces in general Riemannian ambient spaces; however, due to the more complex evolution equation for the second fundamental

[10] Sacksteder, "On hypersurfaces with no negative sectional curvatures".

form, local uniform convexity is no longer a preserved condition in general. On the other hand, *mean convexity* (i.e. $\mathrm{H} > 0$) is seen to be a preserved condition due to the evolution equation

$$(\partial_t - \Delta)\,\mathrm{H} = \left(|\mathrm{II}|^2 + \mathrm{Rc}(\mathsf{N},\mathsf{N})\right)\mathrm{H},$$

where Rc is the ambient Ricci curvature. Similar considerations hold for normal deformations by other functions $F = f(\kappa_1, \ldots, \kappa_n)$ of curvature: while the evolution of the second fundamental form is quite complicated, and does not generally lend itself to straightforward tensor maximum principle arguments, the evolution of the speed F is relatively simple: with respect to any orthonormal frame[11],

$$\partial_t F = \frac{\partial F}{\partial \mathrm{II}_{ij}}\left(\nabla_i\nabla_j F + \left[\mathrm{II}^2_{ij} + \mathrm{Rm}(\mathsf{N}, e_i\,\mathsf{N}, e_j)\right] F\right)\;; \qquad (3.16)$$

so the scalar maximum principle certainly ensures that positivity of the speed is a preserved condition, so long as the coefficients $\frac{\partial F}{\partial \mathrm{II}_{ij}}$ are positive definite[12]. Andrews exploited this fact to construct a deformation which does indeed contract suitably locally convex hypersurfaces to round points.

[11] A smooth function of the eigenvalues of a matrix which is symmetric under permutations induces a smooth function of the matrix coefficients which is symmetric under conjugation by orthogonal matrices, and *vice versa*. See Ball, "Differentiability properties of symmetric and isotropic functions".

[12] This is also a requirement for the short-time existence of solutions, since these coefficients determine the parabolicity of the linearization of the flow.

Theorem 3.15 (Andrews[13]). *Let (N^{n+1}, h) be a Riemannian manifold satisfying* $\sup_N |\mathrm{Rm}| < \infty$, $\sup_N |\nabla \mathrm{Rm}| < \infty$, *and*

$$\sec \geq -K, \quad K \in [0, \infty).$$

Let $X_0 : M^n \to N^{n+1}$, $n \geq 2$, be a compact hypersurface satisfying

$$\mathrm{II} \geq \sqrt{K} g.$$

There exists a unique maximal solution $X : M^n \times [0, T) \to N^{n+1}$ to the flow

$$\partial_t X = -\left(\frac{1}{\kappa_1 - \sqrt{K}} + \cdots + \frac{1}{\kappa_n - \sqrt{K}}\right)^{-1}\mathsf{N} \qquad (3.17)$$

by the harmonic mean of the shifted principal curvatures $\kappa_i - \sqrt{K}$. The maximal time T is finite and there exists $p \in N^{n+1}$ such that $X_t \to p$ as $t \to T$. After rescaling an ambient neighbourhood of p by $(2n(T-t))^{-\frac{1}{2}}$, the resulting rescaled hypersurfaces converge uniformly in the smooth topology to an embedding of the unit sphere in Euclidean space. In particular, $X_0(M^n)$ is a smooth n-sphere and bounds in N^{n+1} a smooth $(n+1)$-ball.

[13] Andrews, "Contraction of convex hypersurfaces in Riemannian spaces"

Sketch of the proof. Since the speed function

$$F \doteqdot \left(\frac{1}{\kappa_1 - \sqrt{K}} + \cdots + \frac{1}{\kappa_n - \sqrt{K}}\right)^{-1}$$

satisfies $\frac{\partial F}{\partial \mathrm{II}_{ij}} > 0$ on any hypersurface satisfying $\mathrm{II} > \sqrt{K} g$, an analogous argument to that of Theorem 1.3 shows that any compact hypersurface satisfying $\mathrm{II} > \sqrt{K} g$ may be evolved for a short time according to (3.17). These hypersurfaces necessarily satisfy $\mathrm{II} > \sqrt{K}g$. In fact, since

$$\begin{aligned}\left(\partial_t - \frac{\partial F}{\partial \mathrm{II}_{ij}}\nabla_i\nabla_j\right) F &= \frac{\partial F}{\partial \mathrm{II}_{ij}}\left(\mathrm{II}^2_{ij} + \mathrm{Rm}(\mathsf{N}, e_i\, \mathsf{N}, e_j)\right) F \\ &= \frac{\partial F}{\partial \kappa_i}\left(\kappa_i^2 + \mathrm{Rm}(\mathsf{N}, e_i\, \mathsf{N}, e_i)\right) F \\ &\geq \frac{\partial F}{\partial \kappa_i}\left(\kappa_i^2 - K\right) F \\ &\geq 0,\end{aligned}$$

the maximum principle ensures that lower bounds for F are preserved. Since $\kappa_1 - \sqrt{K} \geq F$, we find that an inequality of the form $\mathrm{II} - \sqrt{K}g \geq \alpha g$, $\alpha > 0$, is preserved. In fact, estimating $\kappa_i + \sqrt{K} \geq F + 2\sqrt{K} \geq F$, we may estimate

$$\begin{aligned}\left(\partial_t - \frac{\partial F}{\partial \mathrm{II}_{ij}}\nabla_i\nabla_j\right) F &\geq \frac{\partial F}{\partial \kappa_i}\left(\kappa_i^2 - K\right) F \\ &= \frac{\partial F}{\partial \kappa_i}\left(\kappa_i - \sqrt{K}\right)\left(\kappa_i + \sqrt{K}\right) F \\ &\geq \frac{\partial F}{\partial \kappa_i}\left(\kappa_i - \sqrt{K}\right) F^2 \\ &= F^3\end{aligned}$$

due to Euler's theorem for homogeneous functions. The ODE comparison principle then ensures that $\min_{M^n \times \{t\}} F$ grows at least like $(C-t)^{-\frac{1}{2}}$, which also ensures that the existence time must be finite.

By a more involved calculation, the ratio $Q \doteqdot \frac{|\mathrm{II} - \sqrt{K}g|}{F}$ can be seen to satisfy

$$\left(\partial_t - \frac{\partial F}{\partial \mathrm{II}_{ij}}\nabla_i\nabla_j\right) Q \leq 2g\left(\frac{\nabla F}{F}, \nabla Q\right) + C,$$

where C depends only on the initial and ambient data (including bounds for the ambient curvature and its derivative). So the ODE comparison principle ensures that the maximum of $\frac{|\mathrm{II} - \sqrt{K}g|}{F}$ grows at most linearly in time, and hence remains bounded along the flow (as the maximal time is finite). Since $F \leq \kappa_1 - \sqrt{K}$, this guarantees that the shifted principal curvatures remain uniformly pinched during the evolution. One consequence of this is that the coefficient matrix $\frac{\partial F}{\partial \mathrm{II}_{ij}}$ remains uniformly bounded from above and below (in the sense of bilinear forms), which ensures that the flow remains uniformly parabolic. Regularity estimates for fully nonlinear parabolic PDE (applied to local graph parametrizations) then guarantee that the second fundamental form blows-up at the final time (cf. Theorem 2.19).

To analyze what happens as the singularity forms, we perform a BLOW-UP procedure: consider a sequence of points x_j and times t_j along which $\lambda_j \doteqdot |\mathrm{II}_{(x_j,t_j)}| = \max_{M^n\times[0,t_j]}|\mathrm{II}|$. Translating time by t_j, rescaling the ambient metric by λ_j^2 and the translated time parameter by λ_j^{-2}, and mapping neighbourhoods of $X(x_j,t_j)$ into $T_{X(x_j,t_j)}N^{n+1}$ via the inverse of the exponential map, we obtain, by choosing a sequence of identifications $T_{X(x_j,t_j)}N^{n+1}\cong\mathbb{R}^{n+1}$, a sequence of time-dependent immersions $X_j : U_j\times I_j\to\mathbb{R}^{n+1}$ of neighbourhoods $U_j\times I_j$ of $(x_j,0)$ into $\mathbb{R}^{n+1}$ which are properly defined in parabolic cylinders $B_{R_j}(0)\times(-R_j^2,0]$ of radius $R_j\sim\lambda_j^{-1}$ and satisfy $X_j(x_j,0)=0$. Since $\lambda_j\to\infty$, X_j is eventually properly defined in any ambient parabolic cylinder $B_R\times(-R^2,0]$. Since $\lambda_j=\sup_{M^n\times[0,t_j]}|\mathrm{II}|$ and the ambient geometry is bounded, the curvature of X_j is maximized at the spacetime origin up to small error which tends to zero as $j\to\infty$. Similarly, since the ratio $\frac{\kappa_1-\sqrt{K}}{\kappa_n-\sqrt{K}}$ is scale invariant, the sequence will eventually be uniformly pinched (without shift), up to an error which tends to zero. By the PDE bootstrapping estimates employed above, bounds for X_j to all orders follow, so we obtain a convergent subsequence (cf. Theorem 2.22) to a complete limit $X : M^n_\infty\times(-\infty,0]\to\mathbb{R}^{n+1}$, which evolves by the harmonic mean of the (unshifted) principal curvatures. By applying the Krylov–Safanov Harnack inequality to the scalar parabolic PDE satisfied by local graphical parametrizations, one also obtains a uniform bound for the second fundamental form from below along the rescaled sequence, which guarantees that the limit is not flat. Since the limit will be uniformly pinched, it must then be compact by Theorem 3.14. Since the maximum of the ratio $Q=\frac{|\mathrm{II}-\sqrt{K}g|}{F}$ is bounded, the limiting process forces it to be constant in time on the limit flow. But then the strong maximum principle ensures that it is constant in space as well, and an analysis of the gradient terms in its evolution equation (which must vanish) then implies that the limit is a shrinking round sphere (cf. the proof of Proposition 3.2). So the solution is asymptotically round after rescaling, at least along some sequence of times.

This information can now be pulled back to the original flow, and bootstrapped to conclude that the hypersurfaces contract to round points. See[14] for further details. □

[14] Andrews, "Contraction of convex hypersurfaces in Euclidean space", "Contraction of convex hypersurfaces in Riemannian spaces".

Combining this with an idea of Gromov[14] yields a simple proof of the classical quarter pinched sphere theorem of Rauch, Berger and Klingenberg[15]. In fact, it yields something slightly stronger.

[14] See Eschenburg, "Local convexity and nonnegative curvature—Gromov's proof of the sphere theorem".

[15] Berger, "Les variétés riemanniennes (1/4)-pincées"; Klingenberg, "Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung"; Rauch, "A contribution to differential geometry in the large".

Theorem 3.16 (Quarter-pinched "twisted" sphere theorem[16])**.** *Let N^n, $n\geq 3$, be a compact, simply connected Riemannian manifold. If $\frac{1}{4}<\sec\leq 1$, then N^n is diffeomorphic to a* TWISTED SPHERE *(a quotient of the union of*

[16] Andrews, "Contraction of convex hypersurfaces in Riemannian spaces".

two discs by a diffeomorphism of their boundaries).

Proof. Choose a point $o \in N^n$ and denote by S^{n-1} the unit sphere in T_oN^n. Consider the family of "exponential spheres" $X : S^{n-1} \times (0, \infty) \to N^n$ defined by $X(z,t) \doteqdot \exp_o(tz)$. These spheres are dynamically related through the equation

$$\partial_t X = \mathsf{N} \, .$$

For small values of t, the maps $X_t \doteqdot X(\cdot, t)$ are smooth embeddings and their images bound a disk. As t varies, the induced metric g and second fundamental form II deform according to

$$\partial_t g = 2\,\mathrm{II} \tag{3.18a}$$

$$\nabla_t \,\mathrm{II} = -\mathrm{II}^2 - \mathrm{Rm}(\cdot, \mathsf{N}, \cdot, \mathsf{N})\,. \tag{3.18b}$$

Using the bounds on the ambient curvature, (3.18b) yields

$$\cot t \le \kappa_i < \tfrac{1}{2}\cot(\tfrac{1}{2}t)\,. \tag{3.19}$$

which by (3.18a) ensures that the spheres are nondegenerate for any $t < \pi$. The strict inequality from above in (3.19) implies that there is some distance $t < \pi$ for which $0 > \kappa_i > -\infty$ for each i, which means that the corresponding exponential sphere is strictly convex in the *outward* direction. But then Theorem 3.15 guarantees that this sphere bounds a disc in N^n. This gives an expression for N^n as a quotient of the union of two discs along a diffeomorphism of their boundaries. □

Since the second part of the proof Theorem 3.16 (application of Theorem 3.15) does not require the pinching assumption, the argument actually yields the following quantitative refinement.

Theorem 3.17 (Dented sphere theorem[17]). *Let N^n, $n \ge 3$, be a compact, simply connected Riemannian manifold with sectional curvatures bounded from below by $-K$ for some $K \ge 0$. Choose $\varepsilon > 0$ such that $\varepsilon \cot(\varepsilon\pi) < -\sqrt{K}$, and $\rho \in [\frac{\pi}{2}, \pi)$ such that $\varepsilon\cot(\varepsilon\rho) = -\sqrt{K}$. If there is a point $o \in N^n$ such that $\varepsilon < \sec \le 1$ on the ball $B_\rho(o)$, then N^n is diffeomorphic to a twisted sphere.*

[17] ibid.

3.4 *Contraction of quadratically pinched submanifolds to round points*

The notion of (local uniform) convexity does not generalize to higher codimension (since the second fundamental form is vector valued). Nonetheless, there have been attempts to find a satisfying generalization of Huisken's theorem. The following result is due to Andrews and Baker.[18]

[18] Andrews and Baker, "Mean curvature flow of pinched submanifolds to spheres".

Theorem 3.18. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+k}$, $n \geq 2$, be the maximal mean curvature flow starting from a smooth, compact immersed submanifold $X_0 : M^n \to \mathbb{R}^{n+k}$ satisfying the inequalities*

$$|\vec{\mathrm{H}}| > 0 \ \ \textit{and} \ \ |\vec{\mathrm{II}}|^2 \leq c_n |\vec{\mathrm{H}}|^2$$

at all points, where

$$c_n \doteqdot \begin{cases} \frac{4}{3n} & \textit{if } 2 \leq n \leq 4 \\ \frac{1}{n-1} & \textit{if } n \geq 4. \end{cases}$$

There exists $p \in \mathbb{R}^{n+k}$ and an $(n+1)$-dimensional subspace L^{n+1} of $\mathbb{R}^{n+k}$ such that

$$\frac{X(\cdot,t) - p}{\sqrt{2n(T-t)}} \to \overline{X}$$

uniformly in the smooth topology as $t \to T$, where $\overline{X} : M^n \to \mathbb{R}^{n+1}$ is an embedding whose image is the unit sphere in L^{n+1}.

Note that the pinching condition is sharp in dimensions $n \geq 4$, in the sense that, for any $\varepsilon > 0$, there exist, for each $n \geq 2$ and $k \geq 2$, aspherical n-dimensional submanifolds of $\mathbb{R}^{n+k}$ which satisfy $|\vec{\mathrm{II}}|^2 \leq (\frac{1}{n-1} + \varepsilon)|\vec{\mathrm{H}}|^2$ (consider products of round spheres).

Sketch of the proof of Theorem 3.18. The argument is similar, in principle, to that of Huisken's theorem: one first shows that the pinching condition is preserved, by applying the scalar maximum principle to the evolution equation for[19] $\frac{|\vec{\mathrm{II}}|^2}{|\vec{\mathrm{H}}|^2}$. One then establishes an a bound for the trace free part of $\vec{\mathrm{II}}$ by applying a Stampacchia iteration argument to the function $\left(|\vec{\mathrm{H}}|^\sigma \left[\frac{|\vec{\mathrm{II}}|^2}{|\vec{\mathrm{H}}|^2} - \frac{1}{n} - \varepsilon\right]\right)^p$ for suitable σ and p (cf. Proposition 3.5). A bound for the derivative of $\vec{\mathrm{II}}$ may then be established using the maximum principle (cf. Proposition 3.10), and this provides a uniform blow-up rate for the curvature by way of Myers' theorem, just as in Proposition 3.12. This is enough to establish convergence to a sphere in the C^0 topology. Bootstrapping arguments then give higher order convergence. □

[19] In low dimensions, the suboptimal pinching condition must be imposed here in order to control terms involving derivatives of curvature using the Kato inequality $|\nabla\vec{\mathrm{II}}|^2 \geq \frac{3}{n+2}|\nabla\vec{\mathrm{H}}|^2$.

A stronger result has been established by Baker and Nguyen[20] when both the dimension and codimension are equal to two.

[20] Baker and Huy The Nguyen, "Codimension two surfaces pinched by normal curvature evolving by mean curvature flow".

3.5 Exercises

Exercise 3.1. Provide details for the proof of Proposition 3.3.

Exercise 3.2. Theorem 3.13 asserts, in particular, the existence of a point p such that $X_t \to p$. Prove this by integrating the mean curvature flow evolution equation and applying the estimate $\mathrm{H} \leq \frac{C}{\sqrt{T-t}}$.

4 Curve shortening flow

A key step in the proof of Huisken's theorem on the convergence of convex hypersurfaces to round points was the observation that "roundness" improves as the curvature blows-up. No such estimate is possible for evolving curves, as, in that case, the second fundamental form has only one component! Fortunately, the planar mean curvature flow, more commonly referred to as the CURVE SHORTENING FLOW, enjoys some additional structure, which actually allows us to prove something far stronger.

4.1 Special properties of mean curvature flow in one space dimension

In one space dimension, the mean curvature and second fundamental form coincide, so the mean curvature flow takes the form

$$\partial_t X = \vec{\kappa}, \tag{4.1}$$

where $\vec{\kappa}$ is the CURVATURE VECTOR of the curve, which upon choosing[1] a (local) unit normal field N, may be expressed in terms of the CURVATURE κ as $\vec{\kappa} = -\kappa\,\mathsf{N}$. This equation is also the one-dimensional special case of a number of other higher dimensional flows (e.g. the Gauss curvature flow and the harmonic mean curvature flow). With this in mind, it is perhaps not surprising that (4.1) displays properties of these higher dimensional flows that are not necessarily shared by the mean curvature flow in higher dimensions.

[1] When a choice of convention is required, the unit tangent and normal vectors T and N will satisfy $\mathsf{N} = -J\,\mathsf{T}$, where J denotes counterclockwise rotation by 90 degrees, and, whenever the curve is understood as the boundary of a region, N will agree with the *outward* pointing unit normal. Unless explicitly stated otherwise, a closed, embedded curve will be understood as the boundary of the *bounded* region which the Jordan–Schoenflies theorem guarantees that it bounds (rather than the *unbounded* region); in other words, such a curve will be traversed in a *counterclockwise* manner.

4.1.1 A simple formula for the enclosed area

By the theorem of turning tangents and the first variation of enclosed area, 2.19, the area enclosed by a closed, embedded curve shortening flow changes at a precise rate: -2π. Integrating this yields

$$\text{area}(M^1, t) = \text{area}(M^1, 0) - 2\pi t, \tag{4.2}$$

a remarkably simple—and useful—formula.

4.1.2 *The turning angle parametrization*

Recall that the TURNING ANGLE of a planar curve $\gamma : M^1 \to \mathbb{R}^2$ is the angle $\theta : M^1 \to \mathbb{R}/2\pi\mathbb{Z}$ that its unit tangent vector $\mathsf{T} \doteqdot \gamma'/|\gamma'|$ makes with the x-axis. I.e.

$$\mathsf{T} = (\cos\theta, \sin\theta)\,.$$

Differentiating this equation with respect to arclength yields

$$-\kappa\,\mathsf{N} = (-\sin\theta, \cos\theta)\theta_s = -\theta_s\,\mathsf{N}\,.$$

That is,

$$\theta_s = \kappa\,. \tag{4.3}$$

In particular, for a convex, locally uniformly convex curve $\Gamma = \partial\Omega$, $\theta : \Gamma \to \mathbb{R}/2\pi\mathbb{Z}$ is a diffeomorphism onto the GAUSS IMAGE[2], $\theta(\Gamma)$. The diffeomorphism provides a convenient reparametrization of the curve, $\gamma \circ \theta^{-1} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ (which we shall also simply denote by γ), called the TURNING ANGLE PARAMETRIZATION.

[2] Note that $\theta(\Gamma) = \mathbb{R}/2\pi\mathbb{Z}$ if Ω is bounded, while $|\theta(\Gamma)| < \pi$ if Ω is unbounded.

Now, if some *family* of convex, locally uniformly convex curves $\gamma : M^1 \times I \to \mathbb{R}^2$ evolves by *curve shortening flow*, then, differentiating the equation

$$\mathsf{N} = (\sin\theta, -\cos\theta)$$

with respect to t yields

$$\nabla\kappa = (\cos\theta, \sin\theta)\theta_t\,;$$

that is,

$$\theta_t = \kappa_s\,. \tag{4.4}$$

Combining (4.3) and (4.4), we find that, with respect to the turning angle parametrization,

$$\gamma_t = \vec{\kappa} - \kappa^{-1}\nabla\kappa\,.$$

Similarly, the curvature (expressed in the turning angle parametrization) satisfies

$$\kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa) \tag{4.5}$$

under locally uniformly convex curve shortening flow.

Now, by Equation (4.3), any locally uniformly convex curve can be reconstructed from its curvature function via the formula

$$(x(\theta), y(\theta)) = \left(x_0 + \int_0^\theta \frac{\cos\theta}{\kappa}\,d\theta, y_0 + \int_0^\theta \frac{\sin\theta}{\kappa}\,d\theta\right)\,. \tag{4.6}$$

This reduces (closed) convex, locally uniformly convex curve shortening flow to the equation (4.5) for the curvature function—a quasilinear, strictly parabolic scalar partial differential equation in one space variable (and periodic boundary condition).

Recall next that the SUPPORT FUNCTION $\sigma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ of a convex subset $\Omega \subset \mathbb{R}^2$ is defined by

$$\sigma(\theta) \doteqdot \sup_{x\in\Omega} x \cdot (\sin\theta, -\cos\theta),$$

which is just the distance[3] from the origin to the boundary of the supporting half-space whose outer unit normal is $(\sin\theta, -\cos\theta)$.

Observe that a convex, locally uniformly convex curve $\Gamma = \partial\Omega$ can be reconstructed from its support function via the TURNING ANGLE PARAMETRIZATION[4] $\gamma : \theta(\Gamma) \to \mathbb{R}^2$, which is given by

$$\gamma \doteqdot \sigma\,\mathsf{N} + \sigma_\theta\,\mathsf{T},$$

where $\mathsf{N}(\theta) \doteqdot (\sin\theta, -\cos\theta)$ and $\mathsf{T}(\theta) \doteqdot (\cos\theta, \sin\theta)$. Differentiating this equation with respect to s yields

$$\mathsf{T} = \gamma_s = (\sigma_{\theta\theta} + \sigma)\kappa\,\mathsf{T},$$

and hence

$$\kappa = (\sigma_{\theta\theta} + \sigma)^{-1}. \tag{4.7}$$

Now, if some *family* of convex, locally uniformly convex planar curves $\{\Gamma_t = \partial\Omega_t\}_{t\in I}$ evolves by *curve shortening flow*, then, under the (time-dependent) turning angle parametrization,

$$\gamma_t \cdot \mathsf{N} = -\kappa,$$

from which we find

$$\sigma_t = -(\sigma_{\theta\theta} + \sigma)^{-1}. \tag{4.8}$$

This reduces the study of (closed) locally uniformly convex curve shortening flow to a strictly parabolic scalar partial differential equation in one space variable (with periodic boundary condition).[5]

4.1.3 *Entropy formulae*

The GAGE–HAMILTON ENTROPY[6] of a convex, locally uniformly convex planar curve $\Gamma = \partial\Omega$ is defined to be

$$\mathscr{E}(\Gamma) \doteqdot \left(\frac{\operatorname{area}(\Omega)}{\pi}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2\pi}\int_\Gamma \kappa\log\kappa\, ds\right). \tag{4.9}$$

Proposition 4.1 (Monotonicity of the Gage–Hamilton entropy)**.** *If the convex, locally uniformly convex curves $\{\Gamma_t = \partial\Omega_t\}_{t\in I}$ are bounded and evolve by curve shortening, then*

$$\frac{d}{dt}\mathscr{E}(\Gamma_t) \leq 0$$

at all times, with strict inequality unless $\frac{\kappa_t}{\kappa}$ is constant in θ.

[3] Which could be infinite if Ω is unbounded.

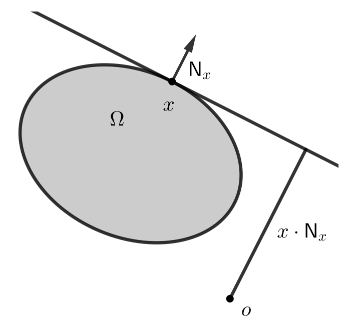


Figure 4.1: For a smooth, strictly convex region Ω, $\sigma(z) = x \cdot \mathsf{N}_x$, where x is the unique point on $\partial\Omega$ satisfying $\mathsf{N}_x = z$.

[4] See Exercise 4.1.

[5] These phenomena are not, strictly speaking, specific to curve shortening flow: on a locally uniformly convex hypersurface, the unit normal vector provides a local diffeomorphism into the sphere, which is global if the hypersurface is convex; the inverse then provides a nice parametrization (the GAUSS MAP PARAMETRIZATION), with respect to which the curvature and support functions satisfy analogous scalar parabolic partial differential equations under mean curvature flow, albeit on on S^n. The curve case does have some unique advantages, however: first, $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, so the relevant partial differential equations can be interpreted classically; 2. due to (4.3), *every* smooth curve can be decomposed into flat and locally uniformly convex pieces, so the local description of a smooth curve by turning angle is essentially general; and 3. in the context of solitons, for example, the partial differential equations for the curvature and support function reduce to ordinary differential equations.

[6] Compare this to the NASH ENTROPY, $-\int u\log u$, of a positive function u, introduced by Nash, "Continuity of solutions of parabolic and elliptic equations".

Proof. In the turning angle parametrization, $\gamma : \mathbb{R}/2\pi\mathbb{Z} \times I \to \mathbb{R}^2$,

$$\begin{aligned}\kappa_t &= \partial_t(\sigma_{\theta\theta} + \sigma)^{-1}\\ &= -(\sigma_{\theta\theta} + \sigma)^{-2}(\sigma_{t\theta\theta} + \sigma_t)\\ &= \kappa^2(\kappa_{\theta\theta} + \kappa),\end{aligned}$$

and

$$ds = \kappa\, d\theta.$$

Thus,

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_t} \kappa \log \kappa\, ds &= \frac{d}{dt}\int_{\mathbb{R}/2\pi\mathbb{Z}} \log \kappa\, d\theta\\ &= \int_{\mathbb{R}/2\pi\mathbb{Z}} \frac{\kappa_t}{\kappa}\, d\theta.\end{aligned}$$

Since

$$\begin{aligned}\partial_t \frac{\kappa_t}{\kappa} &= \partial_t\big(\kappa(\kappa_{\theta\theta} + \kappa)\big)\\ &= \kappa_t(\kappa_{\theta\theta} + \kappa) + \kappa(\kappa_{t\theta\theta} + \kappa_t)\\ &= \left(\frac{\kappa_t}{\kappa}\right)^2 + \kappa\left(\kappa\frac{\kappa_t}{\kappa}\right)_{\theta\theta} + \kappa^2\frac{\kappa_t}{\kappa}\\ &= 2\left(\frac{\kappa_t}{\kappa}\right)^2 + \kappa^2\left(\frac{\kappa_t}{\kappa}\right)_{\theta\theta} + 2\kappa_\theta\left(\frac{\kappa_t}{\kappa}\right)_\theta \qquad (4.10)\\ &= 2\left(\frac{\kappa_t}{\kappa}\right)^2 + \left(\kappa^2\left(\frac{\kappa_t}{\kappa}\right)_\theta\right)_\theta,\end{aligned}$$

we find that

$$\begin{aligned}\frac{d^2}{dt^2}\int_{\Gamma_t} \kappa \log \kappa\, ds &= \int_{\mathbb{R}/2\pi\mathbb{Z}} \left[2\left(\frac{\kappa_t}{\kappa}\right)^2 + \left(\kappa^2\left(\frac{\kappa_t}{\kappa}\right)_\theta\right)_\theta\right] d\theta\\ &= 2\int_{\mathbb{R}/2\pi\mathbb{Z}} \left(\frac{\kappa_t}{\kappa}\right)^2 d\theta.\end{aligned}$$

Applying Hölder's inequality and the theorem of turning tangents, we arrive at

$$\frac{d^2}{dt^2}\int_{\Gamma_t} \kappa \log \kappa\, ds \geq \frac{1}{\pi}\left(\frac{d}{dt}\int_{\Gamma_t} \kappa \log \kappa\, ds\right)^2. \qquad (4.11)$$

On the other hand, recalling (4.2), we see that the function

$$\phi(t) \doteqdot \frac{2\pi^2}{\mathrm{area}(\Omega_t)} = \frac{\pi}{\frac{\mathrm{area}(\Omega_0)}{2\pi} - t}$$

satisfies the corresponding ODE

$$\frac{d\phi}{dt} = \tfrac{1}{\pi}\phi^2.$$

Moreover, by Proposition 2.25 (see Exercise 2.5), the flow may be continued until the enclosed area tends to zero; i.e. (by (4.2)) until time $T \doteqdot \frac{\mathrm{area}(\Omega_0)}{2\pi}$. This means that

$$\phi(t) \to \infty \text{ as } t \to T,$$

and we may thereby deduce, by ODE comparison, that

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_t}\kappa\log\kappa\,ds &\leq \frac{2\pi^2}{\operatorname{area}(\Omega_t)}\\ &= -\pi\frac{d}{dt}\log\operatorname{area}(\Omega_t).\end{aligned}$$

Rearranging, we conclude that

$$\frac{d}{dt}\log\mathscr{E}(\Gamma_t)\leq 0.$$

Now, if the inequality is saturated at some time t_0, then we may deduce from (4.11) that is saturated for all $t \leq t_0$. But this guarantees that the Hölder inequality is saturated, which ensures that $\frac{\kappa_t}{\kappa}$ is constant with respect to θ for $t \leq t_0$. □

The FIREY ENTROPY of a convex, locally uniformly convex planar curve $\Gamma = \partial\Omega$ is defined to be

$$\mathscr{F}(\Gamma) \doteqdot \left(\frac{\pi}{\operatorname{area}(\Omega)}\right)^{\frac{1}{2}}\exp\left(\frac{1}{2\pi}\int_{\mathbb{R}/2\pi\mathbb{Z}}\log\sigma\,d\theta\right). \tag{4.12}$$

Proposition 4.2 (Monotonicity of the Firey entropy)**.** *If the convex, locally uniformly convex curves $\{\Gamma_t = \partial\Omega_t\}_{t\in I}$ are bounded, enclose the origin, and evolve by curve shortening, then*

$$\frac{d}{dt}\mathscr{F}(\Gamma_t)\leq 0$$

at all times, with strict inequality unless $\frac{\kappa}{\sigma}$ is constant with respect to θ.

Proof. Observe that

$$\begin{aligned}\frac{d}{dt}\log\mathscr{F}(t) &= \frac{d}{dt}\frac{1}{2\pi}\int_{\mathbb{R}/2\pi\mathbb{Z}}\log\sigma\,d\theta - \frac{1}{2}\frac{d}{dt}\log\operatorname{area}(\Omega_t)\\ &= -\frac{1}{2\pi}\int_{\mathbb{R}/2\pi\mathbb{Z}}\frac{\kappa}{\sigma}\,d\theta + \frac{\pi}{\operatorname{area}(\Omega_t)}\\ &= \frac{\int_{\mathbb{R}/2\pi\mathbb{Z}}d\theta}{\int_{\mathbb{R}/2\pi\mathbb{Z}}\sigma\,d\theta} - \frac{\int_{\mathbb{R}/2\pi\mathbb{Z}}\frac{\kappa}{\sigma}\,d\theta}{\int_{\mathbb{R}/2\pi\mathbb{Z}}d\theta}\\ &= \frac{\int_{\mathbb{R}/2\pi\mathbb{Z}}d\theta\int_{\mathbb{R}/2\pi\mathbb{Z}}d\theta - \int_{\mathbb{R}/2\pi\mathbb{Z}}\sigma\,d\theta\int_{\mathbb{R}/2\pi\mathbb{Z}}\frac{\kappa}{\sigma}\,d\theta}{\int_{\mathbb{R}/2\pi\mathbb{Z}}\sigma\,d\theta\int_{\mathbb{R}/2\pi\mathbb{Z}}d\theta}\\ &= \frac{\int_{\mathbb{R}/2\pi\mathbb{Z}}\int_{\mathbb{R}/2\pi\mathbb{Z}}\left(q(\theta)-q(\omega)\right)\left(q^{-1}(\theta)-q^{-1}(\omega)\right)d\theta\,d\omega}{\int_{\mathbb{R}/2\pi\mathbb{Z}}\sigma\,d\theta\int_{\mathbb{R}/2\pi\mathbb{Z}}d\theta},\end{aligned}$$

where

$$q \doteqdot \frac{\kappa}{\sigma}.$$

The claims follow, since the integrand in the numerator is manifestly nonpositive, and vanishes only if $\frac{\kappa}{\sigma}$ is constant with respect to θ. □

4.1.4 Zero counting

For parabolic equations in one space variable, the maximum principle has the following powerful refinement (whose proof is nontrivial).

Theorem 4.3 (Sturmian Theorem[7]). *Let $u : [0, L] \times [0, T] \to \mathbb{R}$ be a non-constant solution to the equation*

$$u_t = au_{xx} + bu_x + cu$$

satisfying either Dirichlet ($u(\xi, t) = 0$), Neumann ($u_x(\xi, t) = 0$) or inhomogeneous ($u(\xi, t) \neq 0$) boundary conditions on $\partial[0, L]$ (possibly mixed), or the periodic boundary condition ($u(0, t) = u(L, t)$ and $u_x(0, t) = u_x(L, t)$). Suppose that the coefficients (each a function of space and time) satisfy

$$a > 0 \ \text{ and } \ a,\, a^{-1},\, a_t,\, a_x,\, a_{xx},\, b,\, b_t,\, b_x,\, c \in L^\infty\,;$$

in the case of Neumann conditions, assume also that $a = 1$ and $b = 0$. If $t \in (0, T]$, then the zero set $\{x \in [0, L] : u(x, t) = 0\}$ is finite, and its magnitude strictly decreases across any time $t \in (0, T)$ at which $u(\cdot, t)$ admits a degenerate zero x (i.e. $u(x, t) = 0$ and $u_x(x, t) = 0$).

[7] Sigurd B. Angenent, "The zero set of a solution of a parabolic equation"

One application of this to curve shortening flow is the following "de-intersection" principle, due to Angenent.[8]

[8] Sigurd B. Angenent, "Nodal properties of solutions of parabolic equations".

Proposition 4.4. *For $j = 1, 2$, let $\gamma_j : M^1_j \times [0, T_j) \to \mathbb{R}^2$ be a pair of proper curve shortening flows with $M^1_1 \cong S^1$. Set $\Gamma^j_t \doteqdot \gamma_j(M^1_j, t)$ for $t \in [0, T_j)$. Unless $\gamma_1|_{[0,T)} \equiv \gamma_2|_{[0,T)}$, where $T \doteqdot \min\{T_1, T_2\}$, the intersection points $\Gamma^1_t \cap \Gamma^2_t$ are finite in number for $t \in (0, T)$ and strictly decrease in number at each time $t \in (0, T)$ there is a degenerate intersection (i.e. a point of first order contact).*

Proof. If $\Gamma^1_0 = \Gamma^2_0$, then $M^1_2 \cong S^1$ and we may conclude that $\Gamma^1_t \equiv \Gamma^2_t$ for all $t \in [0, T)$ by uniqueness of solutions to curve shortening flow (on compact manifolds). Suppose that $\Gamma^1_0 \cap \Gamma^2_0$ is nonempty and let $p \in \Gamma^1_0 \cap \Gamma^2_0$ be an intersection point such that $B_r(p)$ contains points of non-intersection, $q \in (\Gamma^2_0 \setminus \Gamma^1_0) \cup (\Gamma^1_0 \setminus \Gamma^2_0)$, for all $r > 0$. Choose $x_0 \in \gamma_1(\cdot, 0)^{-1}(p)$ (there are finitely many) and set $L \doteqdot d\gamma_1(\cdot, t) T_{x_0} S^1$. By the implicit function theorem, we can find some $\delta > 0$ and smooth functions $u_j : L \times [0, \infty) \to \mathbb{R}$ such that $\operatorname{graph} u_j(\cdot, t) \cap B_\delta(p) \subset \Gamma^j_t$ for all $t \in [0, \delta)$ for each j, where $\operatorname{graph} u_j(\cdot, t) \doteqdot \{p + u_j(x, t)\, \mathsf{N}_j(x_0, 0) : x \in L\}$. Since the curves evolve by curve shortening, the height functions u_j satisfy

$$(u_j)_t = \frac{(u_j)_{xx}}{1 + (u_j)_x^2}$$

in a neighbourhood $[-r, r] \times [0, r^2)$ of $(0, 0)$. But then the difference,

$v \doteqdot u_2 - u_1$, satisfies

$$\begin{aligned} v_t &= \frac{(u_2)_{xx}}{1+(u_2)_x^2} - \frac{(u_1)_{xx}}{1+(u_1)_x^2} \\ &= \int_0^1 \frac{d}{ds} \frac{(su_2+(1-s)u_1)_{xx}}{1+(su_2+(1-s)u_1)_x^2}\, ds \\ &= av_{xx} + bv_x\,, \end{aligned}$$

where

$$a \doteqdot \int_0^1 \frac{ds}{1+(su_2+(1-s)u_1)_x^2}$$

and

$$b \doteqdot -2\int_0^1 \frac{(su_2+(1-s)u_1)_{xx}(su_2+(1-s)u_1)_x^2}{1+(su_2+(1-s)u_1)_x^2}\, ds\,.$$

Since, at least for some short time, v must satisfy either Dirichlet-inhomogeneous or inhomogeneous-inhomogeneous boundary conditions at the boundary of $[-r, r]$, we conclude from Theorem 4.3 that the zero set of v becomes finite in a neighbourhood of 0, and remains so, at least for a short time. By compactness, we conclude that $\Gamma_t^2 \cap \Gamma_t^1$ is a finite set for small, positive times. The same localization, applied now in a neighbourhood of a time-interior intersection point, then shows that the number of intersection points is nonincreasing (and strictly decreases each time there is a degenerate intersection). □

Along a curve shortening flow $\gamma : M^1 \times [0, T] \to \mathbb{R}^2$, $M^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, the evolution equation for the curvature, κ, may be expressed with respect to a local (time-independent) coordinate x as

$$\kappa_t = a\kappa_{xx} + b\kappa_x + c\kappa\,,$$

where

$$a \doteqdot |\gamma_x|^{-2}\,,\;\; b \doteqdot -|\gamma_x|^{-4}\gamma_{xx}\cdot\gamma_x\,,\;\; \text{and}\;\; c = \kappa^2.$$

Differentiating then yields

$$(\kappa_x)_t = a(\kappa_x)_{xx} + (a_x+b)(\kappa_x)_x + (b_x+3c)\kappa_x\,.$$

We thereby obtain the following nodal properties of the curvature (also due to Angenent[9]).

[9] Sigurd B. Angenent, "On the formation of singularities in the curve shortening flow".

Proposition 4.5. *Let $\gamma : M^1 \times [0, T) \to \mathbb{R}^2$ be a curve shortening flow on $M^1 \cong S^1$. Unless γ is a shrinking circle, the inflection points $\{x \in M^1 : \kappa(x, t) = 0\}$ and vertices $\{x \in M^1 : \kappa_s(x, t) = 0\}$ are finite in number for all $t > 0$. The number of inflection points is nonincreasing, and strictly decreases each time $\gamma(\cdot, t)$ admits a degenerate inflection point.*

4.1.5 Monotonicity of the total curvature

Denote the total curvature of a planar curve $\gamma : M^1 \to \mathbb{R}^2$ by

$$\mathscr{K} \doteqdot \int_{M^1} |\kappa|\, ds\,.$$

Altschuler established the following monotonicity formula for $\mathscr{K}$ under curve shortening flow.[10]

[10] S. J. Altschuler, "Singularities of the curve shrinking flow for space curves".

Lemma 4.6 (Total curvature monotonicity formula)**.** *On any curve shortening flow $\gamma : M^1 \times I \to \mathbb{R}$ with $M^1 \cong S^1$,*

$$\frac{d\mathscr{K}}{dt} = -2 \sum_{\{p\in M^1 : \kappa(p,\cdot)=0\}} |\nabla\kappa|\,, \tag{4.13}$$

except possibly at finitely many times. In particular, $\mathscr{K}$ is nonincreasing under curve shortening flow, and strictly decreases unless κ has a consistent sign.

Proof. By Proposition 4.5, either the solution is a shrinking circle (in which case the claim holds trivially), or the inflection points are finite in number and nondegenerate, except possibly at a finite set of times (at which their number strictly decreases). Away from these times, we may split $\Gamma_t = \gamma(M^1, t)$ into a finite, locally constant number $N = N(t)$ of segments, $\{\Gamma_t^j\}_{j=1}^N$, with boundaries $\{a_{i-1}, a_i\}_{i=1}^N$, $a_N = a_0$, on which κ is nonzero and alternates sign, so that, for an appropriate choice of arclength parameter,

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t} |\kappa| &= \sum_{j=1}^{N} (-1)^{j-1} \frac{d}{dt}\int_{\Gamma_t^j} \kappa\, ds\\
&= \sum_{i=1}^{N} (-1)^{j-1} \int_{\Gamma_t^j} (\kappa_t - \kappa^3)\, ds\\
&= \sum_{i=1}^{N} (-1)^{j-1} \int_{\Gamma_t^j} \kappa_{ss}\, ds\\
&= -\kappa_s(a_0) + 2\sum_{j=1}^{N-1} (-1)^{j-1}\kappa_s(a_j) + (-1)^{N-1}\kappa_s(a_N)\\
&= 2\sum_{j=0}^{N-1} (-1)^{j-1}\kappa_s(a_j)\,.
\end{aligned}$$

The claim follows since $(-1)^i \kappa_s(a_i) \geq 0$ for each i. □

4.2 Self-similar solutions

Recall that a planar curve $\gamma : M^1 \to \mathbb{R}^2$ generates a self-similar curve shortening flow if

$$\vec{\kappa} + (\gamma^* V)^{\perp} = 0 \tag{4.14}$$

for some planar vector field V of the form

$$V(x) = \tfrac{\lambda}{2}x + \mu J x - v \tag{4.15}$$

for some parameters[11] $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $v \in \mathbb{R}^2$.

Observe that, when $\mu = 0$, we may express the soliton vector field V as the gradient,

$$V = Df\,, \tag{4.16}$$

of a POTENTIAL FUNCTION $f : \mathbb{R}^2 \to \mathbb{R}$, which is given (modulo an additive constant) by

$$f(x) \doteqdot \tfrac{\lambda}{4}|x|^2 - x \cdot v\,. \tag{4.17}$$

[11] The parameter λ generates dilations, while μJ generates rotations and v generates translations.

Observe that we may rewrite (4.14) as the scalar equation

$$\kappa = \tfrac{\lambda}{2}\gamma \cdot \mathsf{N} - \mu\gamma \cdot \mathsf{T} - v \cdot \mathsf{N}\,. \tag{4.18}$$

On a locally uniformly convex shrinker parametrized by turning angle, this becomes

$$\kappa = \tfrac{\lambda}{2}\sigma\,,$$

from which we deduce that

Proposition 4.7. *the monotonicity formula for the Firey entropy (Proposition 4.2) is saturated precisely on the bounded, convex, self-similarly shrinking solutions.*

We have already seen that the straight lines and Grim Reapers are the only self-similarly translating (planar) curve shortening flows. This is complemented by the following result for shrinkers.[12]

Theorem 4.8. *The shrinking circles are the only bounded, convex, self-similarly shrinking curve shortening flows.*

Sketch of the proof. We may assume that $v = 0$ (this may be arranged by a translation in space), in which case (4.18) becomes

$$(\sigma_{\theta\theta} + \sigma)^{-1} = \tfrac{\lambda}{2}\sigma \tag{4.19}$$

subject to the periodic boundary condition $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Observe that

$$\left[\tfrac{1}{2}(\sigma_\theta^2 + \sigma^2) - \log \sigma^{\frac{2}{\lambda}}\right]_\theta = \sigma_\theta\left(\sigma_{\theta\theta} + \sigma - \tfrac{2}{\lambda}\sigma^{-1}\right)$$
$$= 0\,.$$

That is,

$$e^{\frac{1}{2}(\sigma_\theta^2 + \sigma^2)} = A\sigma^{\frac{2}{\lambda}} \tag{4.20}$$

for some $A > 0$.

[12] A complete classification of shrinkers was given by Abresch and Langer, "The normalized curve shortening flow and homothetic solutions" and Epstein and Weinstein, "A stable manifold theorem for the curve shortening equation" (cf. Andrews, "Classification of limiting shapes for isotropic curve flows"). The expanding case was treated by Urbas, "Complete noncompact self-similar solutions of Gauss curvature flows. II. Negative powers". The general case was treated by Halldorsson, "Self-similar solutions to the curve shortening flow".

In particular, at any critical point of σ,

$$e^{\frac{1}{2}\sigma^2} = A\sigma^{\frac{2}{\lambda}}. \tag{4.21}$$

For fixed λ, this equation can have zero, one or two solutions in the domain $\sigma \in (0, \infty)$, depending on the value of A. Denote by A_0 the critical value of A (i.e. the value for which the equation admits one solution). If $A < A_0$, then (4.21) admits no solutions, which is impossible since, by the four vertex theorem, $\sigma = \frac{2}{\lambda}\kappa$ admits at least four critical points. If $A = A_0$, then, since (4.21) admits only one solution, the value of σ at all critical points (including its maximum and minimum) must agree, which means that σ is constant and the solution is a circle. In case $A > A_0$, denote by σ_- and σ_+ the two solutions to (4.21), with $\sigma_- < \sigma_+$. Since σ_θ is nonzero between σ_- and σ_+, we may solve (4.20) for $\frac{d\theta}{d\sigma}$ in $[\theta_-, \theta_+]$. Integrating, we find that the difference in the turning angle between these points is given by

$$\begin{aligned}\Theta &\doteqdot \frac{1}{\sqrt{2}}\int_{\sigma_-}^{\sigma_+} \frac{d\sigma}{\sqrt{\log A + \log \sigma^{\frac{2}{\lambda}} - \frac{1}{2}\sigma^2}}\\ &= \int_1^r \frac{d\rho}{\sqrt{\frac{r^2-1}{\log r^{2/\lambda}}\log \rho^{\frac{2}{\lambda}} - (\rho^2-1)}},\end{aligned}$$

where $r \doteqdot (\frac{\sigma_+}{\sigma_-})^{\frac{2}{\lambda}}$. By a rotation of the plane, we may arrange that $\theta = 0$ at $\rho = 1$. Now, since equation (4.19) is invariant under orientation reversal, the portion of the curve for $\theta \in [\Theta, 2\Theta]$ is congruent to the portion corresponding to $\theta \in [0, \Theta]$, and so on. Since σ admits at least four critical points, we require at least four of these pieces to complete the curve. But $\Theta > \frac{\pi}{2}$ when $r > 1$ (since Θ is monotone decreasing with respect to r and tends to $\frac{\pi}{2}$ as $r \to \infty$[13]), violating 2π periodicity. □

[13] This may be established by performing the coordinate transformation

$$\rho^\beta = \frac{r^\beta - 1}{2}z + \frac{r^\beta + 1}{2}, \quad \beta = \frac{4}{3},$$

which expresses Θ as an integral in z over a fixed domain; see Andrews, "Classification of limiting shapes for isotropic curve flows", Section 5.

Differentiating (4.14), we find that

$$\nabla\kappa = \kappa V^\top - \mu\,\mathsf{T}. \tag{4.22}$$

Differentiating (4.22) and applying (4.14) and (4.22) then yields

$$-\Delta\kappa = -\nabla_{V^\top}\kappa + \kappa^3 - \frac{\lambda}{2}\kappa. \tag{4.23}$$

In fact, the converse is true.

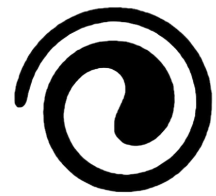
Proposition 4.9. *If a locally uniformly convex curve $\gamma : M^1 \to \mathbb{R}^2$ satisfies (4.23) for some $\lambda \in \mathbb{R}$, with $V^\top$ given by (4.22) for some $\mu \in \mathbb{R}$, then (4.14) holds with V given by (4.15) for some $v \in \mathbb{R}^2$.*

Proof. Consider the ambient vector field

$$U \doteqdot V^\top - \vec{\kappa} - \frac{\lambda}{2}\gamma - \mu J\gamma.$$

Differentiating U and applying (4.22) and (4.23) yields

$$\nabla_s U = 0.$$



So U is constant along γ. That is,

$$\vec{\kappa} + \tfrac{\lambda}{2}\gamma + \mu \mathrm{J}\gamma - V^{\top} = \gamma^* v$$

for some $v \in \mathbb{R}^2$. This is just another way of writing

$$\vec{\kappa} + (\gamma^* V)^{\perp} = 0,$$

where $V = \frac{\lambda}{2}x + \mu \mathrm{J}x - v$. □

On a convex, locally uniformly convex solution to (4.14) which is parametrized by turning angle, equations (4.22) and (4.23) become

$$\kappa(\kappa_{\theta\theta} + \kappa) = \tfrac{\lambda}{2}. \tag{4.24}$$

In particular,

Proposition 4.10. *the monotonicity formula for the Gage–Hamilton entropy (Proposition 4.1) is saturated precisely on the bounded, convex, self-similarly shrinking solutions.*

Given a curve $\gamma : M^1 \to \mathbb{R}$, consider the weighted area functional

$$G(\gamma) \doteqdot \int_{M^1} \mathrm{e}^{-\gamma^* f}\, ds,$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is the gradient soliton potential (for some choice of $v \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$). The gradient soliton equation (4.14) (with $V = Df$) is the Euler–Lagrange equation for G.

Proposition 4.11. *If M^1 is compact and $\{\gamma_\varepsilon : M^1 \to \mathbb{R}\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a smooth variation of $\gamma = \gamma_0$, then*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} G(\gamma_\varepsilon) = -\int_{M^1} (\vec{\kappa} + \gamma^* Df)\cdot \vec{F}\,\mathrm{e}^{-\gamma^* f}\, ds,$$

where $\vec{F}$ is the variation field. Thus, γ is a stationary point of G if and only if it satisfies (4.14) *with $V = Df$.*

Proof. This as an easy consequence of the first variation formula for the length element. □

Consider now, for some shrinker $\gamma : M^1 \to \mathbb{R}^2$, the associated self-similarly *shrinking* curve shortening flow $\sqrt{-t}\phi^*_{\log\sqrt{-t}}\gamma$. This curve shortening flow will satisfy

$$\vec{\kappa}_{\gamma_t} + \tfrac{\lambda(t)}{2}\gamma_t^{\perp} = 0,$$

where $\lambda(t) \doteqdot \frac{1}{-t}$. So the potential function (after adding a normalizing constant) is given by

$$f(x,t) = \lambda(t)\tfrac{|x|^2}{4} + \log(4\pi) = \tfrac{|x|^2}{-4t} + \log(4\pi).$$

Observe that the density function

$$h \doteqdot \lambda e^{-f} = \tfrac{1}{-4\pi t}e^{-\frac{|x|^2}{-4t}}$$

is then the fundamental solution to the planar CONJUGATE HEAT EQUATION[14]

$$(\partial_t - \Delta)^* v = 0$$

on $\mathbb{R}^2$, where

$$(\partial_t - \Delta)^* \doteqdot -(\partial_t + \Delta)$$

is the CONJUGATE HEAT OPERATOR.

[14] So named because a smooth function u satisfies the heat equation in $\Omega \times (a,b) \subset \mathbb{R}^2 \times \mathbb{R}$ if and only if

$$\int_a^b\int_\Omega u(\partial_t - \Delta)^*\varphi\, d\mathcal{L}\, dt = 0$$

for every smooth function φ which is compactly supported in $\Omega \times (a,b)$.

4.3 *The differential Harnack inequality*

The classical heat equation exhibits a remarkable property, known as the (matrix) DIFFERENTIAL HARNACK INEQUALITY, which states that any positive solution $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ must satisfy

$$\nabla^2 \log u + \frac{I}{2t} \geq 0. \tag{4.25}$$

In fact, the inequality must be strict, unless u is a constant multiple of the (self-similar) fundamental solution, $\rho(x,t) \doteqdot (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-x_0|^2}{4t}}$ for some x_0. Integrating the trace of (4.25) along spacetime curves of the form $t \mapsto (\gamma(t), t)$, with γ a geodesic joining points x_1 and x_2, yields the classical HARNACK INEQUALITY:

$$(4\pi t_2)^{\frac{n}{2}}\, u(x_2,t_2) \geq (4\pi t_1)^{\frac{n}{2}}\, u(x_1,t_1) \exp\left(-\frac{|x_2 - x_1|^2}{4(t_2 - t_1)}\right), \tag{4.26}$$

for any x_2, x_1 and any $t_2 > t_1$.

For an ANCIENT SOLUTION[15] $u : \mathbb{R}^n \times (-\infty,\infty) \to \mathbb{R}$, performing a series of time-translations yields the stronger inequality

$$\nabla^2 \log u \geq 0.$$

[15] I.e. a solution whose temporal domain I has an infinite past: $I = (-\infty, \omega)$, $\omega \leq \infty$.

Again, we have strict inequality, except in the exceptional circumstance that $\nabla^2 \log u = 0$; that is, u is a constant multiple of the travelling wave solution, $u(x,t) = e^{(x+tv)\cdot v}$ for some $v \in \mathbb{R}^n$.

Observe that, by (4.22) and (4.23), a locally uniformly convex, self-similarly expanding curve shortening flow must satisfy

$$\frac{\kappa_t}{\kappa} = \frac{|\nabla\kappa|^2}{\kappa^2} - \frac{1}{2t},$$

while a locally uniformly convex, self-similarly translating curve shortening flow must satisfy

$$\frac{\kappa_t}{\kappa} = \frac{|\nabla\kappa|^2}{\kappa^2}.$$

Theorem 4.12 (Differential Harnack inequality[16])**.** *Along any locally uniformly convex curve shortening flow $\gamma : M^1 \times [0,T) \to \mathbb{R}^2$ on a compact one-manifold,*

$$\frac{\partial_t\kappa}{\kappa} - \frac{|\nabla\kappa|^2}{\kappa^2} + \frac{1}{2t} \geq 0. \tag{4.27}$$

Moreover, if (4.27) holds along a (not necessarily compact) proper, locally uniformly convex curve shortening flow $\gamma : M^1 \times [0,T) \to \mathbb{R}^2$, then it holds with strict inequality, unless $\gamma : M^1 \times [0,T) \to \mathbb{R}^2$ is a self-similarly expanding solution.

Along any locally uniformly convex, ANCIENT[17] *curve shortening flow $\gamma : M^1 \times (-\infty,T) \to \mathbb{R}^2$ on a compact one-manifold,*

$$\frac{\partial_t\kappa}{\kappa} - \frac{|\nabla\kappa|^2}{\kappa^2} \geq 0. \tag{4.28}$$

Moreover, if (4.28) holds along a (not necessarily compact) proper, locally uniformly convex, ancient curve shortening flow $\gamma : M^1 \times (-\infty,T) \to \mathbb{R}^2$, then it holds with strict inequality, unless $\gamma : M^1 \times (-\infty,T) \to \mathbb{R}^2$ is a self-similarly translating solution.

[16] Andrews, "Harnack inequalities for evolving hypersurfaces"; Richard S. Hamilton, "Harnack estimate for the mean curvature flow"

[17] I.e. one having an infinite past.

Proof. Consider the functions

$$Q \doteqdot \partial_t \log\kappa - |\nabla\log\kappa|^2 \quad\text{and}\quad P \doteqdot 2t(\partial_t\log\kappa - |\nabla\log\kappa|^2) + 1.$$

Note that $P \equiv 0$ if and only if $\gamma : M^1 \times I \to \mathbb{R}^2$ is a self-similarly expanding solution and $Q \equiv 0$ if and only if $\gamma : M^1 \times I \to \mathbb{R}^2$ is a self-similarly translating solution.

Observe that, with respect to the *turning angle parametrization*[18],

$$Q = \partial_t\log\kappa \quad\text{and}\quad P = 2t\partial_t\log\kappa + 1.$$

[18] Since $\theta_s = \kappa$ and $\theta_t = \kappa_s$, the chain rule implies that

$$\partial_t(f\circ\theta) = f_t + f_\theta\kappa_s = f_t + f_s\frac{\kappa_s}{\kappa}.$$

Recalling the computation (4.10), we find that

$$\partial_t Q = \kappa^2 Q_{\theta\theta} + 2\kappa\kappa_\theta Q_\theta + 2Q^2,$$

and hence

$$\partial_t P = \kappa^2 P_{\theta\theta} + 2\kappa\kappa_\theta P_\theta + 2QP.$$

Since $P|_{t=0} = 1 > 0$, the maximum principle implies that $P \geq 0$ for positive times, which yields (4.27). The rigidity case is a consequence of the strong maximum principle (which implies $P \equiv 0$, i.e. $\partial_t\kappa = \nabla_{\nabla\log\kappa}\kappa + \frac{1}{2t}\kappa$) and Proposition 4.9.

The inequality (4.28) now follows by time-translating (4.27) for a sequence of times approaching minus infinity, and the rigidity case is again a consequence of the strong maximum principle (which implies $Q \equiv 0$, i.e. $\partial_t\kappa = \nabla_{\nabla\log\kappa}\kappa$) and Proposition 4.9. $\square$

Note that, by continuity, smooth limits of curve shortening flows on compact one-manifolds satisfy the differential Harnack inequality.

Corollary 4.13 ((Integral) Harnack inequality)**.** *Along any locally uniformly convex curve shortening flow* $\gamma : M^1 \times [0, T) \to \mathbb{R}^2$ *on a compact one-manifold,*

$$\sqrt{t_2}\kappa(x_2, t_2) \geq \sqrt{t_1}\kappa(x_1, t_1)\exp\left(\frac{d^2(x_1, x_2, t_1)}{-4(t_2 - t_1)}\right)$$

for any $x_1, x_2 \in M^2$ *and any* $0 < t_1 < t_2 < T$*.*

Proof. If $\gamma : [t_1, t_2] \to M^n$ is a minimizing g_{t_1}-geodesic joining $x_1 = \gamma(t_1)$ to $x_2 = \gamma(t_2)$, then $|\gamma'| = \frac{d(x_1,x_2,t_1)}{t_2-t_1}$, so the differential Harnack inequality (4.27) and Young's inequality yield

$$\begin{aligned}
\frac{d}{dt}\log\kappa(\gamma(t), t) &= \frac{\kappa_t}{\kappa} + \frac{\nabla_{\gamma'}\kappa}{\kappa} \\
&\geq \frac{1}{2t} + \frac{|\nabla\kappa|^2}{\kappa^2} - \frac{|\nabla\kappa|}{\kappa}\frac{d(x_1, x_2, t_1)}{t_2 - t_1} \\
&\geq \frac{1}{2t} - \frac{d^2(x_1, x_2, t_1)}{4(t_2 - t_1)^2}.
\end{aligned}$$

Integrating from time t_1 to t_2 yields the claim. □

4.4 *The monotonicity formula for Huisken's functional*

Given a planar curve shortening flow $\gamma : M^1 \times I \to \mathbb{R}^2$, $M^1 \cong S^1$, the Huisken functional $\mathscr{G}$ is defined by

$$\mathscr{G}_{(p_0,t_0)}(M^1, t) \doteqdot \frac{1}{\sqrt{4\pi(t_0 - t)}}\int_{M^1} \mathrm{e}^{\frac{|\gamma(x,t)-p_0|^2}{4(t-t_0)}}\, ds_t(x). \tag{4.29}$$

Observe that this is simply $\frac{1}{\sqrt{2(t_0-t)}}$ times the functional G from Proposition 4.11 (in the shrinking case), evaluated along the curve shortening flow. Observe that $\mathscr{G}$ is invariant under parabolic rescaling about (p_0, t_0), and is thus constant in time along a self-similarly shrinking curve shortening flow which is centred at (p_0, t_0).

Define the density

$$\begin{aligned}
\Psi_{(p_0,t_0)}(x, t) &\doteqdot \frac{1}{\sqrt{4\pi(t_0 - t)}}\mathrm{e}^{\frac{|\gamma(x,t)-p_0|^2}{4(t-t_0)}} \\
&= \sqrt{4\pi(t_0 - t)}\Phi_{(p_0,t_0)}(\gamma(x, t), t),
\end{aligned}$$

where

$$\Phi_{(p_0,t_0)}(p, t) \doteqdot \frac{1}{4\pi(t_0 - t)}\mathrm{e}^{\frac{|p-p_0|^2}{4(t-t_0)}}$$

is the fundamental solution to the planar conjugate heat equation based at (x_0, t_0).

Proposition 4.14 (Monotonicity formula for Huisken's functional[19]). *Given any* $(p_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}$ *and any planar curve shortening flow* $\gamma : M^1 \times I \to \mathbb{R}^2$, $M^1 \cong S^1$,

$$\frac{d}{dt} \mathscr{G}_{(p_0,t_0)}(M^1, t) = -\int_{M^1} \left| \vec{\kappa} + \frac{(\gamma - p_0)^\perp}{2(t_0 - t)} \right|^2 \Psi_{(p_0,t_0)}\, ds \tag{4.30}$$

for all $t \in I \cap (-\infty, t_0)$. *In particular,* $\mathscr{G}_{(p_0,t_0)}(M^1, t)$ *is nonincreasing over* $I \cap (-\infty, t_0)$, *and strictly decreases unless* γ *is self-similarly shrinking about*[20] (p_0, t_0).

Proof. Without loss of generality, we may take $(p_0, t_0) = (0, 0)$. Set $\Phi \doteqdot \Phi_{(0,0)}$, $\Psi \doteqdot \Psi_{(0,0)}$ and $\mathscr{G} \doteqdot \mathscr{G}_{(0,0)}$. Note that Φ satisfies the conjugate planar heat equation

$$-(\partial_t + \Delta_{\mathbb{R}^2})\Phi = 0.$$

We claim that

$$-(\partial_t + \Delta - \kappa^2)\Psi = \left| \vec{\kappa} + \frac{\gamma^\perp}{-2t} \right|^2 \Psi. \tag{4.31}$$

Indeed,

$$\begin{aligned} \Psi_t &= \tfrac{-2\pi}{\sqrt{-4\pi t}}\Phi + \sqrt{-4\pi t}\,(\Phi_t + D\Phi \cdot \gamma_t) \\ &= \tfrac{1}{2t}\Phi + \sqrt{-4\pi t}\,(\Phi_t - \kappa D_{\mathsf{N}}\Phi) \end{aligned}$$

and

$$\begin{aligned} \Delta\Psi &= \Psi_{ss} \\ &= \sqrt{-4\pi t}\,(D_{\mathsf{T}}\Phi)_s \\ &= \sqrt{-4\pi t}\left(D^2\Phi(\mathsf{T},\mathsf{T}) - \kappa D_{\mathsf{N}}\Phi\right) \\ &= \sqrt{-4\pi t}\left(\Delta_{\mathbb{R}^2}\Phi - D^2\Phi(\mathsf{N},\mathsf{N}) - \kappa D_{\mathsf{N}}\Phi\right). \end{aligned}$$

So

$$(\partial_t + \Delta)\Psi = \tfrac{1}{2t}\Psi - \sqrt{-4\pi t}\left(D^2\Phi(\mathsf{N},\mathsf{N}) + 2\kappa D_{\mathsf{N}}\Phi\right).$$

Since Φ satisfies

$$D^2 \log\Phi = \frac{\mathrm{I}}{2t} \quad\text{and}\quad D_{\mathsf{N}}\log\Phi = \frac{\gamma\cdot\mathsf{N}}{2t},$$

this becomes

$$\begin{aligned} (\partial_t + \Delta)\Psi &= \frac{\Psi}{2t} - \sqrt{-4\pi t}\left(\frac{\Phi}{2t} + \frac{(D_{\mathsf{N}}\Phi)^2}{\Phi} + 2\kappa D_{\mathsf{N}}\Phi\right) \\ &= -\Psi\left(\frac{(D_{\mathsf{N}}\Phi)^2}{\Phi^2} + 2\kappa\frac{D_{\mathsf{N}}\Phi}{\Phi}\right) \\ &= \kappa^2\Psi - \Psi\left|\vec{\kappa} - \frac{(D\Phi)^\perp}{\Phi}\right|^2 \\ &= \kappa^2\Psi - \Psi\left|\vec{\kappa} + \frac{\gamma^\perp}{-2t}\right|^2, \end{aligned}$$

[19] Huisken, "Asymptotic behavior for singularities of the mean curvature flow"

[20] I.e. the spacetime translated solution $\tilde{\gamma}(x,t) \doteqdot \gamma(x, t+t_0) - p_0$ is a self-similarly shrinking solution.

which is Equation (4.31). We conclude, using the divergence theorem and the area evolution, that

$$\begin{aligned}\frac{d}{dt}\mathscr{G}(M^1,t) &= \int_{M^1}\left(\Psi_t-\kappa^2\Psi\right)ds\\ &= \int_{M^1}(\partial_t+\Delta-\kappa^2)\Psi\,ds\\ &= -\int_{M^1}\left|\vec{\kappa}+\frac{\gamma^\perp}{-2t}\right|^2\Psi\end{aligned}$$

as claimed. □

4.5 *Noncollapsing*

Roughly speaking, a sequence of embedded curves $\Gamma_j=\partial\Omega_j$, $\Omega_j \underset{\text{open}}{\subset} \mathbb{R}^2$, is said to COLLAPSE if, modulo translation and scaling, their interior regions Ω_j degenerate as $j\to\infty$, with their curvature remaining bounded. One precise way to quantify this is to ask for a sequence of points $x_j\in\Gamma_j$ such that

$$\overline{r}_j(x_j)\sup_{B^2_{j\overline{r}_j(x_j)}(x_j)}|\kappa_j|\le j^{-1}, \tag{4.32}$$

where $\overline{r}(x)$ denotes the INSCRIBED RADIUS of $\partial\Omega$ at $x\in\partial\Omega$—the radius of the largest disc contained in Ω whose boundary passes through the boundary point x.

Note that $\overline{r}\kappa$ is scale invariant. Thus, if (4.32) holds, then, at the scale of the *curvature*, the inscribed radius degenerates to zero. Since $\kappa\le\overline{r}^{-1}$, with strict inequality only if the boundary of the disc $B^2_{\overline{r}}(x-\overline{r}\,\mathsf{N}(x))$ meets $\partial\Omega$ at some other point $y\in\partial\Omega\setminus\{x\}$, this means that two (intrinsically distant) portions of the boundaries are coming together. On the other hand, at the scale of the *inscribed radius*, the curvature is tending towards zero in arbitrarily large regions, and at this scale the regions converge to a strip of width two.

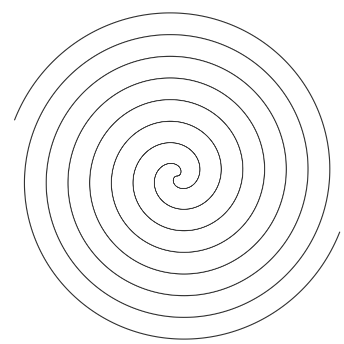


Figure 4.2: On an Archimedean spiral, $\overline{r}\sim 1$ but $\kappa\sim\frac{1}{d}$, where d is the distance to the origin. Thus, far from the origin, $\overline{r}\kappa\sim 0$, and hence (4.32) holds, after passing to a subsequence, along any sequence of points x_j tending to infinity.

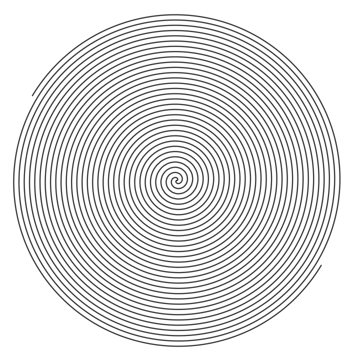


Figure 4.3: At the scale of the curvature of a point very far from the origin, the inscribed radius is very small but the curvature is ~ 1 at distance ~ 1 from the origin.

Example 5. Consider the constant sequence $\Gamma_j=\Gamma$, where

$$\Gamma=\operatorname{graph}(x\mapsto\log\sec x)$$

is the Grim Reaper curve. If $(x_j, y_j=\log\sec x_j)\in\Gamma_j$ is a sequence of points with $x_j\to\pm\frac{\pi}{2}$, then, on the one hand, $r_j\doteqdot\overline{r}(x_j,y_j)\to\frac{\pi}{2}$ as $j\to\infty$. On the other hand, since $\cos x_j\to 0$ as $j\to\infty$, we may pass to a subsequence so that $\cos(x_j) = o(\mathrm{e}^{-j\frac{\pi}{2}})$, and hence

$$\begin{aligned}\sup_{B^2_{jr_j}(x_j,y_j)} \kappa &\le \sup_{B^2_{j\frac{\pi}{2}}(x_j,y_j)} \kappa \\ &\le \sup_{[x_j - j\frac{\pi}{2}, x_j + j\frac{\pi}{2}]\times[y_j - j\frac{\pi}{2}, y_j + j\frac{\pi}{2}]} \kappa \\ &= \kappa(\arccos(\mathrm{e}^{j\frac{\pi}{2}}\cos x_j)) \\ &= \mathrm{e}^{j\frac{\pi}{2}}\cos x_j \\ &= o(1) \;\text{ as }\; j\to\infty.\end{aligned}$$

So the sequence is collapsing. ■

4.5.1 The inscribed and exscribed curvature estimates

Andrews proved that the inscribed radius is pointwise nondecreasing, relative to the scale of the curvature, under curve shortening flow.

Proposition 4.15 (Interior noncollapsing[21]). *Along any convex, locally uniformly convex curve shortening flow* $\{\Gamma_t = \partial\Omega_t\}_{t\in[0,T)}$, $\Omega_t \underset{\text{bounded, convex}}{\subset} \mathbb{R}^2$, *the* INSCRIBED CURVATURE $\overline{k} \doteqdot \overline{r}^{-1}$ *satisfies*

$$(\partial_t - \Delta)\overline{k} \le \kappa^2 \overline{k}$$

in the VISCOSITY SENSE[22]. *In particular,*

$$\overline{k} \le K\kappa, \;\; \text{where} \;\; K \doteqdot \max_{\Gamma_0} \frac{\overline{k}}{\kappa}.$$

Equivalently,

$$\overline{r} \ge \delta\kappa^{-1}, \;\; \text{where} \;\; \delta \doteqdot \min_{\Gamma_0} \overline{r}\kappa.$$

[21] Andrews, "Noncollapsing in mean-convex mean curvature flow"

[22] This is a weak formulation of the differential inequality $(\partial_t - \Delta)u \le \kappa^2 u$ which applies to any continuous function. It asserts that, at any point $(x_0, t_0) \in M^1 \times (0, T)$, any smooth function $\varphi : M^1 \times [0, T) \to \mathbb{R}$ which touches $\overline{k}$ from above at (x_0, t_0), in the sense that $\varphi \ge \overline{k}$ on a backward spacetime neighbourhood $U \times (t_0 - \delta, t_0]$ of (x_0, t_0) with equality at (x_0, t_0), satisfies

$$(\partial_t - \Delta)\varphi \le \kappa^2\varphi \;\text{ at }\; (x_0, t_0).$$

Proof. According to the definition of the inscribed radius (of some boundary $\Gamma = \partial\Omega$) $\overline{r}$, the value taken by the inscribed curvature $\overline{k}$ at a point $x \in \partial\Omega$ is equal to the infimum of the curvatures of discs which are contained in Ω and touch $\partial\Omega$ at x. Observe that this is equivalent to the supremum over all points $y \in \Gamma \setminus \{x\}$ of the curvature of the unique disc which touches Γ at x and passes through y. A short planar geometry exercise reveals this to be

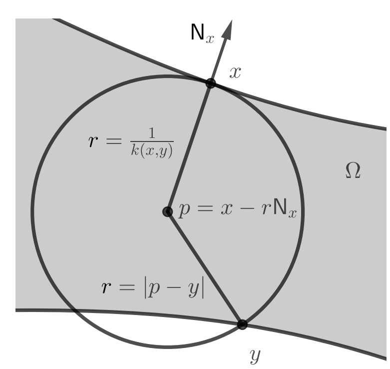


Figure 4.4: Equating $r^2 = |x - r\,\mathsf{N}_x - y|^2$ yields $k(x, y) = r^{-1} = \frac{2(x-y)\cdot\mathsf{N}_x}{|x-y|^2}$.

$$\overline{k}(x) = \sup_{y\in\Gamma\setminus\{x\}} k(x, y),$$

where the function k is defined on $\Gamma \times \Gamma \setminus D$, $D \doteqdot \{(x, x) : x \in \Gamma\}$, by

$$k(x, y) \doteqdot \frac{2(x - y)\cdot \mathsf{N}(x)}{|x - y|^2}.$$

Given a parametrization $\gamma : M^1 \times [0,T) \to \mathbb{R}^2$ for our curve shortening flow, we define

$$k(x,y,t) \doteqdot \frac{2(\gamma(x,t)-\gamma(y,t))\cdot \mathsf{N}(x,t)}{|\gamma(x,t)-\gamma(y,t)|^2},$$

so that the inscribed curvature at a point $\gamma(x,t) \in \Gamma_t$ is given by

$$\overline{k}(x,t) \doteqdot \sup_{y\in M^1\setminus\{x\}} k(x,y,t).$$

Note that (see Exercise 4.6)

$$\overline{k}(x,t) \geq \lim_{y\to x} k(x,y,t) = \kappa(x,t).$$

We claimed that $\overline{k}$ satisfies the differential inequality

$$(\partial_t - \Delta)\overline{k} \leq \kappa^2 \overline{k} \tag{4.33}$$

in the viscosity sense. To see this, let $\varphi : M^1 \times [0,T) \to \mathbb{R}$ be a smooth function which touches $\overline{k}$ from above at a point $(x_0,t_0) \in M^1 \times (0,T)$. There are two cases to consider. Assume first that $\overline{k}(x_0,t_0) = \kappa(x_0,t_0)$. In that case, φ touches κ from above at (x_0,t_0), and hence, at that point,

$$0 \geq (\partial_t - \Delta)(\varphi - \kappa) = (\partial_t - \Delta)\varphi - \kappa^3 \geq (\partial_t - \Delta)\varphi - \kappa^2\varphi$$

as claimed.

Suppose then that $\overline{k}(x_0,t_0) > \kappa(x_0,t_0)$. Then we can find $y_0 \in M^1$ such that $\overline{k}(x_0,t_0) = k(x_0,y_0,t_0)$. It follows that the function $(x_0,y_0,t_0) \mapsto \varphi(x_0,t_0)$ touches k from above at (x_0,y_0,t_0), and hence, at that point,

$$0 \geq \big(\partial_t - (\partial_x + \Lambda\partial_y)^2\big)(\varphi - k) = (\partial_t - \Delta)\varphi - \big(\partial_t - (\partial_x + \Lambda\partial_y)^2\big)k$$

for any choice of $\Lambda \in \mathbb{R}$, where, for a smooth function u defined on a neighbourhood in $M^1 \times M^1 \times [0,T)$ of the point (x_0,y_0,t_0), ∂_x and ∂_y denote (counterclockwise oriented) arclength derivatives in the corresponding variable.

Using the subscript x or y to denote projection onto the corresponding factor, and setting $d \doteqdot |\gamma_x - \gamma_y|$ and $w \doteqdot (\gamma_x - \gamma_y)/d$, consider

$$\partial_t k = \frac{2}{d^2}\left((-\kappa_x \mathsf{N}_x + \kappa_y \mathsf{N}_y)\cdot(\mathsf{N}_x - kdw) + \nabla\kappa_x \cdot (dw)\right),$$

$$(\partial_x + \Lambda\partial_y)k = \frac{2}{d^2}\left((\mathsf{T}_x - \Lambda\,\mathsf{T}_y, \mathsf{N}_x - kdw) + \kappa_x \mathsf{T}_x \cdot (dw)\right)$$

and

$$\begin{aligned}(\partial_x + \Lambda\partial_y)^2 k = \frac{2}{d^2}\Big(&(-\kappa_x \mathsf{N}_x + \Lambda^2\kappa_y \mathsf{N}_y)\cdot(\mathsf{N}_x - kdw)\\ &+ (\mathsf{T}_x - \Lambda\,\mathsf{T}_y)\cdot(\kappa_x \mathsf{T}_x - k(\mathsf{T}_x - \Lambda\,\mathsf{T}_y))\\ &+ (\nabla\kappa_x - \kappa_x^2 \mathsf{N}_x)\cdot(dw) + \kappa_x \mathsf{T}_x \cdot(\mathsf{T}_x - \Lambda\,\mathsf{T}_y)\\ &- 2(\mathsf{T}_x - \Lambda\,\mathsf{T}_y)\cdot(dw)(\partial_x + \Lambda\partial_y)k\Big).\end{aligned}$$

Since $k(x_0, \cdot, t_0)$ is maximized at y_0,

$$0 = \partial_y k = -\frac{2}{d^2} \mathsf{T}_y \cdot (\mathsf{N}_x - k dw)$$

at (x_0, y_0, t_0).

Since

$$|\mathsf{N}_x - k dw|^2 = 1 - 2k(dw) \cdot \mathsf{N}_x + d^2 k^2 = 1,$$

we find that $\mathsf{N}_y = \pm(\mathsf{N}_x - k dw)$ at (x_0, y_0, t_0). We claim that

$$\mathsf{N}_y = \mathsf{N}_x - k dw.$$

Indeed, at (x_0, y_0, t_0), the ball of radius $\frac{1}{k}$ centred at $x - \frac{1}{k}\mathsf{N}_x$, which touches Γ from the interior at x, also touches Γ from the interior at y (see Figure 4.5), so that

$$y - k^{-1}\mathsf{N}_y = x - k^{-1}\mathsf{N}_x .$$

Rearranging yields the claim.

Observe also that the tangent line at y_0 is the reflection of the tangent line at x_0 across the perpendicular bisector of the line $\mathbb{R}w$. Thus,

$$-\mathsf{T}_y = \mathsf{T}_x - 2(\mathsf{T}_x \cdot w)w,$$

and hence

$$2(\mathsf{T}_x \cdot w)(\mathsf{T}_y \cdot w) = \mathsf{T}_x \cdot \mathsf{T}_y + 1.$$

Recalling the gradient identities, we thus obtain, at (x_0, y_0, t_0),

$$\begin{aligned}(\partial_t - \Delta)\varphi &\leq \big(\partial_t - (\partial_x + \Lambda\partial_y)^2\big)k \\ &= \kappa_x^2 k - 2\frac{(\partial_x k)^2}{k - \kappa_x} \\ &\quad + \frac{2}{d^2}\Big(\kappa_y - \kappa_x + k - \kappa_x + 2\Lambda(k - \kappa_x) - \Lambda^2(k - \kappa_y)\Big).\end{aligned}$$

Taking $\Lambda = -1$ then yields

$$\begin{aligned}(\partial_t - \Delta)\varphi &\leq \kappa^2\varphi - 2\frac{(\nabla\varphi)^2}{\varphi - \kappa} \\ &\leq \kappa^2\varphi\end{aligned}$$

as claimed.

The Proposition now follows from the maximum principle (albeit in the context of viscosity solutions): it suffices to prove that the inequality

$$\overline{k} - K\kappa - \varepsilon e^{(C_\sigma+1)t} \leq 0$$

holds on $M^1 \times [0, \sigma]$ for any $\varepsilon > 0$ and $\sigma \in (0, T)$, where

$$K \doteqdot \max_{M^1 \times \{0\}} \frac{\overline{k}}{\kappa}$$

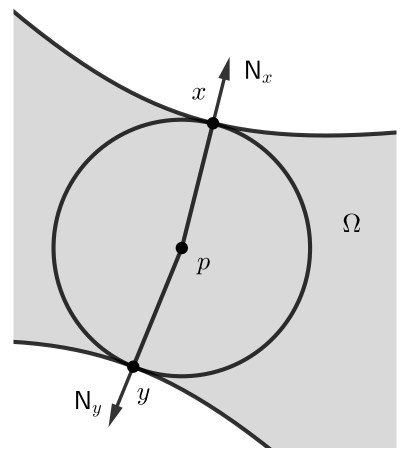


Figure 4.5: If $\overline{k}(x) = k(x, y)$, then the inscribed ball at x of radius $\overline{k}^{-1}(x)$ also makes first order contact with $\partial\Omega$ at y.

and $C_\sigma \doteqdot \max_{M^1\times[0,\sigma]} \kappa^2$. Suppose then, to the contrary, that there exist for some $\varepsilon > 0$ and $\sigma \in (0, T)$ some $(x_0, t_0) \in M^1 \times [0, \sigma]$ such that

$$\overline{k}(x_0, t_0) - K\kappa(x_0, t_0) - \varepsilon \mathrm{e}^{(C_\sigma+1)t_0} = 0.$$

Since the inequality is strict at the initial time, we may take $t_0 > 0$ to be the first time that equality is attained. But then the smooth function

$$\varphi \doteqdot K\kappa + \varepsilon \mathrm{e}^{(C_\sigma+1)t}$$

touches $\overline{k}$ from above at (x_0, t_0), and must therefore satisfy, at (x_0, t_0),

$$\begin{aligned}
0 &\geq (\partial_t - \Delta)\varphi - \kappa^2\varphi \\
&= K(\partial_t - \Delta)\kappa + \varepsilon(C_\sigma + 1)\mathrm{e}^{(C_\sigma+1)t} - \kappa^2 \left(K\kappa + \varepsilon \mathrm{e}^{(C_\sigma+1)t}\right) \\
&= \varepsilon(C_\sigma + 1)\mathrm{e}^{(C_\sigma+1)t} - \varepsilon\kappa^2\mathrm{e}^{(C_\sigma+1)t} \\
&\geq \varepsilon \mathrm{e}^{(C_\sigma+1)t} \\
&> 0,
\end{aligned}$$

which is absurd. We conclude that

$$\overline{k} \leq K\kappa,$$

which is equivalent to the claim. □

In fact, reversing the orientation of the curves in the preceding proof yields a corresponding *exterior* noncollapsing estimate: if we define the CIRCUMSCRIBED RADIUS $\underline{r}$ to be the radius of the smallest disc which encloses[23] Ω and touches $\partial\Omega$ at x, then we obtain the following.

[23] Equivalently, the radius of the largest disc-complement which lies in $\mathbb{R}^2 \setminus \overline{\Omega}$ and touches $\partial\Omega$ at x.

[24] Andrews, "Noncollapsing in mean-convex mean curvature flow"

Proposition 4.16 (Exterior noncollapsing[24]). *Along any convex, locally uniformly convex curve shortening flow, $\{\Gamma_t = \partial\Omega_t\}_{t\in[0,T)}$, $\Omega_t \underset{\text{bounded, convex}}{\subset} \mathbb{R}^2$, the* EXSCRIBED CURVATURE $\underline{k} \doteqdot \underline{r}^{-1}$ *satisfies*

$$(\partial_t - \Delta)\underline{k} \geq \kappa^2\underline{k}$$

in the viscosity sense. In particular,

$$\underline{k} \geq \delta\kappa, \quad \text{where} \quad \delta \doteqdot \min_{\Gamma_0} \frac{\underline{k}}{\kappa}.$$

Equivalently,

$$\underline{r} \leq D\kappa^{-1}, \quad \text{where} \quad D \doteqdot \max_{\Gamma_0} \underline{r}\kappa.$$

4.5.2 *The chord-arc estimate*

Observe that, on any round circle of radius r, the chord-distance d and arc-length ℓ are related by

$$\frac{d}{2r} \equiv \sin\left(\frac{\ell}{2r}\right)$$

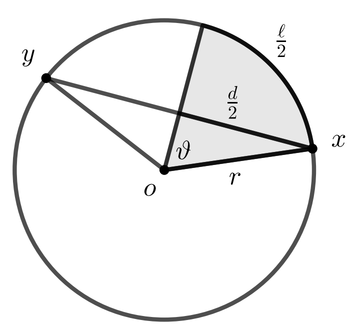


Figure 4.6: On a round circle of radius r, $\frac{d}{2} = r\sin\vartheta$ and $\frac{\ell}{2} = r\vartheta$.

or, equivalently,

$$\sin\left(\frac{\pi\ell}{\mathrm{L}}\right) \equiv \frac{\pi d}{\mathrm{L}}, \tag{4.34}$$

where $\mathrm{L} \doteqdot 2\pi r$ is the total length.

Define the CHORD-ARC CONSTANT of a regular Jordan curve $\Gamma = \partial\Omega \subset \mathbb{R}^2$ to be

$$\mathcal{C}(\Gamma) \doteqdot \sup \frac{\mathrm{L}}{\pi d} \sin\left(\frac{\pi\ell}{\mathrm{L}}\right),$$

where the supremum is taken over all "off-diagonal" pairs $(x, y) \in \Gamma \times \Gamma$; i.e. $x \neq y$.

Obviously, the chord-arc constant of a round circle is one. Moreover, since

$$\frac{\mathrm{L}}{\pi d} \sin\left(\frac{\pi\ell}{\mathrm{L}}\right) \to 1 \ \text{ as } \ \ell \to 0,$$

$\mathcal{C}(\Gamma)$ is always at least one. In fact, $\mathcal{C}(\Gamma) > 1$ unless Γ is a round circle.[25]

[25] See, for example, Andrews, Chow, et al., *Extrinsic geometric flows*, Lemma 3.9.

Huisken proved that the chord-arc constant does not increase under curve shortening flow.[26]

[26] Huisken, "A distance comparison principle for evolving curves".

Proposition 4.17. *Along any curve shortening flow $\gamma : M^1 \times [0, T) \to \mathbb{R}^2$ of simple, closed, planar curves $\Gamma_t = \gamma(M^1, t)$,*

$$\frac{d}{dt}\mathcal{C}(\Gamma_t) \leq 0$$

in the viscosity sense[27] *whenever $\mathcal{C}(\Gamma_t) > 1$. In particular,*

$$\mathcal{C}(\Gamma_t) \leq \mathcal{C}(\Gamma_0)$$

for all $t \in [0, T)$.[28]

[27] This is a weak formulation of the differential inequality $\frac{du}{dt} \leq 0$ which applies to any continuous function. It asserts, for every $t_0 \in (0, T)$, that every smooth function $\varphi : [0, T) \to \mathbb{R}$ which touches u from above at t_0, in the sense that $u \leq \varphi$ for t in a backward neighbourhood $(t_0 - \delta, t_0]$ of t_0 with equality at t_0, satisfies $\frac{d\varphi}{dt}(t_0) \leq 0$.

[28] A sharp version of this estimate was established by Andrews and Bryan, "Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem" by replacing the sine function in the definition of $\mathcal{C}$ with a certain modulus which improves with time. The resulting estimate is sharp enough to establish precise control on the geometry of the evolving curves as the maximal time is approached.

Proof. Set $\mathrm{L}(t) \doteqdot \text{length}(\Gamma_t)$ and define a function $Z : (M^1 \times M^1) \setminus D \to \mathbb{R}$, where $D \doteqdot \{(x, x) : x \in M^1\}$, by

$$Z(x, y, t) \doteqdot \frac{\mathrm{L}(t)}{\pi d(x, y, t)} \sin\left(\frac{\pi\ell(x, y, t)}{\mathrm{L}(t)}\right).$$

If $\mathcal{C}(\Gamma_{t_0}) < 1$, then we can find $(x_0, y_0) \in (M^1 \times M^1) \setminus D$ such that $\mathcal{C}(\Gamma_{t_0}) = Z(x_0, y_0, t_0)$. Since $\xi \mapsto \sin(\xi)$ is even about $\xi = \frac{\pi}{2}$ and Z is symmetric in x and y, we may arrange that $\ell(x_0, y_0, t_0) \leq \frac{\mathrm{L}(t_0)}{2}$, and that $\partial_x\ell = -1$ and $\partial_y\ell = +1$, where ∂_x and ∂_y denote counterclockwise oriented arclength derivatives in the respective factors. Thus, if φ is an upper barrier for $\mathcal{C}(\Gamma_t)$ at t_0, then

$$\varphi \geq \mathcal{C}(\Gamma_t) \geq Z$$

for $t \le t_0$ and $(x,y) \in (M^1 \times M^1) \setminus D$, with equality at (x_0, y_0, t_0), and hence, at that point,

$$
\begin{aligned}
\partial_t \varphi &\le \partial_t Z \\
&= \frac{\mathsf{L}_t}{\mathsf{L}} \left(\frac{\mathsf{L}}{\pi d} \sin\left(\frac{\pi \ell}{\mathsf{L}}\right) - \frac{\ell}{d} \cos\left(\frac{\pi \ell}{\mathsf{L}}\right) \right) - \frac{1}{d} Z w \cdot (\kappa_x \, \mathsf{N}_x - \kappa_y \, \mathsf{N}_y) \\
&\quad + \frac{\ell_t}{d} \cos\left(\frac{\pi \ell}{\mathsf{L}}\right),
\end{aligned} \tag{4.35}
$$

where $w \doteqdot \frac{1}{d}(y - x)$. Since (x_0, y_0) is a critical point of $Z(\cdot,\cdot,t_0)$, we also have

$$0 = \partial_x Z = \frac{1}{d}\left(Z(w \cdot \mathsf{T}_x) - \cos\left(\frac{\pi \ell}{\mathsf{L}}\right)\right)$$

and

$$0 = \partial_y Z = -\frac{1}{d}\left(Z(w \cdot \mathsf{T}_y) - \cos\left(\frac{\pi \ell}{\mathsf{L}}\right)\right),$$

and hence

$$Z(w \cdot \mathsf{T}_x) = \cos\left(\frac{\pi \ell}{\mathsf{L}}\right) = Z(w \cdot \mathsf{T}_y) \tag{4.36}$$

at (x_0, y_0, t_0). It follows that, at this point, either $\mathsf{T}_y = \mathsf{T}_x$ or the line in the direction of w bisects the angle between T_x and T_y; i.e.

$$\mathsf{T}_y = 2(w \cdot \mathsf{T}_x) w - \mathsf{T}_x \,. \tag{4.37}$$

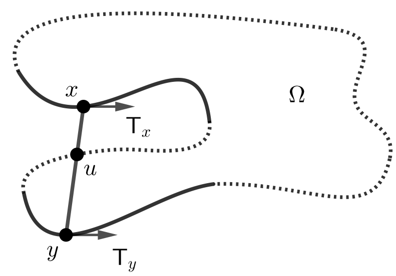


Figure 4.7: If a pair of points with parallel tangent vectors can be found, then their chord must intersect the curve in a third point.

We claim that the latter must be the case. Indeed, if $\mathsf{T}_x = \mathsf{T}_y$ at (x_0, y_0, t_0), then the curve Γ_{t_0} must intersect the chord joining $\gamma(x_0, t_0)$ and $\gamma(y_0, t_0)$ in a third point, $\gamma(u_0, t_0)$. But then, since $Z(\cdot,\cdot,t_0)$ is maximized at (x_0, y_0) and $\xi \mapsto \sin(\xi)$ is even about $\xi = \pi/2$ and strictly concave for $\xi \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$
\begin{aligned}
(dZ)(x_0, y_0, t_0) &= \left(d(x_0, u_0, t_0) + d(u_0, y_0, t_0)\right) Z(x_0, y_0, t_0) \\
&\ge (dZ)(x_0, u_0, t_0) + (dZ)(u_0, y_0, t_0) \\
&= \frac{\pi}{\mathsf{L}(t_0)} \left[\sin\left(\frac{\pi \ell(x_0, u_0, t_0)}{\mathsf{L}(t_0)}\right) + \sin\left(\frac{\pi \ell(u_0, y_0, t_0)}{\mathsf{L}(t_0)}\right) \right] \\
&> \frac{\pi}{\mathsf{L}(t_0)} \sin\left(\frac{\pi(\ell(x_0, u_0, t_0) + \ell(u_0, y_0, t_0))}{\mathsf{L}(t_0)}\right) \\
&= \frac{\pi}{\mathsf{L}(t_0)} \sin\left(\frac{\pi \ell(x_0, y_0, t_0)}{\mathsf{L}(t_0)}\right) \\
&= (dZ)(x_0, y_0, t_0),
\end{aligned}
$$

which is absurd. This establishes (4.37).

Next, consider the second variation coefficients

$$\partial_x^2 Z = -\frac{1}{d} Zw \cdot (\kappa_x \mathsf{N}_x) - \frac{1}{d^2} Z\big(1 - (w \cdot \mathsf{T}_x)^2\big) - \frac{\pi}{d\mathrm{L}} \sin\left(\frac{\pi \ell}{\mathrm{L}}\right),$$

$$\partial_x \partial_y Z = \frac{1}{d^2} Z \left(\mathsf{T}_x \cdot \mathsf{T}_y - (w \cdot \mathsf{T}_x)(w \cdot \mathsf{T}_y)\right) + \frac{\pi}{d\mathrm{L}} \sin\left(\frac{\pi \ell}{\mathrm{L}}\right)$$

and

$$\partial_y^2 Z = \frac{1}{d} Zw \cdot (\kappa_y \mathsf{N}_y) - \frac{1}{d^2} Z\big(1 - (w \cdot \mathsf{T}_y)^2\big) - \frac{\pi}{d\mathrm{L}} \sin\left(\frac{\pi \ell}{\mathrm{L}}\right).$$

Since (x_0, y_0) is a local maximum of[29] $Z(\cdot, \cdot, t_0)$,

$$\begin{aligned} 0 &\geq (\partial_x - \partial_y)^2 Z \\ &= -\frac{1}{d} Zw \cdot (\kappa_x \mathsf{N}_x - \kappa_y \mathsf{N}_y) - \frac{4\pi^2}{d\mathrm{L}} \sin\left(\frac{\pi \ell}{\mathrm{L}}\right), \end{aligned} \tag{4.38}$$

due to some cancellation of terms upon applying (4.37). Putting (4.35) and (4.38) together, we find that

$$\begin{aligned} \partial_t \varphi \leq & -\frac{1}{\pi d}\left(\sin\left(\frac{\pi \ell}{\mathrm{L}}\right) - \frac{\pi \ell}{\mathrm{L}} \cos\left(\frac{\pi \ell}{\mathrm{L}}\right)\right) \int \kappa^2 \, ds \\ & + \frac{4\pi}{d\mathrm{L}} \sin\left(\frac{\pi \ell}{\mathrm{L}}\right) - \frac{1}{d} \cos\left(\frac{\pi \ell}{\mathrm{L}}\right) \int_x^y \kappa^2 \, ds \end{aligned}$$

Since $\ell \leq \frac{\mathrm{L}}{2}$, the trigonometric terms are all nonnegative and

$$\sin\left(\frac{\pi \ell}{\mathrm{L}}\right) \geq \frac{\pi \ell}{\mathrm{L}} \cos\left(\frac{\pi \ell}{\mathrm{L}}\right).$$

Hölder's theorem, in the form

$$4\pi^2 = \left(\int 1 \cdot \kappa \, ds\right)^2 \leq \int 1 \, ds \int \kappa^2 \, ds = \mathrm{L} \int \kappa^2 \, ds,$$

then implies that

$$\partial_t \varphi \leq \frac{1}{d\ell}\left(\frac{4\pi^2 \ell^2}{\mathrm{L}^2} - \ell \int_x^y \kappa^2 \, ds\right) \cos\left(\frac{\pi \ell}{\mathrm{L}}\right).$$

If we denote by ϑ the angle between T_x and T_y, then, since w bisects T_x and T_y, (4.36) implies that

$$\cos\left(\frac{\pi \ell}{\mathrm{L}}\right) = Z \cos\left(\frac{\vartheta}{2}\right)$$

at (x_0, y_0, t_0). Since $\xi \mapsto \cos \xi$ is monotone decreasing for $\xi \in [0, \frac{\pi}{2}]$ and, by hypothesis, $Z(x_0, y_0, t_0) < 1$, we find that

$$\vartheta > \frac{2\pi \ell}{\mathrm{L}}$$

[29] Observe that the variation $\partial_x - \partial_y$ keeps the vector w (which bisects T_x and T_y) constant, producing an optimal second variation.

at (x_0, y_0, t_0).

On the other hand, applying Hölder's inequality similarly as above we find that

$$\vartheta^2 = \left(\int_x^y 1 \cdot \kappa \, ds\right)^2 \leq \int_x^y 1 \, ds \int_x^y \kappa^2 \, ds = \ell \int_x^y \kappa^2 \, ds .$$

We conclude that

$$\frac{d}{dt}\varphi \leq 0$$

at (x_0, y_0, t_0), which establishes the first claim.

To prove the second claim, it suffices to establish that

$$\mathcal{C}(\Gamma_t) - \mathcal{C}(\Gamma_0) - \varepsilon(1+t) \leq 0$$

for all $t \in [0, T)$ for any $\varepsilon > 0$. Note that the inequality holds strictly at time $t = 0$ for any positive ε. Suppose then that some $\varepsilon > 0$ and $t_0 \in (0, T)$ can be found such that

$$\mathcal{C}(\Gamma_t) - \mathcal{C}(\Gamma_0) - \varepsilon(1+t) \leq 0$$

for all $t \leq t_0$, but with equality at time $t = t_0$. But then the function

$$\varphi(t) \doteqdot \mathcal{C}(\Gamma_0) + \varepsilon(1+t)$$

is an upper support for $\mathcal{C}$ at time $t = t_0$, and hence

$$0 \geq \frac{d}{dt}\varphi = \varepsilon > 0,$$

which is absurd. □

4.5.3 *The isoperimetric estimate*

Define the RELATIVE ISOPERIMETRIC CONSTANT of a regular[30] Jordan curve $\Gamma = \partial\Omega$, $\Omega \underset{\text{open}}{\subset} \mathbb{R}^2$, to be

$$\mathcal{I}(\Gamma) \doteqdot \inf_{\Lambda} \text{relength}(\Lambda),$$

where the infimum is taken over all SEPARATING ARCS Λ—simple, regular embeddings of $[0, 1]$ into $\overline{\Omega}$ with boundary on $\Gamma = \partial\Omega$ and interior in Ω which separate Ω into two regions[31], Ω_1 and Ω_2—and the RELATIVE LENGTH of a separating arc Λ is defined by

$$\text{relength}(\Lambda) = \frac{\text{length}(\Lambda)}{\text{length}(\overline{\Lambda})},$$

where the COMPARISON ARC, $\overline{\Lambda}$, is the (unique up to rigid motion) shortest arc which separates the disc $\overline{\Omega}$ of the same area as Ω into regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ of the same areas as Ω_1 and Ω_2, respectively.

[30] Of class at least C^1.

[31] Necessarily, by the Schoenflies theorem.

Obviously, the relative isoperimetric constant of a round circle is one. Moreover, since

$$\text{relength}(\Lambda) \to 1 \text{ as } \text{length}(\Lambda) \to 0,$$

relative isoperimetric constant cannot exceed one on any regular Jordan curve Γ. In fact, $\mathcal{I}(\Gamma) < 1$ unless Γ is a round circle.

Hamilton proved that the relative isoperimetric constant of a regular Jordan curve does not decrease under curve shortening flow.[32]

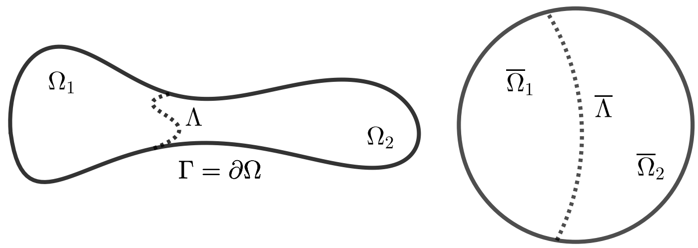


Figure 4.8: Given an arc, Λ, separating Ω into regions, Ω_1 and Ω_2, the comparison arc, $\overline{\Lambda}$, is the shortest arc separating the disc of the same area as Ω into regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ of the same areas as Ω_1 and Ω_2, respectively.

[32] Richard S. Hamilton, "Isoperimetric estimates for the curve shrinking flow in the plane".

Proposition 4.18. *Along any curve shortening flow $\gamma : M^1 \times [0, T) \to \mathbb{R}^2$ of simple, closed, planar curves $\Gamma_t = \gamma(M^1, t)$,*

$$\frac{d}{dt}\mathcal{I}(\Gamma_t) \geq 0$$

in the viscosity sense[33] *whenever $\mathcal{I}(\Gamma_t) < 1$. In particular,*

$$\mathcal{I}(\Gamma_t) \geq \mathcal{I}(\Gamma_0)$$

for all $t \in [0, T)$.[34]

[33] This is a weak formulation of the differential inequality $\frac{du}{dt} \geq 0$ which applies to any continuous function. It asserts, for every $t_0 \in (0, T)$, that every smooth function $\varphi : [0, T) \to \mathbb{R}$ which touches u from below at t_0, in the sense that $u \leq \varphi$ for t in a backward neighbourhood $(t_0 - \delta, t_0]$ of t_0 with equality at t_0, satisfies $\frac{d\varphi}{dt}(t_0) \geq 0$.

[34] A sharp version of this estimate was established by Andrews and Bryan, "A comparison theorem for the isoperimetric profile under curve-shortening flow", sharp enough indeed to establish direct control and on the geometry of the evolving curves as the maximal time is approached.

Sketch of the proof. First note that, given any separating arc Λ for a domain Ω, the first variation formula for the length of a separating arc in the comparison domain $\overline{\Omega}$, subject to the area constraint, guarantees that any comparison arc $\overline{\Lambda}$ has constant curvature and meets the circle $\partial\overline{\Omega}$ orthogonally.

Now, if $\mathcal{I}(\Gamma) < 1$, then (since $\text{relength}(\Lambda) \to 1$ as $\text{length}(\Lambda) \to 0$) a minimizing sequence of separating arcs Λ_j (i.e. $\text{relength}(\Lambda_j) \to \mathcal{I}(\Gamma)$) will have lengths bounded uniformly from below. It is then possible to extract a suitable *weak* limit arc, Λ. Though this limiting arc may not be smooth *a priori*, the vanishing of the first variation of the relative length at Λ ensures that Λ has constant curvature and meets the boundary $\Gamma = \partial\Omega$ orthogonally in the corresponding weak sense, which guarantees that it is smooth (and connected, else a better constant is given by one of the components).

Vanishing of the first variation of relength at Λ also yields

$$\frac{k}{\mathrm{L}} = \frac{\overline{k}}{\overline{\mathrm{L}}},$$

where k and $\overline{k}$ are the (constant) curvatures of Λ and $\overline{\Lambda}$, respectively, and L and $\overline{\mathrm{L}}$ are the lengths of Λ and $\overline{\Lambda}$, respectively.

The nonnegativity of the second variation of the relative length at the minimizing arc guarantees that

$$\lambda \leq \overline{\lambda} \text{ and } \frac{K}{k} \leq \frac{\overline{K}}{\overline{k}},$$

where λ and $\overline{\lambda}$ are the total turning angles of Λ and $\overline{\Lambda}$, respectively, and K and $\overline{K}$ are the respective averages of the curvatures of the boundaries $\partial\Omega$ and $\partial\overline{\Omega}$ at the boundary points of Λ and $\overline{\Lambda}$.

Now suppose that the boundaries $\{\Gamma_t = \partial\Omega_t\}_{t\in[0,T)}$ evolve by curve shortening. Given $t_0 \in (0,T)$, if $\mathcal{I}(\Gamma_{t_0}) < 1$, then we can find some minimizing arc, Λ_{t_0}, as above. Given any variation $\{\Lambda_t\}_{t\in(t_0-\delta,t_0]}$ of Λ_{t_0}, the inequality $\mathrm{relength}(\Lambda_t) \geq \mathcal{I}(\Gamma_t)$ holds for $t \in (t_0-\delta, t_0]$, with equality at time t_0. Thus, if φ is a lower support for $\mathcal{I}(\Gamma_t)$ at time t_0, then $\varphi(t) \leq \mathrm{relength}(\Lambda_t)$ with equality at time t_0, and hence, at time t_0,

$$\frac{d}{dt}\varphi \geq \frac{d}{dt}\,\mathrm{relength}(\Lambda_t).$$

If we construct the variation so that[35]

$$\begin{cases} \partial_t\gamma_{\Lambda_t} = -\kappa_{\Lambda_t}\,\mathsf{N}_{\Lambda_t} & \text{in the interior of } \Lambda_t \\ \mathsf{N}_{\Lambda_t}\cdot\mathsf{N}_{\Gamma_t} = 0 \text{ at } \partial\Lambda_t \end{cases}$$

at time $t = t_0$, then it can be shown that

$$\begin{aligned} \frac{d}{dt}\log\mathrm{relength}(\Lambda_t) &= \frac{\frac{d}{dt}\,\mathrm{length}(\Lambda_t)}{\mathrm{length}(\Lambda_t)} - \frac{\frac{d}{dt}\,\mathrm{length}(\overline{\Lambda}_t)}{\mathrm{length}(\overline{\Lambda}_t)} \\ &= -2\left(\frac{K+k\lambda}{\mathrm{L}} - \frac{\overline{K}+\overline{k}\overline{\lambda}}{\overline{\mathrm{L}}}\right) \\ &= -2\left[\frac{k}{\mathrm{L}}\left(\frac{K}{k}+\lambda\right) - \frac{\overline{k}}{\overline{\mathrm{L}}}\left(\frac{\overline{K}}{\overline{k}}+\overline{\lambda}\right)\right] \end{aligned}$$

at $t = t_0$, which is nonpositive due to the second variation inequalities for the relative length described above. This establishes the first claim.

To prove the second claim, it suffices to establish that

$$\mathcal{I}(\Gamma_t) - \mathcal{I}(\Gamma_0) + \varepsilon(1+t) \geq 0$$

for all $t \in [0,T)$ for any $\varepsilon > 0$. Note that the inequality holds strictly at time $t = 0$ for any positive ε. Suppose then that some $\varepsilon > 0$ and $t_0 \in (0,T)$ can be found such that

$$\mathcal{I}(\Gamma_t) - \mathcal{I}(\Gamma_0) + \varepsilon(1+t) \geq 0$$

for all $t \leq t_0$, but with equality at time $t = t_0$. But then the function

$$\varphi(t) \doteq \mathcal{I}(\Gamma_0) - \varepsilon(1+t)$$

[35] This boundary value problem is known as the FREE BOUNDARY CURVE SHORTENING FLOW. Note that we only require it to hold at time $t = t_0$, however. (So we do not need to solve a backwards heat equation to arrange it!)

is a lower support for $\mathcal{I}$ at time $t = t_0$, and hence

$$0 \leq \frac{d}{dt}\varphi = -\varepsilon < 0,$$

which is absurd. □

4.6 Uniformization of Jordan curves by curve shortening flow

We now have in place all of the ingredients needed to prove that regular Jordan curves shrink to round points under curve shortening flow.[36]

[36] In fact, the tools we have established provide a great many routes to this theorem; here, we outline just one.

4.6.1 Deforming convex curves to round points

We consider first the case of convex initial curves.[37]

[37] It suffices to consider *smooth, locally uniformly convex* initial data, as the estimate of Proposition 2.25 guarantees, by an approximation argument, that the boundary of any bounded, convex set can be evolved continuously by curve shortening flow, becoming immediately smooth and locally uniformly convex at positive times.

Theorem 4.19 (Gage–Hamilton[38]). *Given any convex, locally uniformly convex curve $\gamma_0 : M^1 \to \mathbb{R}^2$, $M^1 \cong S^1$, the maximal curve shortening flow $\gamma : M^1 \times [0, T) \to \mathbb{R}^2$ starting at γ_0 deforms γ_0 through a family of convex curves, $\gamma_t : M^1 \to \mathbb{R}^2$, which converge to a point $p \in \mathbb{R}^2$ after a finite time T, with*

$$\frac{\gamma_t - p}{\sqrt{2(T-t)}} \to \overline{\gamma} \text{ as } t \to T$$

uniformly in the smooth topology, where $\overline{\gamma}$ is an embedding whose image is the unit circle.[39]

[38] Gage and R. S. Hamilton, "The heat equation shrinking convex plane curves"

[39] Observe that, in contrast to the proof of Huisken's theorem (Theorem 3.13), the argument presented here does not provide a *rate* of convergence of the rescaled curves to the shrinking circle. This may be remedied by a *stability argument*; see Exercise 4.8.

Sketch of the proof. We have already seen that convexity is preserved[40] and that $T < \infty$ (Proposition 2.6), so that (by Theorem 2.19)

$$\limsup_{t \to T} \max_{M^1 \times \{t\}} |\kappa| = \infty.$$

[40] This may be viewed as a consequence of Proposition 2.9 and Corollary 2.15. It is also a consequence of uniqueness of solutions and the continued existence of a solution to the PDE (4.8) for the support function whilever $|\kappa| < \infty$.

We shall perform a "blow-up" at the final time to establish the claims. To that end, choose a sequence of times $t_j \to T$ and points $x_j \in M^1$ such that

$$\kappa(x_j, t_j) = \max_{M^1 \times [0, t_j]} \kappa$$

and consider the rescaled curve shortening flows $\gamma_j : M^1 \times I_j \to \mathbb{R}^2$ defined by

$$\gamma_j(x,t) \doteqdot r_j^{-1}\big(\gamma(x, r_j^2 t + t_j) - \gamma(x_j, t_j)\big), \quad I_j \doteqdot [-r_j^{-2} t_j, r_j^{-2}(T - t_j)),$$

where $r_j^{-1} \doteqdot \kappa(x_j, t_j)$. Passing to a subsequence, we may arrange that $r_j^{-2}(T - t_j) \to \omega \in [0, \infty]$ as $j \to \infty$. In fact, $\omega > 0$ since (by applying the ODE comparison principle to the evolution equation for κ) $\max_{M^1 \times \{t\}} \kappa \geq \frac{1}{\sqrt{2(T-t)}}$. By construction, the curvature κ_j of the rescaled flow is bounded by 1 for times $t \le 0$, with equality at the origin at time zero. Since this ensures, by the exterior noncollapsing estimate (Proposition 4.16) and the scale invariance of $\underline{r}\kappa$, that the exscribed radius at the spacetime origin is bounded, we can now deduce from the Bernstein estimates (Proposition 2.18) and the Arzelà–Ascoli Theorem that some subsequence of these rescaled flows converges uniformly in the smooth sense to a limit ancient curve shortening flow, $\gamma_\infty : M^1 \times (-\infty, 0] \to \mathbb{R}^2$.

Next, we observe that the Firey entropy[41], $\mathscr{F}$, is constant on the limit flow. Indeed, since $\mathscr{F}$ is nonincreasing on the original flow, it must take a limit as $t \to T$. Now, since $\mathscr{F}$ is scale invariant, we have, for any $a < b \in (-\infty, 0]$,

$$\mathscr{F}_j(b) - \mathscr{F}_j(a) = \mathscr{F}(r_j^2 b + t_j) - \mathscr{F}(r_j^2 a + t_j)$$

for all j sufficiently large. But both $r_j^2 a + t_j$ and $r_j^2 a + t_j$ tend to T as $j \to \infty$, so the right hand side tends to zero, and we conclude that $\mathscr{F}$ is indeed constant on the limit flow.

It follows from Proposition 4.2 that the limit is a self-similarly shrinking solution, which must be the shrinking circle by Theorem 4.8. We conclude that the flow does indeed approach a shrinking circle after rescaling by the maximum of the curvature, at least along some sequence of times $t_j \to T$ (and after performing some sequence of translations). Note that we must then have $\kappa \sim 1/\sqrt{2(T-t)}$ since $\min \kappa \le 1/\sqrt{2(T-t)} \le \max \kappa$. At this point, the full statement of the theorem may be established via a series of bootstrapping arguments. □

[41] Alternatively, we could invoke the Gage–Hamilton entropy and Proposition 4.1 here.

Corollary 4.20. *The shrinking spheres and the static lines are the only convex ancient curve shortening flows which are noncollapsing.*

Sketch of the proof. Let $\{\Gamma_t\}_{t \in (-\infty, \omega)}$, $\Gamma_t = \partial\Omega_t$, (without loss of generality, $\omega \in \{0, \infty\}$) be a noncollapsing convex ancient curve shortening flow and consider the rescaled flows $\{\Gamma_t^\lambda\}_{t \in (-\infty, 0)}$ defined by $\Gamma_t^\lambda \doteqdot \lambda \Gamma_{\lambda^{-2}t}$. By the local curvature estimate of Proposition 2.25 and the interior noncollapsing hypothesis, the rescaled flows converge to a limit ancient flow $\{\Gamma_t^\infty\}_{t \in (-\infty, 0)}$, $\Gamma_t^\infty = \partial\Omega_t^\infty$, along some sequence of scales $\lambda_j \searrow 0$. By Huisken's monotonicity formula[42] and the fact that the Gaussian length is uniformly bounded on the space of convex curves, the Huisken functional is constant on $\{\Gamma_t^\infty\}_{t \in (-\infty, 0)}$, which must therefore be a self-similarly shrinking solution (by the rigidity case of Huisken's monotonicity formula) and hence either a static line or the shrinking circle, by Theorem 4.8. Now, if $\{\Gamma_t^\infty\}_{t \in (-\infty, 0)}$ is a static line, then the Huisken functional must be constant. Indeed, taking the limit as $\lambda \to \infty$ of $\{\lambda(\Gamma_{\lambda^{-2}t + t_0} - p_0)\}_{t \in (-\infty, 0]}$ for any $t_0 < \omega$ and $p_0 \in \Gamma_{t_0}$ yields a static line. The claim follows since the Huisken

[42] It is not obvious that Huisken's monotonicity formula can be legitimately applied when the flow is noncompact; but it can (at least when the flow is convex).

functional is monotone and invariant under spacetime translation and parabolic scaling.

We conclude that $\{\Gamma_t\}_{t\in(-\infty,\omega)}$ is a self-similarly shrinking solution, and hence a static line. Similarly, if $\{\Gamma^\infty_t\}_{t\in(-\infty,0)}$ is a shrinking cirlce, then $\{\Gamma_t\}_{t\in(-\infty,\omega)}$ must also be a shrinking circle, since by the Gage–Hamilton theorem we obtain the shrinking circle after blowing up at the final time. □

4.6.2 *Deforming regular Jordan curves to round points*

Using the Gage–Hamilton theorem (via Corollary 4.20), we may now prove convergence to a round point for the curve shortening flow of any (sufficiently regular) simple closed curve.

Theorem 4.21 (Grayson[43]). *Given any simple, closed curve $\gamma_0 : M^1 \to \mathbb{R}^2$, the maximal curve shortening flow $\gamma : M^1 \times [0,T) \to \mathbb{R}^2$ starting at γ_0 deforms γ_0 through a family of simple closed curves, $\gamma_t : M^1 \to \mathbb{R}^2$, which converge to a point, $p \in \mathbb{R}^2$, after a finite time T, with*

$$\frac{\gamma_t - p}{\sqrt{2(T-t)}} \to \overline{\gamma} \;\; as \;\; t \to T$$

uniformly in the smooth topology, where $\overline{\gamma}$ is an embedding whose image is the unit circle.

[43] Grayson, "The heat equation shrinks embedded plane curves to round points."

Sketch of the proof. We have already seen that embeddedness is preserved (Proposition 2.9) and that $T < \infty$ (Proposition 2.6), so that (by Theorem 2.19)

$$\limsup_{t\to T} \max_{M^1\times\{t\}} |\kappa| = \infty.$$

We shall perform a "blow-up" at the final time to establish the claims. Recall that (by the ODE comparison principle) $\max_{M^1\times\{t\}} |\kappa| \geq \frac{1}{\sqrt{2(T-t)}}$.

Assume first that $\max_{M^1\times\{t\}} |\kappa| \leq \frac{C}{\sqrt{2(T-t)}}$ (the expected rate of blow-up). Given any sequence of times $t_j \nearrow T$, choose points $x_j \in M^1$ such that

$$r_j^{-1} \doteqdot \max_{M^1\times\{t_j\}} |\kappa| = |\kappa(x_j,t_j)|$$

and consider the rescaled curve shortening flows $\gamma_j : M^1 \times I_j \to \mathbb{R}^2$ defined by

$$\gamma_j(x,t) \doteqdot r_j^{-1}\big(\gamma(x, r_j^2 t + t_j) - \gamma(x_j,t_j)\big)\,,\;\; I_j \doteqdot [-r_j^{-2}t_j, r_j^{-2}(T-t_j))\,.$$

Passing to a subsequence, we may arrange that $r_j^{-2}(T-t_j) \to \omega \in (0,\infty)$ as $j \to \infty$. Observe that the curvature κ_j of the rescaled flow satisfies

$$\begin{aligned}
|\kappa_j(x,t)| &= |r_j\kappa(x, r_j^2 t + t_j)| \\
&\leq \frac{Cr_j}{\sqrt{T - t_j - r_j^2 t}} \\
&= \frac{C}{\sqrt{r_j^{-2}(T-t_j) - t}} \\
&\to \frac{C}{\sqrt{\omega - t}} \text{ as } j \to \infty.
\end{aligned}$$

Thus, by Theorem 2.22, some subsequence of these rescaled flows converges locally uniformly in the smooth sense to an ancient limit curve shortening flow, $\gamma_\infty : M^1_\infty \times (-\infty, 1) \to \mathbb{R}^2$.

We claim that the limit flow is convex. To see this, observe that the total curvature $\mathscr{K}(t)$ is constant in the limit. Indeed, since $\mathscr{K}$ is positive and nonincreasing, it must take a limit as $t \to T$ (on the original flow). But then, since $\mathscr{K}$ is scale invariant,

$$\mathscr{K}_j(b) - \mathscr{K}_j(a) = \mathscr{K}(r_j^2 b + t_j) - \mathscr{K}(r_j^2 a + t_j)$$

for any $a, b \in (-\infty, \omega)$ on the rescaled flows (for j sufficiently large). But both $r_j^2 a + t_j$ and $r_j^2 b + t_j$ tend to T as $j \to \infty$. So the right hand side tends to zero, and we conclude that $\mathscr{K}$ is indeed constant. But then Lemma 4.6 implies that the limit flow has nonnegative curvature, and hence strictly positive curvature by the strong maximum principle. Since the turning angle is equal to 2π, we conclude that the limit flow is indeed convex.

Since Proposition 4.17[44] ensures that the limit flow is noncollapsing, we may now conclude from Corollary 4.20 that it is the shrinking circle. It follows that the flow does indeed approach a shrinking circle after rescaling by $r(t) \doteqdot \sqrt{T-t}$, at least along some sequence of times $t_j \to T$ (and after performing some sequence of translations). At this point, the full statement of the theorem may be established via a series of bootstrapping arguments.

[44] Or, alternatively, Proposition 4.18.

We have not yet proved that that $|\kappa|\sqrt{T-t}$ remains bounded. Suppose then that, to the contrary,

$$\limsup_{t \nearrow T} \max_{M^1 \times \{t\}} |\kappa|\sqrt{T-t} = \infty.$$

For each j, choose $(x_j, t_j) \in M^1 \times [0, T)$ so that

$$(T - j^{-1} - t_j)|\kappa|(x_j, t_j) = \max_{M^1 \times [0, T - j^{-1}]} (T - j^{-1} - t)|\kappa|$$

and set $r_j^{-2} \doteqdot |\kappa|(x_j, t_j)$. Consider the rescaled flows $\gamma_j : M^1 \times [\alpha_j, \omega_j) \to$

$\mathbb{R}^2$ defined by

$$\gamma_j(x,t) \doteqdot r_j^{-1}\big(\gamma(x, r_j^2 t + t_j) - \gamma(x_j, t_j)\big),$$
$$[\alpha_j, \omega_j) \doteqdot \big[-r_j^{-2}t_j, r_j^{-2}(T - j^{-1} - t_j)\big).$$

Observe in this case that

$$\alpha_j \to -\infty, \ \ \omega_j \to \infty,$$

and

$$|\kappa_j(x,t)|^2 = |r_j^1\kappa(x, r_j^2 t + t_j)|^2 \le \frac{T - j^{-1} - t_j}{T - j^{-1} - r_j^2 t + t_j} = \frac{\omega_j}{\omega_j - t},$$

which is uniformly bounded on any compact time interval for j sufficiently large. Thus, by Theorem 2.22, some subsequence of the pointed, rescaled flows $\gamma_j : M^1 \times [\alpha_j, \omega_j) \to \mathbb{R}^2$ must converge to an eternal limit flow $\gamma_\infty : M^1_\infty \times (-\infty, \infty) \to \mathbb{R}^2$. The above argument implies that this limit is convex. Since it has bounded curvature on compact time intervals, it satisfies the differential Harnack inequality. But, by construction,

$$\kappa \le \limsup_{j\to\infty} \frac{\omega_j}{\omega_j - t} = 1 = \kappa(x_\infty, 0).$$

Thus, at $(x_\infty, 0)$, $\partial_t\kappa = 0$ and $\nabla\kappa = 0$, so the rigidity case of the differential Harnack inequality implies that the limit flow is a self-similarly translating solution, which must therefore be the Grim Reaper by Theorem 1.1 and the curvature normalization at $(x_\infty, 0)$. But the Grim Reaper violates the (scale invariant) lower bound for the relative isoperimetric constant[45] (which passes to the limit as it is scale invariant and lower semi-continuous under local uniform convergence). This completes the proof. □

Figure 4.9: Flip the pages to test Grayson's theorem! (Code by Anthony Carapetis, *Curve shortening demo*. Explore further at a.carapetis.com/csf.)

[45] Or, alternatively, the chord-arc constant.

4.7 *Exercises*

Exercise 4.1. Let $\gamma : \theta(M^1) \to \mathbb{R}^2$ be the turning angle parametrization for a convex, locally uniformly convex planar curve $\Gamma = \partial\Omega$.

(a) Show that $\gamma \cdot \mathsf{N} = \sigma$.

(b) Deduce that $\gamma \cdot \mathsf{T} = \sigma_\theta$.

(c) Conclude that $\gamma = \sigma\,\mathsf{N} + \sigma_\theta\,\mathsf{T}$.

Exercise 4.2. Let $\gamma : M^1 \to \mathbb{R}^2$ be a shrinker; i.e.

$$\vec{\kappa} = \tfrac{1}{2}\gamma^\perp.$$

Suppose that $\vec{\kappa} = 0$ at some point $x_0 \in M^1$; set $p \doteqdot \gamma(x_0)$ and $v \doteqdot \mathsf{T}(x_0)$.

(a) Show that the curve $\beta : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\beta(y) \doteqdot p_0 + (y - y_0)v$$

satisfies $|\beta'| \equiv 1$ and

$$\begin{cases} \vec{\kappa} = \frac{1}{2}\beta^{\perp} \\ (\beta(y_0), \beta'(y_0)) = (p, v). \end{cases}$$

(b) Deduce that $\gamma(M^1) \subset \beta(\mathbb{R})$. In particular, if M^1 is connected and γ is proper, then β is an arclength parametrization of γ.

(c) Conclude that every proper connected shrinker is either a straight line through the origin or locally uniformly convex.

(d) Prove that the same claim is true for expanders.

Exercise 4.3. Let $\gamma : M^1 \to \mathbb{R}^2$ be an expander; i.e.

$$\vec{\kappa} = -\tfrac{1}{2}\gamma^{\perp}.$$

Suppose that M^1 is connected and γ is proper.

(a) Show that $M^1 \cong \mathbb{R}$.

(b) Show that γ is an embedding.

Hint: both parts may be established using the first variation of enclosed area under curve shortening flow.

Exercise 4.4. Let $\gamma(M^1) = \Gamma = \partial\Omega$ be an embedded self-similar curve.

(a) Show that

$$\begin{cases} \operatorname{div} V = \lambda \text{ in } \Omega \\ V \cdot \mathsf{N} = \kappa \text{ on } \partial\Omega. \end{cases} \tag{4.39}$$

or, in the gradient case,

$$\begin{cases} \Delta f = \lambda \text{ in } \Omega \\ D_{\mathsf{N}} f = \kappa \text{ on } \partial\Omega. \end{cases} \tag{4.40}$$

(b) In case Ω is bounded, deduce that $\lambda = \frac{2\pi}{\text{area}(\Omega)}$.

Exercise 4.5. Given a gradient self-similar curve $\gamma : M^1 \to \mathbb{R}^2$, with potential function f, set $\varphi \doteqdot \gamma^* f$. Prove the following identities.

(a) $V^{\top} = \nabla\varphi$.

(b) $\nabla\kappa = \kappa\nabla\varphi$.

(c) $\Delta\varphi = \frac{\lambda}{2} - \kappa^2$ (and hence $\frac{\lambda}{2}\,\text{length}(\gamma) = \int_{M^1} \kappa^2\, ds$ if M^1 is compact).

(d) $\frac{1}{2}\nabla\left(\kappa^2 + |\nabla\varphi|^2 - \lambda\varphi\right) = 0$.

Deduce that, on the corresponding self-similar solution to curve shortening flow,

(e) $\partial_t\varphi = |\nabla\varphi|^2$.

(f) $(\partial_t + \Delta)\varphi + \kappa^2 = |\nabla\varphi|^2 + \frac{1}{-2t}$.

(g) $(\partial_t + \Delta - \kappa^2)\psi = 0$, where $\psi \doteqdot \frac{1}{\sqrt{-2t}}\mathrm{e}^{-\varphi}$.

Exercise 4.6. Show that, for any embedded curve Γ,

$$\lim_{y\to x}\frac{2(x-y)\cdot \mathsf{N}(x)}{|x-y|^2} = \kappa(x)\,.$$

Exercise 4.7. Deduce from Propositions 4.15 and 4.16 that, along any convex curve shortening flow $\gamma : M^1 \times [0, T) \to \mathbb{R}^2$, $M^1 \cong S^1$,

$$\rho_\pm(t) \sim \sqrt{2(T-t)} \quad\text{and}\quad \kappa \sim \frac{1}{\sqrt{2(T-t)}}$$

as $t \to T$, where $\rho_-(t)$ resp. $\rho_+(t)$ is the inradius resp. circumradius of $\Gamma_t = \gamma(M^1, t)$ (the radius of the largest disc enclosed by resp. smallest disc enclosing Γ_t).[46]

[46] This gives a slightly different route to the Gage–Hamilton theorem than the one we presented.

Exercise 4.8. Let $\{\tilde\Gamma_{\tilde t}\}_{\tilde t\in[0,T)}$ be a maximal curve shortening flow of simple, closed, planar curves $\tilde\Gamma_{\tilde t} = \partial\tilde\Omega_{\tilde t}$. According to Theorem 4.21, $\tilde\Gamma_{\tilde t}$ becomes convex after some time $\tilde t_0 \in [0, T)$, and shrinks to a point $p_0 \in \mathbb{R}^2$ as $\tilde t \to T$ with circular asymptotic shape. Consider the RESCALED FLOW $\{\Gamma_t = \partial\Omega_t\}_{t\in[-\frac{1}{2}\log(2T),\infty)}$ defined by

$$\Gamma_t = \mathrm{e}^t\left(\Gamma_{\tilde t} - p_0\right), \;\; \mathrm{e}^{-2t} = 2(T-\tilde t)\,.$$

(a) (i) Show that, for $t \geq t_0 \doteqdot -\frac{1}{2}\log(2(T-\tilde t_0))$, the support function σ of the rescaled flow satisfies

$$\partial_t\sigma = F(\sigma) \doteqdot \sigma - (\sigma_{\theta\theta} + \sigma)^{-1}\,.$$

(ii) Show that the linearization of F about the unit circle is given by

$$DF|_1 v = v_{\theta\theta} + 2v\,.$$

(iii) Deduce that

$$\frac{d}{dt}\int_0^{2\pi}(\sigma-1)^2 d\theta \leq 2\int_0^{2\pi}(\sigma-1)\big((\sigma-1)_{\theta\theta} + 2(\sigma-1)\big)\,d\theta + C|\sigma-1|_{L^2}|\sigma-1|^2_{C^2}\,.$$

(iv) "Solve" the linearized equation

$$\partial_t v = v_{\theta\theta} + 2v\,.$$

with 2π-periodic boundary condition by separating variables.

(b) (i) Show that enclosed area is constant under the rescaled flow.

(ii) Show that the enclosed area is given by

$$\text{area}(\Omega_t) = \text{area}(\sigma) \doteqdot \int_0^{2\pi} \sigma(\sigma_{\theta\theta} + \sigma)\, d\theta\,.$$

(iii) Show that the linearization of area about the unit circle is given by

$$D\,\text{area}\,|_1 v = 2\int_0^{2\pi} v\, d\theta\,.$$

(iv) Deduce that

$$\int_0^{2\pi} (\sigma - 1)\, d\theta \le C|\sigma - 1|^2_{C^2}\,.$$

(c) (i) Show that the centre of mass, $q(\Omega_t) \doteqdot \int_{\Omega_t} X\, dX$, is always located at the origin under the rescaled flow.

(ii) Show that q is given by

$$q(\Omega_t) = q(\sigma) \doteqdot \frac{1}{3}\int_0^{2\pi} (\sigma \sin\theta, -\sigma\cos\theta)_\theta \sigma(\sigma_{\theta\theta} + \sigma)\, d\theta\,.$$

(iii) Show that the linearization of q about the unit circle is given by

$$Dq|_1 v = \frac{1}{3}\int_0^{2\pi} v(\cos\theta, \sin\theta)\, d\theta\,.$$

(iv) Deduce that

$$\left|\int_0^{2\pi} (\cos\theta, \sin\theta)(\sigma - 1)\, d\theta\right| \le C|\sigma - 1|^2_{C^2}\,.$$

(d) Writing $\sigma - 1 = \frac{1}{2}A_0(t) + \sum_{j=1}^{\infty}\left(A_j(t)\cos(j\theta) + B_j(t)\sin(j\theta)\right)$, show that

$$\int_0^{2\pi} (\sigma - 1)\left[(\sigma-1)_{\theta\theta} + 2(\sigma - 1)\right] d\theta \le C\left(A_0^2 + A_1^2 + B_1^2\right) - 2|\sigma - 1|^2_{L^2}.$$

(e) Deduce that

$$\frac{d}{dt}|\sigma - 1|^2_{L^2} \le C|\sigma - 1|^4_{C^2} - 2|\sigma - 1|^2_{L^2}\,.$$

(f) Using interpolation, estimate

$$|\sigma - 1|^4_{C^2} \le C_\delta|\sigma - 1|^{4-\delta}_{L^2}$$

for some $\delta \in (0, 2)$.

(g) Conclude that $|\sigma - 1|^2_{L^2}$ decays exponentially once it becomes sufficiently small.

5
Singularities and their analysis

Time is sick
Critical density
Contraction
Singularity
Everything and nothing
Life and death
– King Gizzard & The Lizard Wizard, "Murder of the Universe"

We have seen that finite time singularities will necessarily occur under mean curvature flow on a compact hypersurface. In one space dimension, or in higher dimensions when the initial hypersurface is convex, we were able to deal with finite time singularities by "blowing up" and classifying the possible blow-up limits. As a result, we saw that the mean curvature flow deforms any simple closed curve, or any convex hypersurface, to a round point. One could therefore be forgiven for hoping that mean curvature flow might deform *any* (not necessarily convex) embedding of a sphere to a round point. This turns out to be a little optimistic, however.

Example 6 (A "neckpinch" singularity[1])**.** Consider an embedding of S^2 into $\mathbb{R}^3$ which looks like two large, disjoint round spheres which are far apart but smoothly connected by a long, thin "neck" (as in Figure 5.1, say). This is a very flexible configuration, and it can certainly be arranged that the neck passes through the "hole" in Angenent's doughnut (see Example 4) while the spherical components each enclose very large spheres on each side (of radius 100, say). Since Angenent's doughnut contracts to the origin under mean curvature flow after time one, while the radii of the enclosed spheres remain positive at this time under the flow, the hypersurface must become singular at some earlier time, in accordance with the avoidance principle. ■

[1] Such examples appear to have first been described by Hamilton.

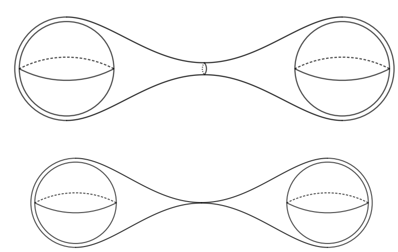


Figure 5.1: A "barbell" configuration. If the "bar" is sufficiently thin compared to the "bells", then it will "pinch off" before the enclosed spheres contract to points.

Example 7 (A "degenerate neckpinch" singularity)**.** In the above example, we could imagine a continuous deformation of the initial surface which shrinks one of the spherical components down to a radius comparable to the neck radius (as in Figure 5.2, say). In this configuration, the small enclosed sphere is no longer a barrier, and it is not unreasonable to expect that the small spherical component of the initial surface is able to contract quickly enough to slip through the neck before the shrinking doughnut pinches it, the solution thereafter becoming convex and shrinking to a round point according to Huisken's theorem.

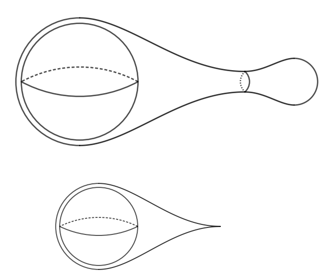


Figure 5.2: An asymmetric barbell configuration. If one of the bells is sufficiently small, it will "pass through" the bar before it pinches off. There is a critical configuration at which the bar pinches off just as the smaller bell is passing through it.

But then there must be a critical stage in the deformation such that for smaller deformations a neckpinch forms, while for larger deformations there is no neckpinch. Somehow, at the critical deformation, the smaller spherical component attempts a run through the neck, but gets caught just as it is about to emerge from the other side. (A rigorous construction of such solutions was undertaken by Angenent and Velázquez.[2]) ■

[2] S. B. Angenent and Velázquez, "Degenerate neckpinches in mean curvature flow".

Example 8 (A "doubly degenerate neckpinch" singularity[3])**.** Imagine now performing this deformation in a symmetric manner, so that *both* bells get caught in the neck as it collapses. In this configuration, the hypersurface does indeed shrink to a point at the singular time, T, but its asymptotic shape cannot be that of a round sphere: for at each time $t < T$, the hypersurface is nonconvex, so the pinching ratio $\inf \kappa_1/\kappa_n$ can be no better than zero at the singular time. ■

[3] S. Altschuler, Sigurd B. Angenent, and Giga, "Mean curvature flow through singularities for surfaces of rotation".

These examples demonstrate that singularities can potentially be quite complicated in dimensions $n \geq 2$, even in the absence of topology. On the other hand, at a neckpinch singularity, most of the hypersurface remains "non-singular" and the flow appears to be performing the opposite of a connected sum. This begs the question, "Can the flow be continued after a singularity, while keeping track of any topological changes at singular times?" Rather than attempting a comprehensive answer to this (very difficult) question, we shall merely present some basic results and tools which suggest that singularities are indeed somewhat "tamable", at least in certain special settings.

We begin by noting the following immediate corollary of Theorem 9.19, which demonstrates the importance of ANCIENT[4] mean curvature flows in the analysis of singularities.

[4] I.e. having an infinite past.

Lemma 5.1. *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a mean curvature flow with $T < \infty$ and $\{(x_k, t_k)\}_{k\in\mathbb{N}}$ a sequence of spacetime points $(x_k, t_k) \in M^n \times [0, T)$ with $t_k \to T$. Suppose that*

1. $r_k^{-2} \doteqdot |\mathrm{II}_{(x_k,t_k)}| \to \infty$ *as* $k \to \infty$*; and*
2. *for every $A < \infty$ some $C < \infty$ can be found such that X is properly defined in $B^{n+1}_{Ar_k}(x_k, t_k) \times (t_k - A^2 r_k^2, t_k]$, and*

$$\sup_{B^{n+1}_{Ar_k}(x_k,t_k)\times(t_k-A^2r_k^2,t_k]} |\mathrm{II}| \leq C r_k^{-2}$$

for every k.

For each k, define the rescaled mean curvature flow $X_k : M^n \times I_k \to \mathbb{R}^{n+1}$ by

$$X_k(x, t) \doteqdot r_k^{-1}\left(X(x, r_k^2 t + t_k) - X(x_k, t_k)\right), \quad I_k \doteqdot [-r_k^{-2} t_k, r_k^{-2}(T - t_k)).$$

There exists a complete ancient pointed mean curvature flow $X_\infty : M^n_\infty \times (-\infty, \omega) \to \mathbb{R}^{n+1}$, $x_\infty \in M^n_\infty$ *such that, after passing to a subsequence, the pointed rescaled mean curvature flows* (X_k, x_k) *converge locally uniformly in the smooth topology to* (X_∞, x_∞). *That is, there exists an exhaustion* $\{U_k\}_{k\in\mathbb{N}}$ *of* M_∞ *by precompact open sets* U_k *satisfying* $\overline{U}_k \subset U_{k+1}$ *and a sequence of diffeomorphisms* $\phi_k : \overline{U}_k \to M$ *with* $\phi_k(x_\infty) = x_k$ *such that* $\phi_k^* X_k \to X_\infty$ *uniformly in the smooth topology on any compact subset of* $M_\infty \times (-\infty, 0]$.

5.1 Curvature pinching improves

We have seen that curvature pinching improves at the onset of singularities under mean curvature flow of *convex* hypersurfaces.[5] This turns out to be a special case of a more general phenomenon.

[5] Recall Proposition 3.6.

5.1.1 Convexity estimate

Recall (from Corollary 2.15) that, in accordance with the maximum principle, MEAN CONVEXITY, $H \geq 0$, is preserved under mean curvature flow on a compact hypersurface (with strict inequality at interior times). We also proved, using the tensor maximum principle, that the tensor inequality $\mathrm{II} \geq \alpha\, H\, g$ is preserved when $\alpha > 0$ (Proposition 3.4). By exactly the same argument, this inequality is also preserved for negative pinching constants.

Proposition 5.2 (Scale invariant lower bounds for the curvature are preserved)**.** *Let* $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a compact, strictly mean convex mean curvature flow. There exists* $\alpha \in \mathbb{R}$ *such that*

$$\mathrm{II} \geq \alpha H g$$

at all times.

We will show that this inequality actually improves at the onset of singularities (cf. the "improvement of roundness" of Proposition 3.6).

Proposition 5.3 (Convexity improves[6])**.** *Let* $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a compact, strictly mean convex mean curvature flow. Given any* $\varepsilon > 0$, *there exists* $C_\varepsilon = C(n, X_0, \varepsilon) < \infty$ *such that*

$$\kappa_1 \geq -\varepsilon\, H - C_\varepsilon$$

at all times.

[6] Huisken and Sinestrari, "Mean curvature flow singularities for mean convex surfaces"

This "improvement of convexity" ensures that singularities in mean convex mean curvature flow of compact hypersurfaces are weakly convex, at least in the sense that, along any sequence of points (x_j, t_j) at which $\mathrm{H}(x_j, t_j) \to \infty$,

$$\frac{\kappa_1}{\mathrm{H}}(x_j, t_j) \geq -\varepsilon - \frac{C_\varepsilon}{\mathrm{H}(x_j, t_j)} \quad \text{for any} \quad \varepsilon > 0,$$

and hence $\liminf_{j\to\infty} \frac{\kappa_1}{\mathrm{H}}(x_j, t_j) \geq 0$.

Sketch of the proof of Proposition 5.3. The proof follows the framework of Huisken's improvement of roundness estimate (Proposition 3.6). The first step is to establish an evolution equation for a suitable "pinching function". We consider first the smallest principal curvature, which—even though it is not necessarily smooth—satisfies

$$(\partial_t - \Delta)\kappa_1 \geq |\mathrm{II}|^2 \kappa_1 \tag{5.1}$$

in the DISTRIBUTIONAL SENSE.[7,8]

Next consider, for any $\varepsilon \in (0, 1)$, the function

$$f_\varepsilon \doteqdot \max\left\{-\kappa_1 - \varepsilon(2L\,\mathrm{H} - |\mathrm{II}|), 0\right\},$$

where L is chosen so that $|\mathrm{II}| \leq L\,\mathrm{H}$ (which can be arranged with L depending only on the initial condition since upper bounds for $|\mathrm{II}|^2/\,\mathrm{H}^2$ are preserved). The purpose of using the term $L\,\mathrm{H} - |\mathrm{II}|$ (instead of simply H, say) is to get our hands on the good quadratic gradient of curvature term in the inequality

$$(\partial_t - \Delta) f_\varepsilon \leq f_\varepsilon \left(|\mathrm{II}|^2 - \gamma \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} \right) \tag{5.2}$$

in the distributional sense, for some $\gamma = \gamma(n, \alpha, \varepsilon)$.

Our goal is now to bound the function

$$f_{\varepsilon,\sigma} \doteqdot \frac{f_\varepsilon}{\mathrm{H}} \mathrm{H}^\sigma$$

for some $\sigma \in (0, 1)$. To that end, we apply (5.2) and Young's inequality to estimate

$$(\partial_t - \Delta) f_{\varepsilon,\sigma} \leq f_{\varepsilon,\sigma} \left(\sigma |\mathrm{II}|^2 - \gamma \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} + \gamma^{-1} \frac{|\nabla f_{\varepsilon,\sigma}|^2}{f_{\varepsilon,\sigma}^2} \right)$$

in the distributional sense[9] for some $\gamma = \gamma(n, \alpha, \varepsilon) > 0$ wherever $f_\varepsilon > 0$. It follows that, for $p \geq p_0(n, \alpha, \varepsilon)$, the function $v^2 \doteqdot (f_{\varepsilon,\sigma})_+^p$ satisfies

$$(\partial_t - \Delta) v^2 \leq v^2 \left(\sigma p |\mathrm{II}|^2 - \gamma p \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} \right) - 2|\nabla v|^2,$$

from which we obtain[10]

$$\begin{aligned} \frac{d}{dt} \int v^2 \, d\mu + \int v^2\, \mathrm{H}^2 \, d\mu \leq \sigma p \int v^2 |\mathrm{II}| \, d\mu \\ - \int \left(\gamma p v^2 \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} + 2|\nabla v|^2 \right) d\mu, \end{aligned} \tag{5.3}$$

[7] This is a weak formulation of the differential inequality $(\partial_t - \Delta)u \geq |\mathrm{II}|^2 u$ which applies to functions which are integrable in space and locally Lipschitz in time. It asserts, for every nonnegative $\varphi \in C^\infty(M^2 \times (0, T))$, that

$$\frac{d}{dt} \int u\varphi \, d\mu \leq \int \left[|\mathrm{II}|^2 u\varphi + u(\partial_t + \Delta - \mathrm{H}^2)\varphi \right] d\mu$$

at almost every time.

[8] The inequality (5.1) also holds in the viscosity sense.

[9] Note that this is exactly the inequality (3.4) we had for our pinching function in the proof of the improvement of roundness estimate!

[10] This is exactly the form of (3.5).

Integrating Simons' identity by parts, similarly as in the proof of Claim 3.8, we may estimate

$$\int |\mathrm{II}|^2 v^2 \, d\mu \leq \delta \int |\nabla v|^2 \, d\mu + C_\delta \int v^2 \frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2} \, d\mu$$

for any $\delta > 0$, where $C_\delta = C_\delta(n, \alpha, \varepsilon, \delta)$, and thereby conclude that

$$\frac{d}{dt} \int v^2 \, d\mu \leq 0,$$

so long as $p \geq p_0(n, \alpha, \varepsilon)$ and $\sigma p^{\frac{1}{2}} \leq \sigma_0(n, \alpha, \varepsilon)$.

Integrating this yields an L^2 estimate for v, which can be bootstrapped to an L^∞ estimate (for some ultimately decided upon $p < \infty$ and $\sigma > 0$) via Huisken–Stampacchia iteration, more or less exactly as in the proof of the improvement of roundness estimate.

The claim then follows by way of Young's inequality. □

5.1.2 *Cylindrical estimates*

There is also an interesting family of preserved local convexity conditions which interpolate between local uniform convexity and mean convexity. Given any $m \in \{1, \ldots, n\}$, a hypersurface is said to be m-CONVEX if

$$\kappa_1 + \cdots + \kappa_m \geq 0$$

at all points.

Proposition 5.4 (Intermediate local convexity is preserved). *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a mean curvature flow on a compact manifold M^n. If the inequality $\kappa_1 + \cdots + \kappa_m > 0$ holds at time $t = 0$, then it holds for all $t \in [0, T)$.*

Sketch of the proof. Since the sum, $f \doteqdot \kappa_1 + \cdots + \kappa_m$, of the smallest m principal curvatures is a concave function of the principal curvatures, it can be shown that

$$(\partial_t - \Delta) f \geq |\mathrm{II}|^2 f$$

in the viscosity sense. The claim is then a consequence of the maximum principle. □

The tensor maximum guarantees that scale invariant upper bounds for the second fundamental form are preserved.

Proposition 5.5 (Scale invariant upper bounds for the curvature are preserved). *Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a compact, strictly mean convex mean curvature flow. There exists $C \in \mathbb{R}$ such that*

$$\mathrm{II} \leq C \mathrm{H} g$$

at all times.

Proof. This is a straightforward application of the tensor maximum principle (cf. Proposition 3.4). □

This inequality also improves at the onset of singularities.

Proposition 5.6 (Cylindrical estimates[11])**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact, strictly $(m+1)$-convex mean convex mean curvature flow. Given any $\varepsilon > 0$, there exists $C_\varepsilon = C(n, X_0, \varepsilon) < \infty$ such that*

$$\kappa_n - \tfrac{1}{n-m}\,\mathrm{H} \le \varepsilon\,\mathrm{H} + C_\varepsilon$$

at all times.[12]

Given $m \in \{0, \dots, n-1\}$, the cylindrical estimate implies that singularities in $(m+1)$-convex mean curvature flow of compact hypersurfaces are, in an asymptotic sense, either *strictly* m-convex, or "m-cylindrical". Indeed, the m-convexity estimate may be rewritten as

$$\sum_{j=m+1}^{n}(\kappa_n - \kappa_j) - \sum_{j=1}^{m}\kappa_j \le \varepsilon\,\mathrm{H} + C_\varepsilon \;\text{ for any }\; \varepsilon > 0.$$

Thus, along any sequence of points (x_j, t_j) at which $\mathrm{H}(x_j, t_j) \to \infty$,

$$\sum_{i=m+1}^{n}\frac{\kappa_n - \kappa_i}{\mathrm{H}}(x_j, t_j) \le \sum_{i=1}^{m}\frac{\kappa_i}{\mathrm{H}}(x_j, t_j) + o(1) \;\text{ as }\; j \to \infty.$$

Since the left hand side is nonnegative, we find, in the limit[13], that $\sum_{i=1}^{m}\frac{\kappa_i}{\mathrm{H}} \ge 0$ with strict inequality unless $\frac{\kappa_{m+1}}{\mathrm{H}} = \cdots = \frac{\kappa_n}{\mathrm{H}}$ and (by improvement of convexity) $0 = \frac{\kappa_1}{\mathrm{H}} = \cdots = \frac{\kappa_m}{\mathrm{H}}$.

Sketch of the proof of Proposition 5.6. Starting with the (distributional) inequality[14]

$$(\partial_t - \Delta)\kappa_n \le |\mathrm{II}|^2\kappa_n\,, \tag{5.4}$$

the proof follows the framework of Propositions 3.6 and 5.3. □

5.2 *Self-similar solutions*

Recall that a hypersurface $X : M^n \to \mathbb{R}^{n+1}$ generates a self-similar mean curvature flow if

$$\vec{\mathrm{H}} + (X^*V)^\perp = 0 \tag{5.5}$$

for some ambient vector field V of the form

$$V(x) = \tfrac{\lambda}{2}x + Ax - v \tag{5.6}$$

for some parameters[15] $\lambda \in \mathbb{R}$, $A \in \mathfrak{so}(n+1)$ and $v \in \mathbb{R}^{n+1}$.

Observe that, when $A = 0$, we may express the soliton vector field V as the gradient,

$$V = Df\,, \tag{5.7}$$

[11] Huisken and Sinestrari, "Mean curvature flow with surgeries of two-convex hypersurfaces"

[12] A natural question that might arise from the statement of Proposition 5.6 is: "Why the constant $\frac{1}{n-m}$?". The inequality $\kappa_n \le \frac{1}{n-m}\mathrm{H}$ describes the smallest convex cone of principal curvatures which contains the "cylindrical ray",

$$\mathrm{C}_m \doteqdot \{(\underbrace{0,\dots,0}_{m\text{-times}}, r^{-1}, \dots, r^{-1}) : r > 0\}.$$

This ray represents the principal curvature n-tuple of a shrinking cylinder with m flat factors, which provide an obstruction to the improvement of upper curvature pinching (the pinching is *constant* on these examples).

At the analytical level, the presence of cylindrical points in the support of a function u are an obstruction to establishing the Poincaré-like inequality

$$\int u^2|\mathrm{II}|^2\,d\mu \le \int\left(\delta|\nabla u|^2 + C_\delta u^2\frac{|\nabla \mathrm{II}|^2}{\mathrm{H}^2}\right)d\mu,$$

which is a key element of the L^2 estimate for the modified pinching function.

On the other hand, the inequality $\kappa_1 + \cdots + \kappa_{m+1} > 0$ describes the *largest* convex cone of principal curvatures which *does not contain* the cylindrical ray C_{m+1}. So Propositions 3.6, 5.3, and 5.6 suggest that the shrinking cylinders (including the shrinking sphere) are *the* obstruction to improving curvature pinching.

[13] We do not assume that the limiting values of the ratios $\frac{\kappa_i}{\mathrm{H}}$ correspond to such ratios of some limiting hypersurface.

[14] The inequality (5.4) also holds in the viscosity sense.

[15] The parameter λ generates dilations, while A generates rotations and v generates translations.

of a POTENTIAL FUNCTION $f : \mathbb{R}^{n+1} \to \mathbb{R}$, which is given (modulo an additive constant) by

$$f(x) \doteqdot \tfrac{\lambda}{4}|x|^2 - x \cdot v. \tag{5.8}$$

Taking the normal component, we may rewrite (5.5) as the scalar equation

$$\mathrm{H} = \tfrac{\lambda}{2} X \cdot \mathsf{N} + AX \cdot \mathsf{N} - v \cdot \mathsf{N}\,. \tag{5.9}$$

Differentiating (5.5), we find that

$$\nabla \mathrm{H} = \mathrm{II}(V^\top) - A\,\mathsf{N}\,. \tag{5.10}$$

Differentiating (5.10) and applying (5.5) and (5.10) then yields

$$-\nabla^2 \mathrm{H} = -\nabla_{V^\top} \mathrm{II} + \mathrm{H}\,\mathrm{II}^2 - \tfrac{\lambda}{2}\,\mathrm{II}\,. \tag{5.11}$$

In fact, the following converse holds.

Proposition 5.7. *If a strictly mean convex convex hypersurface $X : M^n \to \mathbb{R}^{n+1}$ satisfies* (5.11) *for some $\lambda \in \mathbb{R}$, with $V^\top$ given by* (5.10)*, then* (5.5) *holds with V given by* (5.6)*.*

Proof. Consider the ambient vector field

$$U \doteqdot V^\top - \vec{\mathrm{H}} - \tfrac{\lambda}{2} X - AX.$$

Differentiating U and applying (5.10) and (5.13) yields

$$\nabla U = 0.$$

So U is constant along X, which implies the claim. □

Applying Simons' identity to (5.11), we obtain

$$-\Delta\,\mathrm{II} = -\nabla_{V^\top} \mathrm{II} + |\mathrm{II}|^2\,\mathrm{II} - \tfrac{\lambda}{2}\,\mathrm{II}\,. \tag{5.12}$$

Tracing either (5.11) or (5.12), yields

$$-\Delta\,\mathrm{H} = -\nabla_{V^\top} \mathrm{H} + |\mathrm{II}|^2\,\mathrm{H} - \tfrac{\lambda}{2}\,\mathrm{H}\,. \tag{5.13}$$

Theorem 5.8 (Huisken[16]; Colding–Minicozzi[17])**.** *The shrinking spheres are the only compact, embedded, mean convex, self-similarly shrinking mean curvature flows.*

Proof. We only need to consider the $n \geq 2$ case, due to Theorem 4.8. By applying the strong maximum principle to (5.13), we may assume that $\mathrm{H} > 0$. To that end, observe (cf. Proposition 3.5) that

$$\begin{aligned} -\mathrm{e}^{X^*f}\,\mathrm{div}\left(\mathrm{e}^{-X^*f}\nabla\frac{|\mathrm{II}|^2}{\mathrm{H}^2}\right) &= (\nabla_{V^\top} - \Delta)\frac{|\mathrm{II}|^2}{\mathrm{H}^2} \\ &= \nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathrm{II}|^2}{\mathrm{H}^2} - 2\left|\nabla\frac{\mathrm{II}}{\mathrm{H}}\right|^2 \\ &\leq \nabla_{\frac{\nabla \mathrm{H}}{\mathrm{H}}}\frac{|\mathrm{II}|^2}{\mathrm{H}^2}. \end{aligned}$$

[16] Huisken, "Asymptotic behavior for singularities of the mean curvature flow"

[17] The conclusion of the theorem also holds when the *compactness* hypothesis is replaced by *polynomial volume growth*; see Tobias H. Colding and Minicozzi, "Generic mean curvature flow I: generic singularities", Theorem 10.1.

So the maximum principle implies that $\frac{|\mathrm{II}|^2}{\mathrm{H}^2}$ is constant, and hence (by the above calculation)

$$0 \equiv \nabla \frac{\mathrm{II}}{\mathrm{H}} = \frac{1}{\mathrm{H}^2} \left(\mathrm{H} \nabla \mathrm{II} - \nabla \mathrm{H} \otimes \mathrm{II} \right) .$$

That is,

$$\mathrm{H} \nabla \mathrm{II} = \nabla \mathrm{H} \otimes \mathrm{II} \tag{5.14a}$$

Tracing this, we find that

$$\mathrm{H}\, \nabla \mathrm{H} = \mathrm{II}(\nabla \mathrm{H}) . \tag{5.14b}$$

Suppose first that there exists some point x at which $\nabla \mathrm{H}(x) \neq 0$. By (5.14b), $\nabla\, \mathrm{H}$ is an eigenvector of II at x with eigenvalue $\mathrm{H}(x)$. The Codazzi identity and (5.14a) then imply that

$$|\nabla \mathrm{H}|^2\, \mathrm{II}(u) = \mathrm{H}\, \nabla_{\nabla \mathrm{H}} \mathrm{II}(u) = \mathrm{H}\, \nabla_u \mathrm{II}(\nabla \mathrm{H}) = \mathrm{H}\, \nabla_u \mathrm{H} \nabla \mathrm{H} = 0$$

at x for any $u \perp \nabla\, \mathrm{H}(x)$. So $\kappa_i(x) = 0$ for $i = 1, \ldots, n-1$ and $\kappa_n(x) = \mathrm{H}(x)$. It follows that $\frac{|\mathrm{II}|^2}{\mathrm{H}^2} \equiv 1$. Recalling the identity

$$\operatorname{div} X^\top = n - \mathrm{H} X \cdot \mathsf{N},$$

integrating (5.13) yields

$$\begin{aligned}
0 = & -\int \Delta\, \mathrm{H}\, d\mu \\
= & -\int \nabla_{\frac{\lambda}{2} X^\top} \mathrm{H}\, d\mu + \int \left(\mathrm{H}^3 - \tfrac{\lambda}{2}\, \mathrm{H} \right) d\mu \\
= & \tfrac{\lambda}{2} \int \operatorname{div} \left(X^\top \right) \mathrm{H}\, d\mu + \int \left(\mathrm{H}^3 - \tfrac{\lambda}{2}\, \mathrm{H} \right) d\mu \\
= & (n-1) \tfrac{\lambda}{2} \int \mathrm{H}\, d\mu \\
> & \, 0,
\end{aligned}$$

which is absurd.

We conclude that $\nabla\, \mathrm{H} \equiv 0$. The identity (5.13) then implies that $\mathrm{H}^2 \equiv \frac{\lambda}{2}$. We conclude from the Alexandrov theorem (or directly from (5.14a)) that the shrinker is the round sphere of radius $\sqrt{\frac{2}{\lambda}}$. □

Given a hypersurface $X : M^n \to \mathbb{R}^{n+1}$, consider the weighted area functional

$$G(X) \doteqdot \int_{M^n} \mathrm{e}^{-X^* f}\, d\mu_X , \tag{5.15}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is the gradient soliton potential (for some choice of $v \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$). The gradient soliton equation (5.5) (with $V = Df$) is the Euler–Lagrange equation for G.

Proposition 5.9. *If M^n is compact and $\{X_\varepsilon : M^n \to \mathbb{R}\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a smooth variation of $X = X_0$, then*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} G(X_\varepsilon) = -\int_{M^n} (\vec{\mathrm{H}} + X^* Df)\cdot \vec{F}\, \mathrm{e}^{-X^*f}\, d\mu_X,$$

where $\vec{F}$ is the variation field. Thus, X is a stationary point of G if and only if it satisfies (5.5) *with $V = Df$.*

Proof. This as an easy consequence of the first variation formula for the area element. □

Consider now, for some shrinker $X : M^n \to \mathbb{R}^{n+1}$, the associated self-similarly shrinking mean curvature flow $\sqrt{-t}\phi^*_{\log\sqrt{-t}}X$. This mean curvature flow will satisfy

$$\vec{\mathrm{H}}_{X_t} + \tfrac{\lambda(t)}{2} X_t^\perp = 0,$$

where $\lambda(t) \doteqdot \frac{1}{-t}$. So the potential function (after adding a normalizing constant) is given by

$$f(x,t) = \lambda(t)\tfrac{|x|^2}{4} + \tfrac{n+1}{2}\log(4\pi) = \tfrac{|x|^2}{-4t} + \tfrac{n+1}{2}\log(4\pi).$$

Observe that the density function

$$h \doteqdot \lambda^{\frac{n+1}{2}}\mathrm{e}^{-f} = (-4\pi t)^{-\frac{n+1}{2}}\mathrm{e}^{-\frac{|x|^2}{-4t}}$$

is then the fundamental solution to the ambient CONJUGATE HEAT EQUATION[18]

$$(\partial_t - \Delta)^* h = 0$$

on $\mathbb{R}^{n+1}$, where

$$(\partial_t - \Delta)^* \doteqdot -(\partial_t + \Delta)$$

is the CONJUGATE HEAT OPERATOR.

[18] So named because a smooth function u satisfies the heat equation in $\Omega \times (a,b) \subset \mathbb{R}^{n+1}\times\mathbb{R}$ if and only if

$$\int_a^b\int_\Omega u(\partial_t - \Delta)^*\varphi\, d\mathcal{L}\, dt = 0$$

for every smooth function φ which is compactly supported in $\Omega \times (a,b)$.

5.3 *The differential Harnack inequality*

Observe that, by (5.10) and (5.13), a locally uniformly convex, self-similarly expanding mean curvature flow must satisfy

$$\frac{\partial_t \mathrm{H}}{\mathrm{H}} = \frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2} - \frac{1}{2t},$$

while a locally uniformly convex, self-similarly translating mean curvature flow must satisfy

$$\frac{\partial_t \mathrm{H}}{\mathrm{H}} = \frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2}.$$

Theorem 5.10 (Differential Harnack inequality[19])**.** *Along any locally uniformly convex mean curvature flow $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$, M^n compact,*

$$\frac{\partial_t \mathrm{H}}{\mathrm{H}} - \frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2} + \frac{1}{2t} \geq 0. \tag{5.16}$$

Moreover, if (5.16) *holds along a (not necessarily compact) proper, locally uniformly convex mean curvature flow $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$, then it holds with strict inequality, unless $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a self-similarly expanding solution.*

Along any locally uniformly convex, ancient mean curvature flow $X : M^n \times (-\infty,T) \to \mathbb{R}^{n+1}$, M^n compact,

$$\frac{\partial_t \mathrm{H}}{\mathrm{H}} - \frac{|\nabla \mathrm{H}|^2}{\mathrm{H}^2} \geq 0. \tag{5.17}$$

Moreover, if (5.17) *holds along a (not necessarily compact) proper, locally uniformly convex, ancient mean curvature flow $X : M^n \times (-\infty,T) \to \mathbb{R}^{n+1}$, then it holds with strict inequality, unless $X : M^n \times (-\infty,T) \to \mathbb{R}^{n+1}$ is a self-similarly translating solution.*

[19] Andrews, "Harnack inequalities for evolving hypersurfaces"; Richard S. Hamilton, "Harnack estimate for the mean curvature flow"

Sketch of the proof. Consider the functions

$$Q \doteqdot \partial_t \log \mathrm{H} - |\nabla \log \mathrm{H}|^2 \quad \text{and} \quad P \doteqdot 2t(\partial_t \log \mathrm{H} - |\nabla \log \mathrm{H}|^2) + 1.$$

Note that $P \equiv 0$ if and only if $X : M^n \times I \to \mathbb{R}^{n+1}$ is a self-similarly expanding solution and $Q \equiv 0$ if and only if $X : M^n \times I \to \mathbb{R}^{n+1}$ is a self-similarly translating solution.

With respect to the *Gauss map parametrization*, these functions take a simpler form:

$$Q = \partial_t \log \mathrm{H} \quad \text{and} \quad P = 2t\partial_t \log \mathrm{H} + 1.$$

Using the identities

$$\mathrm{II}^{-1} = \overline{\nabla}^2 \sigma + \sigma \mathrm{I}$$

(as a tensor on S^n of type $(1,1)$) and

$$\partial_t \sigma = -\mathrm{H},$$

where $\sigma : S^n \times I \to \mathbb{R}$ is the support function of the solution, and $\overline{\nabla}$ and $\overline{\Delta}$ are the covariant derivative and Laplacian with respect to the round metric, $\overline{g}$, it is not difficult to derive the identities

$$\partial_t Q = \overline{\Delta} Q + 2\overline{g}\big(\overline{\nabla} \log \mathrm{H}, \overline{\nabla} Q\big) + 2Q^2,$$

and

$$\partial_t P = \overline{\Delta} P + 2\overline{g}\big(\overline{\nabla} \log \mathrm{H}, \overline{\nabla} P\big) + 2QP.$$

Since $P|_{t=0} = 1 > 0$, the maximum principle implies that $P \geq 0$ for positive times, which yields the first claim. The strong maximum principle then yields the second. The third and fourth follow by time-translating the estimates for a sequence of times approaching minus infinity. □

Note that, by continuity, smooth (noncompact) limits of mean curvature flows on compact manifolds satisfy the differential Harnack inequality. Hamilton's argument also applies directly in the noncompact case, so long as the solution has bounded curvature on compact time intervals. As such,

Corollary 5.11. *any eternal convex mean curvature flow which attains its (spacetime) curvature maximum evolves by translation.*

Corollary 5.12 ((Integral) Harnack inequality)**.** *Along any locally uniformly convex mean curvature flow $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$, M^n compact,*

$$\sqrt{t_2}\,\mathrm{H}(x_2, t_2) \geq \sqrt{t_1}\,\mathrm{H}(x_1, t_1)\exp\left(\frac{d^2(x_1, x_2, t_1)}{-4(t_2 - t_1)}\right)$$

for any $x_1, x_2 \in M^n$ and any $0 < t_1 < t_2 < T$.

Proof. Integrate the differential Harnack inequality (5.16) along spacetime geodesics (as in Corollary 4.13). □

5.4 The monotonicity formula for Huisken's functional

Given a mean curvature flow $X : M^n \times I \to \mathbb{R}^{n+1}$, M^n compact, the HUISKEN FUNCTIONAL $\mathscr{G}$ is defined by

$$\mathscr{G}_{(p_0,t_0)}(M^n, t) \doteqdot \big(4\pi(t_0 - t)\big)^{-\frac{n}{2}} \int_{M^n} \mathrm{e}^{\frac{|X(x,t)-p_0|^2}{4(t-t_0)}}\, d\mu_t(x)\,. \qquad (5.18)$$

Observe that this is simply $(4\pi(t_0 - t))^{-\frac{n}{2}}$ times functional G from Proposition 5.9 (in the shrinking case), evaluated along the mean curvature flow. Observe that $\mathscr{G}$ is invariant under parabolic rescaling about (p_0, t_0), and is thus constant in time along a self-similarly shrinking mean curvature flow which is centred at (p_0, t_0).

Define the density

$$\begin{aligned}\Psi_{(p_0,t_0)}(x, t) &\doteqdot \big(4\pi(t_0 - t)\big)^{-\frac{n}{2}} \mathrm{e}^{\frac{|X(x,t)-p_0|^2}{4(t-t_0)}} \\ &= \sqrt{4\pi(t_0 - t)}\,\Phi_{(p_0,t_0)}(X(x, t), t)\,,\end{aligned}$$

where

$$\Phi_{(p_0,t_0)}(p, t) \doteqdot \big(4\pi(t_0 - t)\big)^{-\frac{n+1}{2}} \mathrm{e}^{\frac{|p-p_0|^2}{4(t-t_0)}}$$

is the fundamental solution to the ambient conjugate heat equation based at (x_0, t_0).

Proposition 5.13 (Monotonicity formula for Huisken's functional[20]). *Given any* $(p_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}$ *and any mean curvature flow* $X : M^n \times I \to \mathbb{R}^{n+1}$, M^n *compact*[21]

$$\frac{d}{dt}\mathscr{G}_{(p_0,t_0)}(M^n, t) = -\int_{M^n} \left|\vec{\mathrm{H}} + \frac{(X - p_0)^\perp}{2(t_0 - t)}\right|^2 \Psi_{(p_0,t_0)}\, d\mu \qquad (5.19)$$

for all $t \in I \cap (-\infty, t_0)$. *In particular,* $\mathscr{G}_{(p_0,t_0)}(M^n, t)$ *is nonincreasing over* $I \cap (-\infty, t_0)$, *and strictly decreases unless X is self-similarly shrinking about*[22] (p_0, t_0).

Proof. As in the two dimensional case (Proposition 4.14), the claim follows from the pointwise identity

$$(\partial_t + \Delta - \mathrm{H}^2)\Psi_{(p_0,t_0)} = -\left|\vec{\mathrm{H}} + \frac{(X - p_0)^\perp}{2(t_0 - t)}\right|^2 \Psi_{(p_0,t_0)}. \qquad \square$$

5.4.1 *The local regularity theorem*

Given $(p, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$ and $r > 0$ such that $t - r^2 \in I$, the Gaussian area ratio $\Theta_r(p, t)$ of a solution to mean curvature flow $X : M^n \times I \to \mathbb{R}^{n+1}$ is defined by

$$\Theta_r(p, t) \doteqdot \mathscr{G}_{(p,t)}(t - r^2) = \left(4\pi r^2\right)^{-\frac{n}{2}} \int_{M^n} \mathrm{e}^{-\frac{|X-p|^2}{4r^2}}\, d\mu_{t-r^2}. \qquad (5.20)$$

By Huisken's monotonicity formula (Proposition 5.13), $\Theta_r(p, t)$ is nondecreasing in r when M^n is compact.

A simple calculation reveals that $\Theta_r(p, t) \equiv 1$ on a stationary hyperplane solution passing through p. Conversely, the asymptotic Gaussian area ratio

$$\Theta(\infty) \doteqdot \lim_{r\to\infty} \Theta_r(0, 0)$$

of a proper ancient mean curvature flow is always *at least one*, with equality *only* on a stationary hyperplane.[23]

Theorem 5.14 (Local regularity theorem[24]). *Given* $n \in \mathbb{N}$, *there exist constants* $\varepsilon > 0$ *and* $C < \infty$ *with the following property*: *Let* $X : M^n \times I \to \mathbb{R}^{n+1}$ *be a smooth mean curvature flow of embedded hypersurfaces which is properly defined in the spacetime cylinder* $P_r(p_0, t_0) \doteqdot B_r(p_0) \times (t_0 - r^2, t_0]$. *If*

$$\sup_{(p,t)\in P_r(p_0,t_0)} \Theta_r(p, t) < 1 + \varepsilon,$$

where Θ *is the Gaußian area ratio of* X, *then*

$$\sup_{P_{r/2}(p_0,t_0)} |\mathrm{II}| \leq C r^{-1}. \qquad (5.21)$$

[20] Huisken, "Asymptotic behavior for singularities of the mean curvature flow"

[21] By introducing suitable cut-off functions, the compactness hypothesis on M^n can be removed, so long as the evolving immersions are proper, and $\mathscr{G}_{(p_0,t_0)}(M^n, t)$ is finite for all $t \in I$; see Ecker, *Regularity theory for mean curvature flow*, Theorem 4.11. One important situation in which these hypotheses are guaranteed to hold arises when the evolving hypereurfaces are convex; see Bourni, Langford, and Tinaglia, "Convex ancient solutions to mean curvature flow", Claim 2.2.2. A local version of the monotonicity formula was established by Ecker, "A local monotonicity formula for mean curvature flow".

[22] I.e. the spacetime translated solution $\tilde{X}(x, t) \doteqdot X(x, t + t_0) - p_0$ is a self-similarly shrinking solution.

[23] White, "A local regularity theorem for mean curvature flow", Proposition 2.10.

[24] ibid.

Proof. Observe that, by parabolically rescaling and translating in space and time, it suffices to establish the claim when $r = 1$ and $(p_0, t_0) = (0, 0)$. Suppose then that the conclusion fails in this case for all $\varepsilon > 0$ and $C < \infty$. Then for each $j \in \mathbb{N}$ there must exist some smooth mean curvature flow $X_j : M_j^n \times I_j \to \mathbb{R}^{n+1}$ of embedded hypersurfaces which is properly defined in $P_1(0,0)$ and satisfies

$$\sup_{(p,t)\in P_1(0,0)} (\Theta_{X_j})_1(p,t) < 1 + j^{-1},$$

but admits a point $(p_j, t_j) \in P_{1/2}(0,0)$ at which

$$|\mathrm{II}_{X_j}|(p_j, t_j) > j.$$

Following Haslhofer and Kleiner,[25] we seek points $(q_j, s_j) \in P_{3/4}(0,0)$ such that

$$\lambda_j \doteqdot |\mathrm{II}_{X_j}|(q_j, s_j) > j \;\text{ and }\; \sup_{P_{j/10\lambda_j}(q_j,s_j)} |\mathrm{II}_{X_j}| \le 2\lambda_j. \tag{5.22}$$

Now, if $(q_j^0, s_j^0) \doteqdot (p_j, t_j)$ already satisfies (5.22), then it is the point we seek. Otherwise, there is a point $(q_j^1, s_j^1) \in P_{j/10\lambda_j^0}(q_j^0, s_j^0)$ such that $\lambda_j^1 \doteqdot |\mathrm{II}_{X_j}|(q_j^1, s_j^1) > 2\lambda_j$. If (q_j^1, s_j^1) satisfies (5.22), then it is the point we seek. Otherwise, there is a point $(q_j^2, s_j^2) \in P_{j/10\lambda_j^1}(q_j^1, s_j^1)$ such that $\lambda_j^2 \doteqdot |\mathrm{II}_{X_j}|(q_j^1, s_j^1) > 2\lambda_j^1$, etc. Note that

$$\tfrac{1}{2} + \tfrac{j}{10\lambda_j^0}\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\right) < \tfrac{3}{4}.$$

Thus, since $|\mathrm{II}_{X_j}|$ is finite in $\overline{P_1(0,0)}$, the iteration must terminate after a finite number of steps and the final point of the iteration will lie in $P_{3/4}(0,0)$ (and satisfy (5.22)).

With our new sequence of points (q_j, s_j) in hand, consider the flows $\widehat{X}_j : M_j^n \times \widehat{I}_j \to \mathbb{R}^{n+1}$ obtained by shifting q_j to the origin and parabolically rescaling by λ_j. This new sequence satisfies $|\mathrm{II}_{\widehat{X}_j}|(0,0) = 1$ and $\sup_{P_{j/10}(0,0)} |\mathrm{II}_{\widehat{X}_j}| \le 2$. Thus, by Theorem 2.22, we can pass smoothly to a proper ancient limit flow satisfying

$$|\mathrm{II}|(0,0) = 1 \;\text{ and }\; \Theta_1(0,0) = 1.$$

But the latter property implies that the limit is a stationary hyperplane, contradicting the former. □

[25] Haslhofer and Kleiner, "Mean curvature flow of mean convex hypersurfaces".

5.5 *Noncollapsing*

Roughly speaking, a sequence of embedded hypersurfaces $M_j = \partial\Omega_j$, $\Omega_j \underset{\text{open}}{\subset} \mathbb{R}^{n+1}$, is said to COLLAPSE if, modulo translation and scaling, their interior regions Ω_j degenerate as $j \to \infty$, with their curvature remaining bounded. One precise way to quantify this is to ask for a sequence of points $x_j \in \Gamma_j$ such that

$$\bar{r}_j(x_j) \sup_{B^{n+1}_{j\bar{r}_j(x_j)}(x_j)} |\mathrm{II}_j| \le j^{-1}, \tag{5.23}$$

where $\bar{r}(x)$ denotes the INSCRIBED RADIUS of $\partial\Omega$ at $x \in \partial\Omega$—the radius of the largest ball contained in Ω whose boundary passes through the boundary point x.

Note that $\bar{r}\kappa$ is scale invariant. Thus, if (5.23) holds, then, at the scale of the *curvature*, the inscribed radius degenerates to zero. Since $\max_{|v|=0} \mathrm{II}(v,v) \le \bar{r}^{-1}$, with strict inequality only if the boundary of the ball $B^{n+1}_{\bar{r}}(x - \bar{r}\,\mathsf{N}(x))$ meets $\partial\Omega$ at some other point $y \in \partial\Omega \setminus \{x\}$, this means that two (intrinsically distant) portions of the boundaries are coming together. On the other hand, at the scale of the *inscribed radius*, the curvature is tending towards zero in arbitrarily large regions, and at this scale the regions converge to a slab of width two.

Andrews proved that the inscribed radius is pointwise nondecreasing, relative to the scale of the mean curvature, under embedded, mean convex mean curvature flow.

Proposition 5.15 (Interior noncollapsing[26]). *Along any embedded[27], mean convex mean curvature flow $\{M_t = \partial\Omega_t\}_{t\in[0,T)}$, $\Omega_t \subset \mathbb{R}^{n+1}$ bounded, the* INSCRIBED CURVATURE $\bar{k} \doteqdot \bar{r}^{-1}$ *satisfies*

$$(\partial_t - \Delta)\bar{k} \le |\mathrm{II}|^2 \bar{k}$$

in the VISCOSITY SENSE[28]. *In particular,*

$$\bar{k} \le K\,\mathrm{H}, \ \ \textit{where} \ \ K \doteqdot \max_{M_0} \frac{\bar{k}}{\mathrm{H}}.$$

Equivalently,

$$\bar{r} \ge \delta \mathrm{H}^{-1}, \ \ \textit{where} \ \ \delta \doteqdot \min_{M_0} \bar{r}\,\mathrm{H}.$$

Sketch of the proof. Using the inequality (5.4) to treat the diagonal case, the argument proceeds much as in the one-dimensional case (Proposition 4.15). □

In fact, reversing the orientation of the hypersurfaces yields a corresponding *exterior* noncollapsing estimate: if we define the EXSCRIBED CURVATURE $\underline{k}$ at $x \in \partial\Omega$ to be the radius of the largest GENERALIZED BALL (oriented region with constant extrinsic curvature[29]) which encloses Ω and touches $\partial\Omega$ at x, then we obtain the following.

[26] Andrews, "Noncollapsing in mean-convex mean curvature flow"

[27] In fact, it suffices here, and in the followin g theorem, for the evolving hypersurfaces to be *Alexandrov immersed*; see Lambert and Mäder-Baumdicker, "A note on Alexandrov immersed mean curvature flow".

[28] This is a weak formulation of the differential inequality $(\partial_t - \Delta)u \le |\mathrm{II}|^2 u$ which applies to any continuous function. It asserts that, at any point $(x_0, t_0) \in M^n \times (0, T)$, any smooth function $\varphi : M^n \times [0, T) \to \mathbb{R}$ which touches $\bar{k}$ from above at (x_0, t_0), in the sense that $\varphi \ge \bar{k}$ on a backward spacetime neighbourhood $U \times (t_0 - \delta, t_0]$ of (x_0, t_0) with equality at (x_0, t_0), satisfies

$$(\partial_t - \Delta)\varphi \le |\mathrm{II}|^2 \varphi \ \text{ at } \ (x_0, t_0).$$

[29] I.e. a ball (constant positive curvature), a halfspace (constant zero curvature) or a ball-compliment (constant negative curvature).

[30] Andrews, "Noncollapsing in mean-convex mean curvature flow"

Proposition 5.16 (Exterior noncollapsing[30]). *Along any embedded, mean convex mean curvature flow, $\{M_t = \partial\Omega_t\}_{t\in[0,T)}$, $\Omega_t \subset \mathbb{R}^{n+1}$ bounded, the exscribed curvature satisfies*

$$(\partial_t - \Delta)\underline{k} \geq |\mathrm{II}|^2\underline{k}$$

in the viscosity sense. In particular,

$$\underline{k} \geq K\,\mathrm{H}, \quad \text{where} \quad K \doteq \min_{M_0} \frac{\underline{k}}{\mathrm{H}}.$$

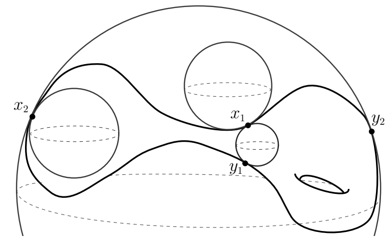


Figure 5.3: The inscribed curvature at x_1 is positive and given by $\overline{k}(x_1) = k(x_1, y_1)$. The exscribed curvature at x_1 is negative and given by $\underline{k}(x_1) = \kappa_1(x_1)$. The inscribed curvature at x_2 is positive and given by $\overline{k}(x_2) = \kappa_n(x_2)$. The exscribed curvature at x_2 is positive and given by $\underline{k}(x_2) = k(x_2, y_2)$.

Thus,

- *if $K > 0$, then the* CIRCUMSCRIBED RADIUS $\underline{r}$ *(at each $x \in \partial\Omega_t$, the radius of the smallest ball which encloses Ω_t and touches $\partial\Omega_t$ at x) satisfies*

$$\underline{r} \leq D\,\mathrm{H}, \quad \text{where} \quad D \doteq K^{-1};$$

- *if $K \leq 0$, then the* EXSCRIBED RADIUS *(at each $x \in \partial\Omega_t$, the radius of the largest ball which lies in $\mathbb{R}^n \setminus \Omega_t$ and touches $\partial\Omega_t$ at x) satisfies*

$$\underline{r} \geq \delta\,\mathrm{H}, \quad \text{where} \quad \delta \doteq -K^{-1}.$$

5.5.1 An estimate for the curvature

Exploiting the interior noncollapsing estimate (Proposition 5.15) in conjunction with the improvement-of-convexity estimate (Proposition 5.3), we shall establish that a compact, mean convex, embedded mean curvature flow is uniformly starshaped about any given point, at the scale of the curvature at that point. This yields the following local curvature estimate (cf. §2.5).

Proposition 5.17 (An estimate for the curvature[31]). *Given any $\Lambda > 0$, and any compact, mean convex, embedded mean curvature flow $\{\partial\Omega_t\}_{t\in[0,T)}$, there exist $K = K(n, \Lambda, \Omega_0) < \infty$ and $C = C(n, \Lambda, \Omega_0) < \infty$ such that*

$$r^{-1} \doteq \mathrm{H}(x,t) \geq K \quad \Longrightarrow \quad \sup_{B_{\Lambda r}(x)\times(t-\Lambda^2 r^2, t]} \mathrm{H} \leq C r^{-1}.$$

[31] We present the argument of Lynch, "Convexity and gradient estimates for fully nonlinear curvature flows", §4-5, which extends ideas of Brendle and Huisken, "A fully nonlinear flow for two-convex hypersurfaces in Riemannian manifolds". There is also a quite different (earlier) argument, due to Haslhofer and Kleiner, "Mean curvature flow of mean convex hypersurfaces", which makes use of *two-sided* noncollapsing and the local regularity theorem.

Sketch of the proof. By the interior noncollapsing estimate (Proposition 5.15), we can find some $\delta = \delta(\Omega_0) > 0$ such that, for any $t \in [0, T)$ and any $x \in \partial\Omega_t$, the ball of radius $\delta\,\mathrm{H}^{-1}(x,t)$ centred at the point $x - \mathrm{H}^{-1}(x,t)\,\mathsf{N}(x,t)$ is enclosed by $\partial\Omega_t$ (and hence also by $\partial\Omega_s$ for all $s < t$, since the flow is monotone). By Proposition 2.24, it therefore suffices to show that, whenever $r^{-1} = \mathrm{H}(x,t)$ is sufficiently large, the trumpet $T_{\frac{1}{2}}(\hat{x}, p, \delta r)$ about the point $\hat{x} \doteq x - r\,\mathsf{N}(x,t)$ is contained in $U_{10\Lambda r}(x,s)$, the connected component of $\Omega_s \cap B_{10\Lambda r}(x)$ which contains $B_{\delta r}(\hat{x})$, for all $p \in U_{1-\Lambda r}(x,s)$ and all $s \in (t - 100\Lambda^2 r^2, t] \Subset [0, T)$ (cf.

Proposition 2.26). Let us denote by $Q(\delta, 10\Lambda)$ the set of pairs (x,t) which *do* satisfy this property.

Suppose then that, contrary to the claim, there is a sequence of spacetime points (x_j, t_j) with $r_j^{-1} \doteqdot \mathrm{H}(x_j,t_j) \to \infty$ such that $(x_j,t_j) \notin Q(\delta, \Lambda)$. By a "point-picking" argument (cf. (5.22) above), we can choose our sequence so that if $t \le t_j$ and $\mathrm{H}(x,t) \ge 4\,\mathrm{H}(x_j,t_j)$, then $(x,t) \in Q(\delta,\Lambda)$. Now, by assumption, we can find some time $s_j \in (t_j - 100\Lambda^2 r^2, t_j]$ and some point $p_j \in U_{10\Lambda r}(x_j, s_j)$ such that the trumpet $T_{\frac{1}{2}}(\hat{x}_j, p_j, \delta r_j)$ intersects $\partial\Omega_{s_j}$ somewhere in the connected component of $\overline{\Omega}_{s_j} \cap B_{10\Lambda r}(x)$ which contains $B_{\delta r}(\hat{x})$. But then, by "moving" the mouthpiece of this trumpet, we can find another trumpet $T_{\frac{1}{2}}(\hat{x}_j, \tilde{p}_j, \delta r_j)$ which does lie in the connected component of $\overline{\Omega}_{s_j} \cap B_{10\Lambda r}(x)$ which contains $B_{\delta r}(\hat{x})$ but makes contact with $\partial\Omega_{s_j}$ at some point, y_j say. At this point, we find that

$$\kappa_1(y_j, s_j) \le -\frac{\delta}{\delta + 100\Lambda^2}\frac{2}{\sqrt{4+\delta^2}} r_j^{-1} \doteqdot -\gamma r_j^{-1}.$$

On the other hand, by mean convexity and preservation of pinching, there is some $C = C(\Omega_0) < \infty$ such that

$$\kappa_1(y_j, s_j) \ge -C\,\mathrm{H}(y_j, s_j),$$

and hence

$$\mathrm{H}(y_j, s_j) \ge \gamma C^{-1} r_j^{-1}.$$

Since $r_j^{-1} = \mathrm{H}(x_j, t_j)$ tends to infinity, this ensures that $\mathrm{H}(y_j, s_j)$ tends to infinity as well, and the improvement-of-convexity estimate then ensures that

$$\frac{\mathrm{H}(x_j,t_j)}{\mathrm{H}(y_j,s_j)} \le -\gamma^{-1}\frac{\kappa_1(y_j,s_j)}{\mathrm{H}(y_j,s_j)} \to 0 \ \text{ as } \ j \to \infty.$$

I.e. $\mathrm{H}(y_j, s_j)$ tends to infinity *faster* than $\mathrm{H}(x_j, t_j)$. In particular, we eventually have $\mathrm{H}(y_j, s_j) \ge 4\mathrm{H}(x_j, t_j)$, and hence, after passing to a subsequence, $(y_j, s_j) \in Q(\delta, \Lambda)$ for all j. Thus (by Proposition 2.24 and the Bernstein estimates), if we rescale by $\delta\mathrm{H}^{-1}(y_j, s_j)$ about the spacetime points (y_j, s_j) and rotate so that the normal at the spacetime origin is e_{n+1}, we can (after passing to a subsequence) take a smooth limit in the spacetime cylinder $B_{\delta^{-1}\Lambda}(-e_1) \times (-\delta^{-2}\Lambda^2, 0]$.

This limit flow will have nonnegative principal curvatures (by the improvement-of-convexity estimate) and positive mean curvature (by the strong maximum principle, since it is nonnegative by construction and positive at the spacetime origin). Moreover, since its smallest principal curvature will vanish at the spacetime origin, it must split off a line (in accordance with Proposition 3.2).

Since the mean curvature at (y_j, s_j) dominates the mean curvature at (x_j, t_j), it can be shown that the rescaled trumpets which touch the rescaled solutions from the inside at the spacetime origin converge (after passing to a further subsequence) to a limit hypersurface, Σ, which is either a round cone of positive aperture (in case the distance from the origin to the vertex of the rescaled trumpet remains bounded) or a hyperplane (in case the distance from the origin to the vertex of the rescaled trumpet does not remain bounded), and touches the limit flow from the inside at the spacetime origin. Note that the ray of the cone through the origin must be a splitting direction for the limit flow.[32]

So the (locally convex, strictly mean convex) limit solution both splits off a line and touches a cone/hyperplane Σ from the outside at the spacetime origin. This is clearly impossible in case Σ is a hyperplane, but is not immediately a contradiction in case Σ is a cone (since a small piece of a cylinder can touch a cone from the outside); but the convergence may actually be extended along the whole ray. Indeed, since the cross sections in the limit lie outside of corresponding sections of the cone, Proposition 2.24 can be exploited to estimate the curvature in a neighbourhood of the ray (in terms of the inradius of the corresponding conic section). This results in the anticipated contradiction. □

[32] In case the vertex of the limit cone is the origin, the tangent hyperplane to the limit solution at the spacetime origin still contains one of the rays of the limit cone.

5.6 Exercises

Exercise 5.1. Let $X : M^n \to \mathbb{R}^{n+1}$, $n \geq 2$, be a compact, mean convex, self-similarly shrinking mean curvature flow.

(a) Show, using improvement of convexity (Proposition 5.3), that $X : M^n \to \mathbb{R}^{n+1}$ satisfies $\kappa_1 \geq 0$.

(b) Deduce, using the splitting theorem (Proposition 3.2), that $X : M^n \to \mathbb{R}^{n+1}$ satisfies $\kappa_1 > 0$.

(c) Conclude, using improvement of roundness (Proposition 3.6), that $X : M^n \to \mathbb{R}^{n+1}$ is umbilic, and hence the shrinking sphere.

Exercise 5.2. Prove that a curve in the halfplane $\Sigma \doteqdot \{(x,r) : r > 0\}$ generates an axially symmetric shrinker in $\mathbb{R}^{n+1}$ (via rotation about the x-axis) if and only if it is a critical point of the length functional in the metric

$$\sigma \doteqdot r^{2(n-1)} \mathrm{e}^{-\frac{x^2+r^2}{4}} (dx^2 + dr^2) .$$

Exercise 5.3. Let $\{\partial\Omega_t\}_{t\in[\alpha,\omega]}$ be a compact, convex and locally uniformly convex mean curvature flow.

(a) (i) Show (using (5.16)) that

$$\partial_t \left(\sqrt{2(t-\alpha)}\, \mathrm{H} \right) \geq 0 \tag{5.24}$$

with respect to the Gauss map parametrization.

(ii) Deduce, or show using (5.17), that

$$\partial_t \mathrm{H} \geq 0 \tag{5.25}$$

(with respect to the Gauss map parametrization) if $\{\partial\Omega_t\}_{t\in[\alpha,\omega]}$ extends to an ancient solution.

(b) Denote the support function of Ω_t by $\sigma(\cdot,t)$.

(i) Show, using (5.24), that[33]

$$\mathrm{H}(\cdot,t) \leq \frac{\sigma(\cdot,t)-\sigma(\cdot,\omega)}{2(\omega-t)}\left(1+\sqrt{\frac{\omega-\alpha}{t-\alpha}}\right).$$

(ii) Deduce, or show using (5.25), that

$$\mathrm{H}(\cdot,t) \leq \frac{\sigma(\cdot,t)-\sigma(\cdot,\omega)}{\omega-t}$$

if $\{\partial\Omega_t\}_{t\in[\alpha,\omega]}$ extends to an ancient solution.

(c) Conclude that

$$\max_{\left\{z\in S^2:\sigma(z,t)-\sigma(z,\omega)\leq C\sqrt{\omega-t}\right\}} \mathrm{H}(\cdot,t) \leq \frac{2C}{\sqrt{\omega-t}}, \tag{5.26}$$

so long as $t \geq \omega - \frac{\omega-\alpha}{2}$, say.

[33] Compare this with Tso's estimate (Proposition 2.23).

6
Towards a classification of ancient solutions

Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a maximal mean curvature flow on a compact manifold M^n. By Theorem 2.19, we know that $T < \infty$ and $\limsup_{t \nearrow T} \max_{M^n \times \{t\}} |\mathrm{II}| \to \infty$. If we choose (x_j, t_j) so that $\lambda_j \doteqdot |\mathrm{II}_{(x_j,t_j)}| = \max_{M^n \times [0, T-j^{-1}]} |\mathrm{II}|$, then the mean curvature flows $X_j : M^n \times I_j \to \mathbb{R}^{n+1}$ defined by

$$X_j(x,t) \doteqdot \lambda_j \left(X(x, \lambda_j^{-2} t + t_j) - X(x_j, t_j) \right), \quad I_j \doteqdot [-\lambda_j^2 t_j, 0]$$

will satisfy $|\mathrm{II}| \leq 1$ and $|\mathrm{II}_{(x_j,t_j)}| = 1$. By the compactness theorem (Theorem 2.22), we can then find a complete *ancient* (subsequential) limit flow $X_\infty : M^n_\infty \times (-\infty, 0] \to \mathbb{R}^{n+1}$, on which

1. $|\mathrm{II}| \leq K < \infty$.

   If the original flow is mean convex, then we will also have

2. $\mathrm{II} \geq 0$ and $\mathrm{H} > 0$

due to Proposition 5.3 (and the fact that $|\mathrm{II}| = 1$ at the spacetime origin). But then the differential Harnack inequality[1]

2. $\partial_t \mathrm{H} - \frac{|\nabla \mathrm{H}|^2}{\mathrm{H}} \geq 0$

will also hold.

If the original flow is is mean convex *and embedded*, then (due to Theorem 5.15) we can also arrange that the evolving regions bounded by the flows converge to a limit satisfying

4. $\bar{r} \geq \delta\, \mathrm{H}^{-1}$, $\delta > 0$.

A good understanding of such solutions will thus provide a good understanding of singularity formation in mean convex (embedded) mean curvature flow. Confidence that this is genuine progress towards an understanding of singularity formation can be taken from the following classical theorem of Hirschman.[2]

Theorem 6.1 (Appell's theorem[3])**.** *Any positive ancient solution u to the heat equation on $\mathbb{R}^n$ satisfying $u(x,0) = \mathrm{e}^{o(|x|)}$ must be constant.*

[1] Hamilton showed that the argument sketched in Theorem 5.10 may still be applied when M^n is noncompact, so long as the flow has bounded curvature on compact time intervals. See Richard S. Hamilton, "Harnack estimate for the mean curvature flow".

[2] This is the caloric counterpart of Liouville's theorem for harmonic functions. As for Liouville's theorem, the hypotheses are necessary—consider the solutions e^{x_1+t} and $|x|^2 + 2nt$, for example. Note that Widder's theorem guarantees that a positive solution to the heat equation on $\mathbb{R}^n \times [\alpha, \omega)$ can be extended uniquely (amongst positive solutions) to $\mathbb{R}^n \times [\alpha, \infty)$. See Widder, "Positive solutions of the heat equation".

[3] Appell, "Sur l'équation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$ et la Théorie de la chaleur"; Hirschman, "A note on the heat equation"

We will present an overview of the landscape and structure of convex ancient solutions to mean curvature flow. The proofs of many of these results are highly technical. In such cases, we either only sketch the arguments, or omit the proof entirely.

6.1 *Ancient solutions in one space dimension*

So far, the only ancient mean curvature flows we have seen in one dimension are solitons. Namely, the static/shrinking line, the shrinking sphere, and the Grim Reaper. There is a further (non-soliton) example[4].

[4] Discovered by Sigurd B. Angenent, "Shrinking doughnuts", and rediscovered by Lukyanov, Vitchev, and A. B. Zamolodchikov, "Integrable model of boundary interaction: the paperclip" and by Nakayama, Iizuka, and Wadati, "Curve lengthening equation and its solutions".

Example 9 (The ancient paperclip)**.** The family $\gamma : \mathbb{R}/2\pi\mathbb{Z} \times (-\infty, 0) \to \mathbb{R}^2$ defined by

$$\gamma(\theta, t) \doteqdot \left(\int_0^\theta \frac{\cos\omega}{\kappa(\omega, t)} d\omega, \int_{\frac{\pi}{2}}^\theta \frac{\sin\omega}{\kappa(\omega, t)} d\omega \right),$$

where

$$\kappa(\theta, t) \doteqdot \sqrt{\cos^2\theta + \frac{1}{e^{-2t} - 1}}, \tag{6.1}$$

is parametrized by turning angle and evolves by curve shortening flow. Indeed, since γ is parametrized by turning angle by construction (so that its curvature is given by κ), we only need to check that

$$\kappa_t = \kappa^2(\kappa_{\theta\theta} + \kappa),$$

which is a straightforward exercise. Since the curvature function is always positive and the turning number is always one, the curves $\Gamma_t \doteqdot \gamma(\mathbb{R}/2\pi\mathbb{Z}, t)$ bound bounded convex regions Ω_t. Note also that κ is increasing in t for fixed θ (as it must, by the differential Harnack inequality in the form (5.25)).

Setting $a^2(t) \doteqdot \frac{1}{e^{-2t}-1}$, observe that

$$x(\theta, t) \doteqdot \int_0^\theta \frac{\cos\omega}{\kappa(\omega, t)} d\omega = \arctan\left(\frac{\sin\theta}{\sqrt{\cos^2\theta + a^2(t)}} \right) \tag{6.2a}$$

$$y(\theta, t) \doteqdot \int_{\frac{\pi}{2}}^\theta \frac{\sin\omega}{\kappa(\omega, t)} d\omega = -t + \log\left(\frac{\sqrt{\cos^2\theta + a^2(t)} - \cos\theta}{\sqrt{1 + a^2(t)}} \right). \tag{6.2b}$$

In particular,

$$\cos x = e^t \cosh y. \tag{6.3}$$

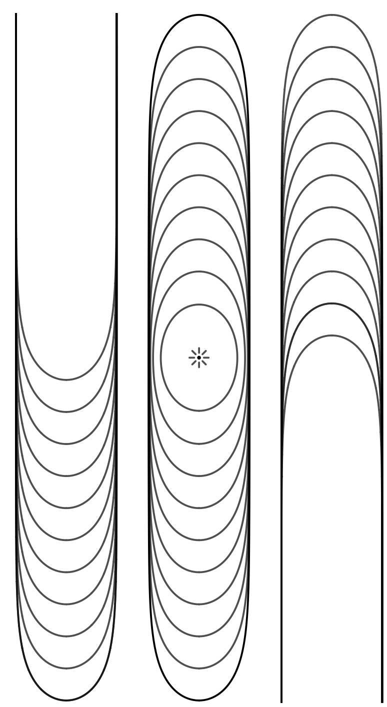


Figure 6.1: The ancient paperclip alongside its asymptotic Grim Reapers.

We may read off a number of properties of the solution from these formulae.

- First, Γ_t is symmetric under reflection across both the x- and y-axes; it has exactly four vertices—the points of intersection with the axes (corresponding to $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$).

– Second, as $t \to -\infty$, the enclosed regions approach Π, the vertical strip of width two, in the sense that Ω_t are decreasing in t and $\cup_{t<0}\Omega_t = \Pi$. In fact, the curves $\partial\Omega_t$ approach the boundary of $\partial\Pi$ locally uniformly in the smooth topology as $t \to -\infty$. (This can be checked directly using (6.1) and its derivatives, but we only need to check that $\kappa \to 0$ in any compact subset of the plane, due to the Bernstein estimates and interpolation.)

– Third, the rescaled curves $(-t)^{-\frac{1}{2}}\Gamma_t$ converge to the unit circle (locally uniformly in the smooth topology) as $t \to 0$.

– Fourth, the spacetime translated flows $\{\Gamma_{t+s} - y(\frac{\pi}{2} \pm \frac{\pi}{2}, s)\}_{t\in(-\infty,-s)}$ converge as $s \to -\infty$ (locally uniformly in the smooth topology) to the Grim Reaper $\{G^{\pm}_t \doteqdot \operatorname{graph} y_{\pm}(\cdot, t)\}_{t\in(-\infty,\infty)}$, where $y_{\pm}(x,t) \doteqdot \pm(\log\cos x - t + \log 2)$.

One plausible way to "derive" the ancient paperclip solution is as follows: any convex curve shortening flow $\{\partial\Omega_t\}_{t\in I}$ may be exhibited as a family of level sets of a function $u : \Omega \to \mathbb{R}$ via

$$\Omega_t = \{(x,y) \in \mathbb{R}^2 : u(x,y) \geq t\}.$$

By Exercise 1.2, the function u must satisfy the level set flow equation (1.12), which in two dimensions may be rewritten as

$$u_{xx}u_y^2 + u_{yy}u_x^2 + u_x^2 + u_y^2 = 2u_{xy}u_xu_y. \tag{6.4}$$

Observe that, under the Ansatz

$$\nabla u(x,y) = (F(x), G(y)), \tag{6.5}$$

the equation (6.4) separates: the terms on the right hand side vanish, and rearranging yields

$$\frac{F_x + 1}{F^2} + \frac{G_y + 1}{G^2} = 0.$$

So, up to interchanging x and y, there must be some constant $\lambda \geq 0$ such that

$$\frac{G_y + 1}{G^2} = \lambda^2 = -\frac{F_x + 1}{F^2}.$$

These equations admit the solution

$$F(x) = -\lambda^{-1}\tan(\lambda(x - x_0)) \quad \text{and} \quad G(y) = -\lambda^{-1}\tanh(\lambda(y - y_0)).$$

Integrating then yields

$$u(x,y) - u_0 = \lambda^{-2}\log\left(\frac{\cos(\lambda(x - x_0))}{\cosh(\lambda(y - y_0))}\right).$$

Fixing $\lambda = 1$, $(x_0, y_0) = (0,0)$, and $u_0 = 0$ yields the paperclip solution (in level set form). The parameter u_0 corresponds to time-translations, while (x_0, y_0) corresponds to spatial translations and λ to parabolic dilations. ■

The ancient paperclip example completes the list of convex ancient curve shortening flows!

Theorem 6.2 (Classification of convex ancient curve shortening flows[5])**.** *Every maximal, convex ancient curve shortening flow* $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ *is either*

- *a shrinking round circle,*
- *a static halfplane or strip,*
- *a Grim Reaper, or*
- *an ancient paperclip.*

[5] Bourni, Langford, and Tinaglia, "Convex ancient solutions to curve shortening flow"; Daskalopoulos, R. Hamilton, and Sesum, "Classification of compact ancient solutions to the curve shortening flow"

Sketch of the proof. By the strong maximum principle, our convex ancient curve shortening flow $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ has positive curvature everywhere unless it is flat, and hence the boundary of a static half-plane or a strip. We may therefore assume that the curvature is everywhere positive.

Next, we claim that $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ satisfies the differential Harnack inequality (4.28). We only proved this (in Proposition 4.12) in case M^1 is compact, but the argument can also be applied in the noncompact case (in all dimensions) if the solution has bounded curvature on compact time intervals.[6,7] To prove this, fix any $t_0 < \omega$ and consider any sequence of points $x_j \in \partial\Omega_{t_0}$ such that $|x_j| \to \infty$ as $j \to \infty$. Observe that, by convexity of Ω_{t_0}, the sequence of translates $\Omega_{t_0} - x_j$ subconverges in the Hausdorff sense to a limit convex set which contains a line, and hence splits off a line. This limit can only be a halfplane or a strip. If it is a strip, then, by a straightforward application of convexity, the strip must have half-width at least equal to the inradius r of Ω_{t_0}. But then Proposition 2.25 can be applied to bound the curvature of $\partial\Omega_t$ by $\sim r^{-1}$ on a backwards time interval from t_0 of length $\sim r^2$. (In fact, the Bernstein estimates and the theorem of turning tangents now imply that the curvature tends to *zero* at infinity.)

[6] Richard S. Hamilton, "Harnack estimate for the mean curvature flow".

[7] In fact, a more sophisticated argument removes even this hypothesis—the idea is to approximate any convex mean curvature flow by compact mean curvature flows; see Bourni, Langford, and Lynch, "Collapsing and noncollapsing in convex ancient mean curvature flow"; Daskalopoulos and Saez, "Uniqueness of entire graphs evolving by mean curvature flow"; X.-J. Wang, "Convex solutions to the mean curvature flow".

By (5.26), the curvature grows at most like $\frac{1}{\sqrt{-t}}$ in origin centred balls of radius $\sim \sqrt{-t}$ as $t \to -\infty$. It follows that the rescaled flows $\lambda\partial\Omega_{\lambda^{-2}t}$ have uniformly bounded curvature in any compact subset of $\mathbb{R}^{n+1} \times (-\infty, 0)$ for $\lambda > 0$ sufficiently small. If the enclosed regions do not degenerate, then the compactness theorem (Theorem 2.22) ensures that we can find a sequence $\lambda_j \to 0$ and a convex ancient curve shortening flow $\{\partial\Omega^\infty_t\}_{t\in(-\infty,0)}$ (called a BLOW-DOWN of $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$) such that $\{\lambda_j\partial\Omega_{\lambda_j^{-2}t}\}_{t\in(-\infty,\lambda_j^2\omega)\cap(-\infty,0)}$ tends to $\{\partial\Omega^\infty_t\}_{t\in(-\infty,0)}$ locally uniformly in the smooth topology as $j \to \infty$.

Now, it can be shown that the Gaussian area functional is bounded by a dimensional constant on the space of convex subsets of $\mathbb{R}^{n+1}$, and this guarantees that Huisken's functional $\mathscr{G}$ (being monotone by

Proposition 5.13) takes a limit along $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ as $t \to -\infty$. By invariance under parabolic rescaling, we can then establish that $\mathscr{G}$ is constant on the limit flow (cf. the proofs of Theorems 4.19 and 4.21). The rigidity case of the monotonicity formula then guarantees that the limit is a self-similarly shrinking solution. If it has compact timeslices, then it must be the shrinking circle by Theorem 4.8. In that case, the original flow must also have compact timeslices, so that $\omega < \infty$ and the rescaled flows $\{\lambda\partial\Omega_{\lambda^{-2}t+\omega}\}_{t\in(-\infty,\omega)}$ converge to the shrinking circle as $\lambda \to \infty$, in accordance with the Gage–Hamilton theorem (Theorem 4.19). Since $\mathscr{G}$ is also invariant under time translation, we may now conclude that $\mathscr{G}$ is constant on the ancient solution $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$, which must then be the shrinking circle. If, instead, the blow-down is noncompact, then it must be a stationary line (since the support function, equal to twice the curvature by the shrinker equation, tends to zero at infinity). We now proceed as in the compact case: since we can blow-up at any interior time to obtain a line, we again find that $\mathscr{G}$ is constant, and thereby conclude from the rigidity case of the monotonicity formula that $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ is a stationary line.

We still need to consider the situation in which the interiors of the rescaled flows $\{\lambda\partial\Omega_{\lambda^{-2}t}\}_{t\in(-\infty,\lambda^2\omega)}$ degenerate as $\lambda \to 0$. In that case, a technical argument of X.-J. Wang (which we shall sketch in §6.2 below) shows that $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ must actually (remarkably!) be confined to a static strip region. Up to a rotation and a parabolic rescaling, we may arrange that the smallest strip enclosing $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ is $\Pi \doteqdot \{(x,y) \in \mathbb{R}^2 : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$.

Let $t \mapsto p(t) \in \Omega_t$ be a continuous choice of "tip" for $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$; i.e. a point whose normal is either e_2 or $-e_2$. Since the differential Harnack inequality (in the form of (5.25)) ensures that the curvature is monotone increasing in time in the turning angle parametrization, the compactness theorem ensures that the spacetime translated flows $\{\partial\Omega_{t+s} - p(s)\}_{s\in(-\infty,\omega-s)}$ admit a limit $\{\Gamma_t\}_{t\in(-\infty,\infty)}$ along some sequence of times $s_j \to -\infty$. It is not hard to show that this limit must evolve by translation and therefore be either a Grim Reaper or a stationary line. In fact, since its normal is vertical at the spacetime origin and it is contained in a strip of width π, $\{\Gamma_t\}_{t\in(-\infty,\infty)}$ must be a vertically translating Grim Reaper of scale at most one.

We claim that the scale of any asymptotic Grim Reaper is actually equal to one. Indeed, if this were not the case, then the scale would be uniformly *less* than one, $r < 1$, say. But then, due to the differential Harnack inequality, the curvature at the tip would be uniformly *more* than one—at least r^{-1}. Integrating the differential Harnack inequality, this implies that $|p(t) - p(0)| \gtrsim r^{-1}t$, which guarantees that the area A enclosed by Ω_t and the x-axis grows like $A \gtrsim -\frac{\pi}{2}(1 + r^{-1})t$. But this violates the first variation of area (which imposes $A \sim -\pi t$).

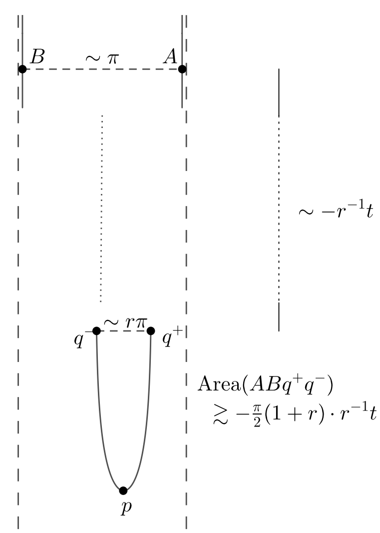


Figure 6.2: If the scale of the asymptotic Grim Reaper at the lower tip is r, then the area enclosed by the time t slice below the x-axis is $\gtrsim -\frac{\pi}{2}(1 + r^{-1})t$.

This shows that our solution $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ is very close to either a Grim Reaper or to the paperclip solution when $t \sim -\infty$ (recall that the latter is very close to two oppositely oriented Grim Reapers when $t \sim -\infty$). These "unique asymptotics" can be bootstrapped into uniqueness using the maximum principle, in the form of Alexandrov's method of moving planes. □

6.2 *The slab dichotomy and its consequences*

Let $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ be a maximal convex ancient mean curvature flow in $\mathbb{R}^{n+1}$ and consider the limit region $\Omega_{-\infty} \doteqdot \cup_{t\in(-\infty,\omega)}\Omega_t$. Since the regions $\{\Omega_t\}_{t\in(-\infty,\omega)}$ are monotone decreasing with respect to inclusion, $\Omega_t \to \Omega_{-\infty}$ locally uniformly in the Hausdorff sense. It follows that the mean curvature flows $\{\partial\Omega^s_t\}_{t\in(-\infty,-s)}$ defined by $\Omega^s_t \doteqdot \Omega_{s+t}$ converge locally uniformly in the Hausdorff sense to the static flow, $\{\partial\Omega_{-\infty}\}_{t\in(-\infty,\infty)}$. But the differential Harnack inequality and the Bernstein estimates ensure that the convergence is actually smooth, which implies that $\{\partial\Omega_{-\infty}\}_{t\in(-\infty,\infty)}$ is a smooth mean curvature flow. Since it is static, we conclude that $\partial\Omega_{-\infty}$ is either flat or empty, and hence $\Omega_{-\infty}$ is either the whole space, a half-space, or a slab (the region between two distinct parallel hyperplanes).

Theorem 6.2 shows that all three cases are possible. But note that the half-plane is rigid in that the only examples whose limiting set is a half-plane are the half-planes themselves. X.-J. Wang proved that this is true in all dimensions. His theorem is a consequence of the following remarkable estimate.

Lemma 6.3 (Slab estimate[8]). *There exist $\beta = \beta(n) > 0$ and $R = R(n) < \infty$ with the following property. Let $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ be a convex ancient mean curvature flow in $\mathbb{R}^{n+1}$ which contains the spacetime origin; i.e. $0 < \omega$ and $0 \in \Omega_0$. If $\Omega_{-1} \cap B_R(0) \subset \{x \in \mathbb{R}^{n+1} : -\beta < x_{n+1} < \beta\}$, then there is a fixed slab in $\mathbb{R}^{n+1}$ which contains Ω_t for all $t \in (-\infty, 0]$.*

[8] X.-J. Wang, "Convex solutions to the mean curvature flow", Lemma 2.7

Sketch of the proof. We may represent $\partial\Omega_t$ as a pair of graphs, $\partial\Omega_t = \operatorname{graph} g_+(\cdot,t) \cup \operatorname{graph} g_-(\cdot,t)$, over the projection of Ω_t onto the hyperplane $\{x_{n+1} = 0\}$, with $\partial\Omega^+_t \doteqdot \operatorname{graph} g_+(\cdot,t)$ lying "above" $\partial\Omega^-_t \doteqdot \operatorname{graph} g_-(\cdot,t)$. The hypothesis is then essentially that the vertical displacements, $\max g_+(\cdot,t)$ and $-\min g_-(\cdot,t)$, increase by at most $\beta \sim 0$ in a ball of radius $R \gg 0$ as time moves backwards from 0 to -1.

Using the fact that planar slicings of $\partial\Omega_t$ evolve faster than curve shortening flow, and hence exhibit superlinear enclosed area growth in backwards time, it can be shown that the horizontal displacements increase by at least $\sim \beta^{-1} \gg 0$ over the same backwards time interval. Exploiting convexity of Ω_t (which implies concavity of g_+ and $-g_-$ in the space variable), this then ensures that (taking $R \gg \beta^{-1}$) the gradients $Dg_{\pm}(\cdot, t)$ are controlled by ~ 1 in a ball of radius $\sim \beta^{-1}$ over the backwards time interval $[-1, -\frac{1}{2}]$.

Now, concavity of the vertical displacements g_+ and $-g_-$ with respect to the time variable (which is a consequence of the differential Harnack inequality for convex ancient solutions) implies that $g_+(\cdot, t) - g_-(\cdot, t)$ at most doubles during the next unit of backwards time; so the above arguments can be repeated with the constant 2β. On the other hand integrating the gradient bound ensures that the mean curvature $\mathrm{H}(\cdot, t)$ cannot be very large *on average* in a ball of radius $\sim \beta^{-1}$ during this time interval. The Fubini theorem then ensures that we can find a point x in this ball at which the mean curvature $\mathrm{H}(x, \cdot)$ is small on average over our time interval. Since the mean curvature controls the rate of change of the displacement, this integrates to an improved displacement bound. When applied very carefully, the process can be iterated (with summably decaying loss), resulting in the desired estimate $g_+(0, t) - g_-(0, t) \le O(1)$. □

Corollary 6.4 (Slab dichotomy[9]). *Let $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ be a convex, locally uniformly convex ancient mean curvature flow in $\mathbb{R}^{n+1}$. Either*

- *$\{\Omega_t\}_{t\in(-\infty,\omega)}$ exhausts all of space, or*
- *$\{\Omega_t\}_{t\in(-\infty,\omega)}$ is confined to a fixed slab region.*

[9] ibid., Corollary 2.2

Sketch of the proof. As in the proof of Theorem 6.2, the estimate (5.26) and the compactness theorem (Theorem 2.22) ensure that the rescaled mean curvature flows $\{\lambda\partial\Omega_{\lambda^{-2}t}\}_{t\in(-\infty,\lambda^2\omega)}$ take a limit along some subsequence $\lambda_j \to 0$.

If the enclosed regions do not degenerate, then Huisken's monotonicity formula (Proposition 5.13) guarantees that limit will be a convex self-shrinking mean curvature flow, and hence a shrinking cylinder by Theorem 6.11 below (cf. Theorem 5.8). We readily conclude from this that $\{\Omega_t\}_{t\in(-\infty,\omega)}$ exhausts all of space.

On the other hand, if the enclosed regions degenerate, then the limit must be a hyperplane of multiplicity two, and the hypotheses of the slab estimate can certainly be arranged for j sufficiently large. So $\{\lambda_j\partial\Omega_{\lambda_j^{-2}t}\}_{t\in(-\infty,\lambda_j^2\omega)}$ is confined to a fixed slab for such j, and hence $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ must also. □

A convex ancient solution whose enclosed regions exhaust all of space is said to be ENTIRE.

6.2.1 A nontrivial example

So far, our only examples of ancient solutions are either solitons with a high degree of symmetry (obtained by reduction to an ODE) or the ancient paperclip solution (an explicit non-soliton solution obtained by imposing an *ad hoc* ansatz on the level set flow equation). White[10] provided the first truly "parabolic" (in the sense of PDE methods) construction of an ancient mean curvature flow.

[10] White, "The nature of singularities in mean curvature flow of mean-convex sets".

Theorem 6.5 (The ancient Steeden[11])**.** *There exists a non-round entire convex ancient mean curvature flow* $X : S^2 \times (-\infty, 0) \to \mathbb{R}^3$ *on which* $O(2) \times O(1)$ *acts by isometries.*

[11] Steeden are the producers of the iconic Australian Rugby League football (which is more oval than a European football and less pointy than a North American football). Evidently, I am a Rugby League fan; followers of the Rugby Union may prefer the "ancient Gilbert"; followers of Australian Rules Football may prefer the "ancient Sherrin". Followers of American or Canadian football should consider orbifolds.

Sketch of the proof. The idea is to take a limit of "very old" solutions constructed by evolving suitable initial data. We begin by evolving a sequence of ($O(2) \times O(1)$-invariant) smoothly capped cylinders $C_k = S^1 \times [-k, k]$ of radius one and length $2k$. When $k = 0$, the solution is the round sphere of radius one, which shrinks to a point after time ~ 1. For other values of k, C_k still shrinks to a point in time ~ 1 (the initial cylinder is an outer barrier), becoming round in the process (in accordance with Huisken's theorem). After translating time, we can arrange that the final time is $t = 0$. Using srinking spheres as inner barriers, it can be shown that the "perigee" and "apogee" take a fixed amount of time to decrease by $1/2$. So we can parabolically rescale so that, for $k \geq 1$, the "eccentricity" is ~ 2 and the inradius is $\sim 1/2$ at time $t = -1$, and that the initial time α_k goes to $-\infty$ as $k \to \infty$. Since the inscribed radius times the mean curvature is uniformly controlled from below at the initial time, it remains so due to Andrews' noncollapsing estimate (Proposition 5.15). Proposition 5.17 and the Bernstein estimates then ensure that the curvature and its derivatives are uniformly bounded along the sequence. We can now take a limit as $k \to \infty$ using the compactness theorem. Since we ensured that the eccentricity is ~ 2 at time -1 along the sequence, the limit cannot be the shrinking sphere. □

6.2.2 Entire ancient solutions in two space dimensions

The ancient Steeden completes the list of *entire* convex ancient solutions to mean curvature flow in $\mathbb{R}^3$.

Theorem 6.6 (Angenent–Daskalopoulos–Šešum[12], Brendle–Choi[13], X.-J. Wang[14])**.** *Every entire convex ancient mean curvature flow in* $\mathbb{R}^3$ *is one of the following:*

1. *a shrinking sphere.*
2. *a shrinking cylinder.*
3. *a radio-dish soliton.*
4. *an ancient Steeden.*

[12] S. Angenent, Daskalopoulos, and Sesum, "Uniqueness of two-convex closed ancient solutions to the mean curvature flow"

[13] Brendle and K. Choi, "Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$"

[14] X.-J. Wang, "Convex solutions to the mean curvature flow"

6.2.3 The structure of entire convex ancient solutions

X.-J. Wang developed a beautiful structure theory for entire convex ancient solutions to mean curvature flow. We will present a brief overview of his theory.

We first observe that the interior curvature estimate of Proposition 2.25 implies the following rudimentary compactness property for the space of convex ancient solutions.

Proposition 6.7. *Let $\{\partial\Omega^j_t\}_{t\in(-\infty,0]}$ be a sequence of convex ancient mean curvature flows and $x_j \in \Omega^j_{t_j}$, $t_j \in (-\infty, 0]$ a sequence of points and times such that $(x_j, t_j) \to (0,0)$ and*

$$\liminf_{j\to\infty} \mathrm{H}^j(x_j, t_j) > 0.$$

The following are equivalent.

1. *A subsequence of $\{\partial\Omega^j_t\}_{t\in(-\infty,0]}$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty, 0])$.*
2. *There are constants $\rho > 0$ and $C < \infty$ such that, after passing to a subsequence,*
$$\sup_{B_\rho(0)\times[-\rho^2,0]} \mathrm{H}^j \le C$$
for each j.
3. *The sequence $\{\Omega^j_0\}_{j\in\mathbb{N}}$ subconverges in the Hausdorff topology to a convex set of dimension $n+1$.*
4. *After passing to a subsequence, there is an open ball in $\cap_{j\in\mathbb{N}}\Omega^j_0$.*

Sketch of the proof. This is a straightforward consequence of the Bernstein estimates (Proposition 2.21) and Proposition 2.25.[15] □

The following "paraboloid estimate" is a consequence of the slab estimate and Proposition 6.7.[16]

Proposition 6.8 (Paraboloid estimate[17]). *There exists $\eta = \eta(n) > 0$ with the following property. Let $\{\partial\Omega_t\}_{t\in(-\infty,0]}$ be an entire convex ancient mean curvature flow. Given $p_0 \in \partial\Omega_{t_0}$,*

$$B_{\eta\sqrt{t_0-t}}(p_0) \subset \Omega_t \text{ for all } t \le t_0 - \mathrm{H}(p_0, t_0)^{-2}.$$

Proof. Suppose, contrary to the claim, that there is a sequence of entire convex ancient solutions $\{\partial\Omega^i_t\}_{t\in(-\infty,0]}$ with the following properties:

– $\partial\Omega^i_0$ contains the origin.

– There is a sequence of times $t_i \le -\mathrm{H}^i(0,0)^{-2}$ such that

$$\sqrt{-t_i}\,\mathrm{dist}(0, \partial\Omega^i_{t_i}) \to 0 \text{ as } i \to \infty.$$

[15] See, e.g., Bourni, Langford, and Lynch, "Collapsing and noncollapsing in convex ancient mean curvature flow", Proposition 2.2.

[16] The terminology comes from the fact that the inclusion $B_{\eta\sqrt{t_0-t}}(p_0) \subset \Omega_t$ for some $\eta > 0$ is equivalent to the existence of a paraboloid lying below the graph of the ARRIVAL TIME of $\{\Omega_t\}_{t\in(-\infty,0]}$—the function $u : \cup_{t\in(-\infty,0]}\partial\Omega_t \to \mathbb{R}$ defined by

$$u(X) = t \iff X \in \partial\Omega_t.$$

[17] X.-J. Wang, "Convex solutions to the mean curvature flow", Theorem 2.2

Performing a parabolic rescaling by $\sqrt{-t_i}$ for each $i \in \mathbb{N}$, we may assume that $t_i = -1$ and $\mathrm{H}^i(0,0) \geq 1$. Passing to a subsequence, we may assume that Ω^i_{-1} converges locally uniformly in the Hausdorff topology to a closed convex set K.

We consider two cases. First, if K has no interior, then it lies in a hyperplane by convexity. We are assuming $0 \in \Omega^i_{-1}$, so K contains the origin, and up to a rotation we may assume that $K \subset \{x_{n+1} = 0\}$. In particular, given any $\beta > 0$ and $R < \infty$, for all sufficiently large i,

$$\Omega^i_{-1} \cap B_R(0) \subset \{|x_{n+1}| \leq \beta\}.$$

Choosing R sufficiently large and β sufficiently small, the slab estimate (Lemma 6.3) ensures that $\partial\Omega^i_t$ is confined to fixed a slab for all $t \leq 0$, contrary to our assumption.

Suppose instead that K contains an open ball $B_{2\rho}$. Let T be the supremum over all times $t \leq 0$ such that Ω^i_t has a subsequential Hausdorff limit containing $B_{\rho/2}$. The avoidance principle and the fact that $B_{2\rho} \subset K$ ensure that $T > -1$, and by Proposition 6.7, $\{\partial\Omega^i_t\}_{t\in(-\infty,T]}$ subconverges in C^∞_{loc} to a smooth convex ancient solution $\{\partial\Omega_t\}_{t\in(-\infty,T]}$. Since $0 \in \overline{\Omega}^i_T$ and $\mathrm{dist}(0, \partial\Omega^i_{-1}) \to 0$, we have $0 \in \partial\Omega_t$ for all $t \in [-1, T]$. Applying the strong maximum principle to H shows $\partial\Omega_t$ is stationary, and thus consists of a hyperplane or pair of parallel hyperplanes for all $t \leq T$. In this case $B_{2\rho} \subset K = \overline{\Omega}_{-1}$ implies $B_{2\rho} \subset \Omega_T$, hence there is a subsequence in i such that $B_\rho \subset \Omega^i_T$, and unless $T = 0$ we obtain a contradiction to the maximality of T using the avoidance principle. Thus, $\partial\Omega^i_0$ converges in C^∞_{loc} to a hyperplane or pair of parallel hyperplanes, but we rescaled to ensure $\mathrm{H}^i(0,0) \geq 1$, so this is impossible. □

In conjunction with the interior curvature estimate (Proposition 2.25), the paraboloid estimate implies sequential precompactness of the space of entire convex ancient mean curvature flows.

Theorem 6.9 (Precompactness of the space of entire convex ancient solutions[18]). *Let $\{\partial\Omega^i_t\}_{t\in(-\infty,0]}$, $i \in \mathbb{N}$, constitute a sequence of entire convex ancient mean curvature flows. Suppose that $0 \in \partial\Omega^i_0$ for each i and $\mathrm{H}^i(0,0) \to \mathrm{H}_0 \in [0,\infty]$ as $i \to \infty$. After passing to a subsequence, $\{\partial\Omega^i_t\}_{t\in(-\infty,0]}$ converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty,0))$ to a convex ancient mean curvature flow $\{\partial\Omega_t\}_{t\in(-\infty,0)}$. Moreover,*

1. *if $\mathrm{H}_0 = 0$, then the convergence is in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty,0])$ and the limit is a stationary hyperplane of multiplicity one;*

2. *if $\mathrm{H}_0 \in (0,\infty)$, then the convergence is in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1} \times (-\infty,0])$ and the limit $\{\partial\Omega_t\}_{t\in(-\infty,0]}$ is entire;*

[18] See X.-J. Wang, "Convex solutions to the mean curvature flow", Corollary 2.3 or Bourni, Langford, and Lynch, "Collapsing and noncollapsing in convex ancient mean curvature flow", Theorem 4.6 for a proof.

3. *if* $H_0 = \infty$, *then the limit* $\{\partial\Omega_t\}_{t\in(-\infty,0)}$ *is entire,* $\Sigma \doteqdot \cap_{t<0}\Omega_t$ *is an affine subspace of* $\mathbb{R}^{n+1}$, *and* Ω_t *splits as a product* $\Sigma \times \Omega_t^\perp$, *where* $\{\Omega_t^\perp\}_{t\in(-\infty,0)}$ *is a family of bounded convex bodies in* $\Sigma^\perp$.

The following short proof that entire ancient solutions to mean curvature flow are noncollapsing illustrates the utility of Theorem 6.9.

Theorem 6.10. *A convex ancient mean curvature flow* $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ *in* $\mathbb{R}^{n+1}$ *is entire if and only if it is noncollapsing.*[19]

[19] In fact, an application of the strong maximum principle to the evolution of the inscribed curvature guarantees that $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ is collapsing no worse than the shrinking cylinder $\{S^1_{\sqrt{-2t}}\} \times \mathbb{R}^{n-1}$ (i.e. $\bar{r}\,H \geq 1$ at all points).

Proof. It is easy to deduce entirety from noncollapsing.

To establish the reverse implication, suppose that there is a sequence of convex ancient mean curvature flows $\{\partial\Omega_t^i\}_{t\in(-\infty,0]}$ such that $0 \in \partial\Omega_0^i$ and $H_i(0,0) = 1$, but the inscribed radius $\bar{r}_i$ at $(0,0)$ satisfies $\bar{r}_i \to 0$. Part (2) of Theorem 6.9 tells us that $\{\partial\Omega_t^i\}_{t\in(-\infty,0]}$ subconverges in $C^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1}\times(-\infty,0])$ to a convex ancient mean curvature flow $\{\partial\Omega_t\}_{t\in(-\infty,0]}$. In particular, $\liminf_{i\to\infty}\bar{r}_i > 0$, which is a contradiction. □

6.3 *Further examples of convex ancient solutions*

There are a great many further examples[20] of ancient mean curvature flows, even under the assumption of convexity.

[20] The below list is *not* exhaustive.

Example 10 (Generalized Steedens[21]). White's construction (Theorem 6.5) generalizes to spheres S^n of any dimension $n \geq 2$ and any bisymmetry class $O(k)\times O(n+1-k)$, $k = 2,\dots,n$. These examples are convex and entire. ■

[21] Haslhofer and Hershkovits, "Ancient solutions of the mean curvature flow"

Note that, while the symmetry groups $O(k)\times O(n+1-k)$ and $O(\ell)\times O(n+1-\ell)$ agree (up to a congruence of $\mathbb{R}^{n+1}$) when $\ell = n+1-k$, the two corresponding examples in the above construction are *not* congruent (since, for instance, the blow-down of the example with symmetry group $O(k)\times O(n+1-k)$ is the shrinking cylinder $S^{k-1}_{\sqrt{-2(k-1)t}}\times\mathbb{R}^{n-k}$). This begs the question of the whereabouts of the "missing" example: the one corresponding to the symmetry group $O(1)\times O(n)$ (whose blow-down should be $S^0_0\times\mathbb{R}^n$—the hyperplane of multiplicity two).

Example 11 (The ancient pancake[22]). For each $n \geq 2$, there is an $O(1)\times O(n)$-invariant, convex ancient mean curvature flow which exhausts the slab $\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1} : -\frac{\pi}{2} < x_1 < \frac{\pi}{2}\}$. Its "height" $h(t) \doteqdot \sigma(e_1,t)$ grows like

[22] Bourni, Langford, and Tinaglia, "Collapsing ancient solutions of mean curvature flow"; X.-J. Wang, "Convex solutions to the mean curvature flow"

$$h(t) \geq \tfrac{\pi}{2} - O((-t)^{-k}) \ \text{ for any } \ k\in\mathbb{N} \ \text{ as } \ t\to-\infty$$

and its "radius" $r(t) \doteqdot \sigma(e_2,t)$ grows like

$$r(t) = -t + (n-1)\log(-t) + c_n + o(1) \ \text{ as } \ t\to-\infty.$$

This example is constructed by rotating the time $t = -R$ slice of the (horizontally oriented) ancient paperclip solution about the y-axis to obtain an $O(1) \times O(n)$-invariant convex hypersurface, evolving this hypersurface by mean curvature flow to obtain, after time-translation, an "old-but-not-ancient" mean curvature flow $\{\partial\Omega_t^R\}_{t\in[-\alpha_r,0)}$ which shrinks to a round point at time zero in accordance with Huisken's theorem, and (after establishing a number of uniform-in-R estimates) taking a limit as $R \to \infty$. ■

Self-similarly shrinking and translating solutions are "trivial" examples of ancient mean curvature flows. The (mean) convex self-similarly shrinking solutions are relatively easily classified.

Theorem 6.11 (Colding–Minicozzi[23])**.** *The shrinking cylinders $S^k_{\sqrt{-2kt}} \times \mathbb{R}^{n-k}$, $k \in \{1, \ldots, n\}$, are (up to ambient isometries) the only embedded, mean convex, self-similarly shrinking mean curvature flows.*

[23] Tobias H. Colding and Minicozzi, "Generic mean curvature flow I: generic singularities", Theorem 10.1.

On the other hand, there are a great many convex self-similarly translating mean curvature flows. The first nontrivial examples to be constructed were the FLYING WINGS (so named, by Richard Hamilton, for their resemblance to the Northrop and Grumman "flying wing" aircraft).

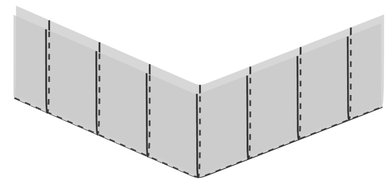


Figure 6.3: A "flying wing" translator.

Example 12 (Flying wings)**.** For each $\theta \in (0, \frac{\pi}{2})$ and $n \geq 2$, there is an $O(1) \times O(n-1)$-invariant convex translator in $\mathbb{R}^{n+1}$ with bulk velocity e_{n+1} which exhausts the slab $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : -\frac{\pi}{2}\sec\theta < x_1 < \frac{\pi}{2}\sec\theta\}$. It is asymptotic to an S^{n-2} family of Grim hyperplanes tilted[24] through angle θ and scaled by $\sec\theta$.

[24] Recall that rotating the Grim hyperplane $\{(x, y, -\log\cos x + t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x \in (-\frac{\pi}{2}, \frac{\pi}{2})\}_{t\in(-\infty,\infty)}$ through angle $\theta \in [0, \frac{\pi}{2})$ in a vertical plane $e \wedge e_{n+1}$, $e \in \mathrm{span}\{e_1, e_{n+1}\}^\perp$, and parabolically rescaling by $\sec\theta$ yields another *unit speed, vertically translating* mean curvature flow.

These examples are obtained (after establishing suitable estimates) by taking a limit of solutions to the Dirichlet problem for the graphical translator PDE over suitable bounded, convex domains which tend to the desired horizontal slab. In fact, they were originally constructed by first performing the Legendre transform to the graphical translator PDE to obtain a certain *fully nonlinear* equation.[25] (The upshot being that solutions to this equation are automatically convex, and are thereby equipped with the *a priori* estimate $|\mathrm{II}| \leq \mathrm{H} \leq 1$ due to the translator equation.) They may also be obtained via a barrier construction and Allard's regularity theorem[26], and there is yet a another approach which exploits the stability of graphical translators as critical points of the energy functional (5.15).[27] (In fact, in this construction, the principal curvatures at the tip are prescribed, rather than the slab width.)

[25] X.-J. Wang, "Convex solutions to the mean curvature flow".

[26] Bourni, Langford, and Tinaglia, "On the existence of translating solutions of mean curvature flow in slab regions"

[27] Hoffman et al., "Graphical translators for mean curvature flow".

The downside of the two latter approaches is that convexity of the constructed examples is unclear. But this can be established *a posteriori* under the bisymmetry condition.[28]

[28] Bourni, Langford, and Tinaglia, "On the existence of translating solutions of mean curvature flow in slab regions"; Spruck and Xiao, "Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature".

That each of these constructions agrees is not obvious—it follows *a* *posteriori* from the fact that each slab admits at most one convex, locally uniformly convex translator.[29]

A quite natural (and more general) construction was later found which directly encodes the convexity and asymptotic Grim Reapers; we will discuss this approach further below, in Example 15. ■

[29] Hoffman et al., "Graphical translators for mean curvature flow".

The flying wing family interpolates between the Grim plane ($\theta = 0$) and the radio-dish ($\theta = \frac{\pi}{2}$). There is also a family of entire analogues of the flying wings, for which the (k-dimensional) radio dish plays the role of the Grim Reaper.

Example 13 (Entire wings[30])**.** For every $n \geq 3$ and $k \in \{2, \ldots, n-1\}$, and each pair of numbers $0 < \lambda < \mu$ satisfying $(n-k)\lambda + k\mu = 1$, there exists an entire, $O(k) \times O(n-k)$-invariant convex translator in $\mathbb{R}^{n+1}$ with bulk velocity e_{n+1} and principal curvatures given by $(\underbrace{\lambda, \ldots, \lambda}_{k\text{-times}}, \mu, \ldots, \mu)$ at its "tip" (the point of intersection with the x_{n+1}-axis). ■

[30] Hoffman et al., "Graphical translators for mean curvature flow"; X.-J. Wang, "Convex solutions to the mean curvature flow"

There is also a family of examples which interpolate between the generalized ancient Steedens.

Example 14 (Deformed Steedens[31])**.** For every $n \geq 3$ and each $k \in \{2, \ldots, n-1\}$, there exists an $(n-k)$-parameter family of entire convex ancient mean curvature flows that are "only" $O(k) \times \underbrace{O(1) \times \cdots \times O(1)}_{(n+1-k)\text{-times}}$-invariant. The blow-down of each member of the family is the shrinking cylinder $S^{k-1}_{-2(k-1)t} \times \mathbb{R}^{n+1-k}$. ■

[31] Du and Haslhofer, "On uniqueness and nonuniqueness of ancient ovals"

There are also a great many non-entire examples. The following construction suggests the existence of examples which decompose into axially congruent Grim hyperplanes in *any* configuration.

Example 15 (Ancient polytopes, flying hyperwings and formations of flying (hyper)wings[32])**.** Given any regular polytope[33] $P \subset \mathbb{R}^n$ (normalized to circumscribe the unit sphere) there exists a convex ancient mean curvature flow $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ whose SQUASH-DOWN

$$\Omega_* \doteqdot \lim_{t\to-\infty} \frac{1}{-t}\Omega_t$$

is equal to $P \times \{0\} \subset \mathbb{R}^{n+1}$. The solution $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ exhausts the slab $\{(x_1, \ldots, x_{n+1}) : -\frac{\pi}{2} < x_{n+1<\frac{\pi}{2}}\}$ and inherits the dihedral symmetries of P; it is also symmetric under reflection across $\mathbb{R}^n \times \{0\}$.

If P is unbounded, then $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ evolves by translation in the direction of the axis of P. In particular, this provides a different construction of the flying wings, but also generates many further examples when $n \geq 3$—"flying hyperwings".

[32] Bourni, Langford, and Tinaglia, "Ancient mean curvature flows out of polytopes"

[33] Note that we allow convex polytopes to be unbounded (any intersection of finitely many halfspaces is permissible).

These examples are constructed by forming the intersection of the convex regions bounded by the time $-R$ slices of the Grim hyperplanes whose linear factors are parallel to the facets of P, evolving the boundary of this region by mean curvature flow to obtain, after a time-translation, an "old-but-not-ancient" mean curvature flow $\{\partial\Omega^R_t\}_{t\in[\alpha_R,0)}$ which reaches the origin at time zero, and (after establishing a number of uniform-in-R estimates) taking a limit as $R\to\infty$.

This construction is slightly more general; it can also be applied to obtain ancient solutions out of any (not necessarily regular) simplex P (including unbounded simplices, which are obtained by removing a face from a bounded simplex). These examples are quite surprising in that they do not, in general, admit any symmetry beyond the reflection symmetry across $\mathbb{R}^n\times\{0\}$.

It is also possible to construct examples out of the semi-frustum (truncation) of an unbounded regular polytope. These examples resemble a family of flying hyperwings appearing from infinity at time minus infinity, coalescing into a single flying hyperwing at time plus infinity (conserving the total exterior angle). They are interesting in that they are eternal but do not evolve by translation.[34]

It seems likely that there should exist examples out of *any circumscribed convex polytope*, but this remains an open problem.

Moreover, while it can be shown that, conversely, the squash-down of any unit scale (non-entire) example must circumscribe the unit ball, *uniqueness* is wide open. ■

A good classification of ancient solutions is thus a very difficult problem in general, even under the assumption of convexity. The two dimensional case may be within reach, however.[35]

6.4 Exercises

Exercise 6.1. Find *all* solutions to curve shortening flow satisfying the Ansatz (6.5).

Exercise 6.2. Let $\{\partial\Omega_t\}_{t\in(-\infty,\omega)}$ be a convex, locally uniformly convex ancient mean curvature flow in $\mathbb{R}^{n+1}$ which is not entire (so that, by Theorem 6.4, it is *confined to* a slab region). Show that $\cup_{t\in(-\infty,\omega)}\Omega_t$ *is* a slab region.

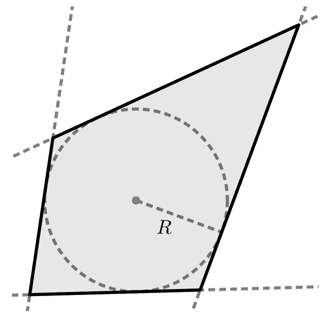


Figure 6.4: Grim planes are placed along each edge of the circumscribed polytope; the boundary of the enclosed region is evolved by mean curvature flow, terminating at the origin at time zero (after a spacetime translation); in the limit as $R\to\infty$, an ancient solution is obtained.

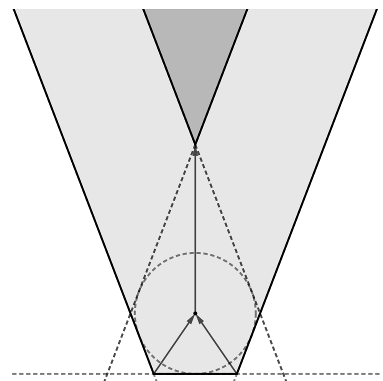


Figure 6.5: A formation of flying (hyper)wings appears at time $-\infty$ and merges into a simpler formation at time $+\infty$, preserving width, total velocity, and total exterior angle.

[34] It is tempting to conjecture, based on Corollary 5.11, that *any* convex, eternal mean curvature flow should evolve by translation (see, e.g. White, "The nature of singularities in mean curvature flow of mean-convex sets", Conjecture 1).

[35] Some very recent progress on the classification of entire convex ancient solutions in $\mathbb{R}^4$ has been announced by K. Choi and Haslhofer, "Classification of ancient noncollapsed flows in $\mathbb{R}^4$" (building on work of K. Choi, Haslhofer, and Hershkovits, "Classification of noncollapsed translators in $\mathbb{R}^4$"; Du and Haslhofer, "On uniqueness and nonuniqueness of ancient ovals" and B. Choi et al., "Classification of bubble-sheet ovals in $\mathbb{R}^4$"). In short: we have already described all possible examples.

7 *Epilogue*

During the last 50 years, in view of great breakthroughs in analysis—say, De Giorgi–Nash, or the regularity theory for harmonic maps, then minimal surfaces—it became possible to use these [nonlinear] structures—minimal surfaces, harmonic maps, constant mean curvature surfaces—to investigate structures in differential geometry that before then could only be investigated with ODE, geodesics, measuring angles, etc. But now nonlinear PDE are understood so well that they could be used as a toolbox in understanding structures in differential geometry. And I think we've only just started—there will be much more that can be done; in particular, I think really elliptic and parabolic theory—Ricci flow is a prime example—has become a fantastic tool in a new relation between analysis and geometry.

– Gerhard Huisken, *Mathmedia Interview: Prof. Gerhard Huisken*

Part II

Ricci flow

When the dimension n is 0 or 1 there is not much to prove. The $n = 2$ result is a special case of uniformization. The first modern work is Smale's[1] *in which he proved PCn, $n \geq 5$... PC4 amounts to Smale's outline with a topological twist.*[2] *But here the tail wags the dog. When you delve into this detail the twist expands to fill your entire field of view... The final case (historically), dimension 3, was proved by Perelman*[3] *using Hamilton's theory of Ricci flow. It is entirely different in outline, more like Beethoven's 9th than a conventional proof, and still stands as the greatest accomplishment of 21st century mathematics.*

– Michael H. Freedman, "Afterword: PC4 at age 40".

[1] Smale, "Generalized Poincaré's conjecture in dimensions greater than four".

[2] Freedman, "The topology of four-dimensional manifolds".

[3] Perelman, "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds.", "Ricci flow with surgery on three-manifolds.", "The entropy formula for the Ricci flow and its geometric applications".

Preamble to Part II

The Ricci flow, introduced by Richard Hamilton[4,5] in 1982, is a deformation process for Riemannian metrics which, in a suitable "gauge", formally resembles the heat equation, and indeed exhibits a number of phenomena which are shared by other diffusion processes. These diffusive properties are highly desirable from the point of view of geometric and topological applications—in principle, the Ricci flow smooths out rough metrics and diffuses their curvature, driving them towards ideal and canonical equilibrium states, thereby restricting the possible topologies which the initial metric can carry. Alas, life is never so straightforward: the Ricci flow equation (suitably interpreted) is degenerate and nonlinear, and suffers singularities in finite time, all of which prevent the direct implementation of this programme. Nonetheless, it has proved itself to be one of the most fruitful tools available to the geometric analyst, leading (famously) to proofs of the Poincaré and Thurston conjectures, amongst manifold further important advances.

These motivations aside, the Ricci flow is the canonical heat equation for Riemannian metrics, and gives rise to many remarkable and beautiful geometric structures (e.g. solitons, ancient solutions) and analytic features (e.g. gradient-like structures, differential Harnack inequalities, pseudolocality) and as such is a fascinating area of study for topologists, geometers, and analysts alike.

We shall present here an introduction to the Ricci flow leading up to the foundations of some modern developments.[6] We assume the reader has some basic familiarity with partial differential equations and Riemannian geometry. For background, the reader may refer, for instance, to the books of Olver[7] and Chavel.[8]

[4] Richard S. Hamilton, "Three-manifolds with positive Ricci curvature".

[5] Remarkably, the Ricci flow equation also arises independently in the context of quantum field theory—as the first order (or "one loop") approximation of the renormalization group equation for nonlinear sigma models. In this context, it was actually discovered slightly before Hamilton's foundational work, by Friedan, "Nonlinear models in $2+\varepsilon$ dimensions".

[6] There are now a number of excellent resources on the Ricci flow, including (and by no means limited to) the books of Andrews and Hopper, *The Ricci flow in Riemannian geometry*, Chow and Knopf, *The Ricci flow: an introduction*, Chow, Lu, and Ni, *Hamilton's Ricci flow* and Morgan and Tian, *Ricci flow and the Poincaré conjecture*; the article of Kleiner and Lott, "Notes on Perelman's papers"; and the excellent lectures of Richard Bamler (accessible on Richard's webpage at the time of writing), each of which this part has drawn upon to some degree.

[7] Olver, *Introduction to partial differential equations*.

[8] Chavel, *Riemannian geometry*.

8
The fundamentals

A smooth one-parameter family $\{g_t\}_{t\in I}$ of smooth Riemannian metrics g_t on a smooth[1] n-manifold M^n EVOLVES BY/SATISFIES/IS A RICCI FLOW[2] if

$$\frac{dg_t}{dt} = -2\mathrm{Rc}_{g_t}\,, \tag{8.1}$$

where Rc_{g_t} is the Ricci tensor associated to g_t and the time derivative is understood fibrewise, in the usual sense: for any $x \in M$,

$$\left(\frac{dg_t}{dt}\right)_x \doteqdot \lim_{h\to 0}\frac{(g_{t+h})_x - (g_t)_x}{h}\,.$$

If we introduce local coordinates $\{x^i : U \to \mathbb{R}\}_{i=1}^n$ in some region $U \subset M$, then for each $x \in U$ we may represent $(g_t)_x$ and $(\mathrm{Rc}_t)_x$ as

$$(g_t)_x = g_{ij}(x,t)dx^i \otimes dx^j \;\text{ and }\; (\mathrm{Rc}_{g_t})_x = \mathrm{Rc}_{ij}(x,t)dx^i\otimes dx^j\,,$$

and we see that[3]

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= -2\mathrm{Rc}_{ij}\\ &= -2g^{k\ell}\mathrm{Rm}_{ikj\ell}\\ &= g^{k\ell}\left(\frac{\partial^2 g_{ij}}{\partial x^k\partial x^\ell} + \frac{\partial^2 g_{k\ell}}{\partial x^i\partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^i \partial x^\ell} - \frac{\partial^2 g_{i\ell}}{\partial x^k\partial x^j}\right)\\ &\quad + \frac{1}{2}g^{k\ell}g^{mn}\Big[\Big(\partial_k g_{jn} + \partial_j g_{kn} - \partial_n g_{kj}\Big)\left(\partial_i g_{m\ell} + \partial_m g_{i\ell} - \partial_\ell g_{im}\right)\\ &\qquad\qquad - \left(\partial_i g_{jn} + \partial_j g_{in} - \partial_n g_{ij}\right)\partial_m g_{k\ell}\Big]\,,\end{aligned} \tag{8.2}$$

a system of nonlinear second order partial differential equations. Unappealing, certainly, but it does have the redeeming feature that it is weakly parabolic (which explains the choice of sign on the right hand side).

We can make this a little nicer (and gain some very important intuition) by being more selective in our choice of "gauge": at any time t, about any point $x \in M$, the existence of a g_t-HARMONIC COORDINATE

[1] Henceforth, we shall stop using the qualifier "smooth" so irritatingly often, leaving it for the most part to the reader to decide how regular they wish a given object to be in order to make sense of a given statement.

[2] In fact, we shall soon replace this by a more abstract definition, which may appear more complicated at first but has many advantages. The two definitions are equivalent in the sense that there is a canonical bijection between their solutions.

[3] Note that we follow the convention $\mathrm{Rm}(X,Y,Z,W) \doteqdot g(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z, W)$ for the Riemann curvature tensor.

CHART can be established using standard results on the existence and regularity of solutions to elliptic partial differential equations. These are coordinates satisfying

$$\Delta_{g_t} x^i = 0,$$

where Δ_{g_t} is the Laplace–Beltrami operator induced by g_t; as such, in g_t-harmonic coordinates, the Ricci flow system can be seen to take the form

$$\frac{\partial g_{ij}}{\partial t} = \Delta_{g_t}(g_{ij}) + \text{terms of lower order} \tag{8.3}$$

at time t. This suggests that **we should view the Ricci flow as a kind of geometric heat equation for Riemannian metrics** (and also provides justification for the factor of 2 on the right hand side of the equation). We shall soon see that it is quite right to do so, but before pursuing this further, let us first establish some additional useful intuition, this time more geometric.

8.1 *Invariance properties*

The Ricci flow is invariant under certain canonical operations[4], in the sense that these operations take one solution and produce another.

[4] The following list is not intended to be exhaustive.

8.1.1 *Pullback by diffeomorphisms*

If $\{g_t\}_{t\in I}$ is a Ricci flow on M^n and $\phi : N^n \to M^n$ is a diffeomorphism, then (since the Ricci curvature is invariant under diffeomorphisms)

$$\left(\frac{d}{dt}\phi^* g_t\right)_x = \left(\phi^* \frac{d}{dt} g_t\right)_x = -2\phi^*(\mathrm{Rc}_{g_t}) = -2(\mathrm{Rc}_{\phi^* g_t})_x.$$

That is, $\{\phi^* g_t\}_{t\in I}$ is a Ricci flow on N^n. This is not at all surprising.

On the other hand, if we allow the diffeomorphism to change with time[5]), then we pick up an extra term due to the chain rule:

[5] We shall always assume the group property $\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1+t_2}$ for one-parameter families of diffeomorphisms ϕ_t.

$$\frac{d}{dt}\phi_t^* g_t = -2\mathrm{Rc}_{\phi_t^* g_t} + \mathcal{L}_V(\phi_t^* g_t),$$

where V is the vector field defined by

$$V(\phi_t(x)) \doteqdot \frac{d}{dt}(t \mapsto \phi_t(x)).$$

The converse of this statement is that if g_t satisfies the equation

$$\frac{d}{dt} g_t = -2\mathrm{Rc}_{g_t} + \mathcal{L}_V g_t$$

for some vector field V, then the family of metrics $\phi_{-t}^* g_t$ satisfies Ricci flow, where ϕ_t is the flow of V.

8.1.2 *Time translations*

If $\{g_t\}_{t\in I}$ is a Ricci flow on M and $\tau \in \mathbb{R}$, then clearly $\{g_{t+\tau}\}_{t\in I-\tau}$ is a Ricci flow on M.

8.1.3 *Parabolic rescaling*

If $\{g_t\}_{t\in I}$ is a Ricci flow on M and $\lambda > 0$, then (since the Ricci tensor is scale invariant)

$$\left(\frac{d}{dt}\lambda^2 g_{\lambda^{-2}t}\right)_x = -2\big(\mathrm{Rc}_{g_{\lambda^{-2}t}}\big)_x = -2\big(\mathrm{Rc}_{\lambda^2 g_{\lambda^{-2}t}}\big)_x.$$

That is, $\{\lambda^2 g_{\lambda^{-2}t}\}_{t\in\lambda^2 I}$ is a Ricci flow on M.

8.1.4 *Orthogonal sums with flat factors*

If $\{g_t\}_{t\in I}$ is a Ricci flow on M and $k \in \mathbb{N}$, then $\{g_t + g_{\mathbb{R}^k}\}_{t\in I}$ is a Ricci flow on $M \times \mathbb{R}^k$.

8.1.5 *Quotients and lifts*

Let $q : N^n \to M^n = N^n/G$ be a quotient map (induced by a proper and free action of a subgroup $G \subset \mathrm{Diff}(N^n)$). If $\{g_t\}_{t\in I}$ evolves by Ricci flow on M^n, then the lifts $\{q^* g_t\}_{t\in I}$ evolve by Ricci flow on N^n. Conversely, if $\{g_t\}_{t\in I}$ evolves by Ricci flow on N^n and each $\phi \in G$ is an isometry of each g_t, then $\{g_t\}_{t\in I}$ descends to a Ricci flow on M^n.

8.2 *Invariant solutions (a.k.a. self-similar solutions/solitons)*

The continuous symmetries of Ricci flow (diffeomorphism, time translation and scaling) give rise to special types of solutions: those that evolve purely by some combination of these symmetries. There are three primary types (but more generally one might consider combinations of these motions).

8.2.1 *Steady self-similar solutions*

A solution $\{g_t\}_{t\in\mathbb{R}}$ to Ricci flow on a manifold M is called a STEADY SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ of M such that

$$\phi_\varepsilon^* g_{t-\varepsilon} = g_t$$

for all ε and t. Differentiating with respect to ε at $\varepsilon = 0$, we find that such a solution must satisfy the equation

$$0 = \mathrm{Rc}_{g_t} + \tfrac{1}{2}\mathcal{L}_V g_t$$

for all t.

Conversely, if a Riemannian manifold (M, g) satisfies

$$0 = \mathrm{Rc} + \tfrac{1}{2}\mathcal{L}_V g$$

for some vector field V, whose flow is ϕ, then the family of metrics $\{g_t \doteqdot \phi_t^* g\}_{t\in\mathbb{R}}$ satisfies

$$\frac{d}{dt}g_t = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g_{t+\varepsilon} = \mathcal{L}_V g_t = \mathcal{L}_V \phi_t^* g = \phi_t^* \mathcal{L}_V g = -2\phi_t^* \mathrm{Rc}_g = -2\mathrm{Rc}_{g_t}.$$

8.2.2 *Shrinking/expanding self-similar solutions*

A solution $\{g_t\}_{t\in I}$ to Ricci flow on a manifold M is called a HOMOTHETIC SELF-SIMILAR SOLUTION if there is a one-parameter family of diffeomorphisms $\{\phi_t\}_{t\in I}$ of M such that

$$\mathrm{e}^{2\varepsilon}\phi_\varepsilon^* g_{\mathrm{e}^{-2\varepsilon}t} = g_t$$

for all $t \in I$ and ε such that $\mathrm{e}^{-2\varepsilon}t \in I$. Differentiating with respect to ε at $\varepsilon = 0$, we find that such a solution must satisfy the equation

$$0 = g_t + 2t\mathrm{Rc}_{g_t} + \tfrac{1}{2}\mathcal{L}_V g_t$$

for all t. There are two cases: if $I = (-\infty, 0)$, then $\{g_t\}_{t\in(-\infty,0)}$ is called a SHRINKING SELF-SIMILAR SOLUTION. If $I = (0, \infty)$, then $\{g_t\}_{t\in(0,\infty)}$ is called an EXPANDING SELF-SIMILAR SOLUTION.

Conversely, if a Riemannian manifold (M, g) satisfies

$$0 = g \pm \mathrm{Rc} + \tfrac{1}{2}\mathcal{L}_V g$$

for some vector field V, then the family of metrics $\{g_t\}_{\pm t\in(0,\infty)}$ defined respectively by $g_t \doteqdot \pm 2t\phi^*_{\log\sqrt{\pm t}}g$ satisfies

$$\frac{d}{dt}g_t = \pm 2\phi^*_{\log\sqrt{\pm t}}(g + \tfrac{1}{2}\mathcal{L}_V g) = -2\mathrm{Rc}_{g_t}.$$

8.2.3 *Examples: Einstein metrics*

Recall that a Riemannian manifold (M, g) is EINSTEIN if

$$\mathrm{Rc} = (n-1)\lambda g$$

for some $\lambda \in \mathbb{R}$ ($\lambda \in \{-1, 0, 1\}$ modulo scaling). Einstein metrics provide examples of "trivial" soliton Ricci flows: if $\lambda = 0$, e.g. $(M^n, g) = (\mathbb{R}^n, g_{\mathbb{R}^n})$, then $\{g_t = g\}_{t\in(-\infty,\infty)}$ is a steady self-similar Ricci flow (in this case STATIC), if $\lambda = -1$, e.g. $(M^n, g) = (H^n, g_{H^n})$, then $\{g_t = 2(n-1)tg\}_{t\in(0,\infty)}$ is an expanding self-similar Ricci flow, and if $\lambda = 1$, e.g. $(M^n, g) = (S^n, g_{S^n})$, then $\{g_t = -2(n-1)tg\}_{t\in(-\infty,0)}$ is a shrinking self-similar Ricci flow.

Observe that the static Ricci flow $t \mapsto g_t \doteqdot g_{\mathbb{R}^n}$ on Euclidean space $\mathbb{R}^n$ (for example) may also be viewed, not quite trivially, as a steady Ricci flow by pulling back along the flow ϕ of any Killing vector field K (since $\mathcal{L}_K g_{\mathbb{R}^n} = 0$). Similarly, we may view Euclidean space as an expanding or shrinking Ricci flow by pulling back along the flow of the conformally Killing radial vector field $X \doteqdot x^i\partial_i$ or its negative (since $\mathcal{L}_X g_{\mathbb{R}^n} = 2g_{\mathbb{R}^n}$).

8.3 Explicit solutions

Certain "explicit" solutions can be constructed "by hand" by imposing suitable symmetry or algebraic ansätze. We present three examples here, but there are a great many more examples which have been discovered by analogous methods.

By imposing a large enough symmetry group, the Ricci flow equation may be reduced to a (possibly highly complicated) system of ordinary differential equations.

Example 16 (The shrinking sphere)**.** We seek a solution to Ricci flow on S^n starting from a round metric, $g_0 = r_0^2 g_{S^n}$. Since we *expect* roundness to be preserved, we suppose *a priori* that the timeslices are always round,

$$g_t = r^2(t) g_{S^n}\,.$$

The Ricci tensor of g_t is then

$$\mathrm{Rc}_{g_t} = \mathrm{Rc}_{r^2 g_{S^n}} = \mathrm{Rc}_{g_{S^n}} = (n-1)g_{S^n}\,,$$

while its time derivative is

$$\frac{d}{dt}g_t = 2rr' g_{S^n}\,.$$

The Ricci flow equation is therefore equivalent to $2rr' = (n-1)$, which is solved by

$$r^2(t) = r_0^2 - 2(n-1)t\,,\quad t \in \left(-\infty, \tfrac{r_0^2}{2(n-1)}\right). \qquad \blacksquare$$

We can play a similar game with self-similar solutions, though in this case—since the time evolution is already trivial—we may relax symmetry by one degree of freedom.

Example 17 (Hamilton's cigar soliton)**.** We seek a two dimensional steady soliton on the plane which is circle fibred. I.e. a metric on $\mathbb{R}^2$ which takes the form

$$g = dr^2 + \psi^2(r)d\theta^2$$

in polar coordinates and satisfies

$$-\mathrm{Rc} = \tfrac{1}{2}\mathcal{L}_V g$$

for some vector field $V = f(r)\partial_r$. In two dimensions, the Ricci curvature is just $\mathrm{Rc} = \mathrm{K}g$, where K is the GAUSS CURVATURE, which in our case is given by $\mathrm{K} = -\frac{\psi_{rr}}{\psi}$. The Lie derivative term is found to be

$$\frac{1}{2}\mathcal{L}_V g = f_r ds^2 + f\frac{\psi_r}{\psi}\psi^2\, d\theta^2.$$

Equating the two, we find that

$$\frac{f_r}{f} = \frac{\psi_r}{\psi}.$$

So $f = 2C\psi$ for some constant C, and hence

$$\psi_{rr} = 2C\psi\psi_r.$$

Equivalently,

$$\psi_r = C\psi^2 + D.$$

Under the polar coordinate compatibility conditions (ψ admits a smooth odd extension about $r = 0$, where $\psi_r = 1$), this is solved by

$$\psi = \lambda \tanh(\lambda^{-1} r).$$

When $\lambda = 1$, the resulting metric

$$g = dr^2 + \tanh^2 r\, d\theta^2$$

is called HAMILTON'S CIGAR. (The parameter λ merely induces a scaling of the "standard" cigar.)

Setting $V = -2\tanh r\partial_r$ yields the flow equation

$$\begin{cases} \dfrac{ds}{dt} = -2\tanh s \\ s(r,0) = r, \end{cases}$$

which is solved by

$$\sinh s(r,t) = \mathrm{e}^{-2t}\sinh r.$$

So Hamilton's cigar gives rise to the Ricci flow

$$g = \frac{\cosh^2 r}{\mathrm{e}^{4t} + \sinh^2 r}(dr^2 + \tanh^2 r\, d\theta^2).$$

Cigar solutions of different scales may be obtained by parabolic rescaling. ■

Example 18 (Bryant's radio-dish soliton)**.** We may play the same game in higher dimensions. We now seek an $O(n)$-invariant metric

$$g = dr^2 + \psi^2(r) g_{S^{n-1}}$$

This body holding me reminds me of my own mortality
Embrace this moment, remember
We are eternal, all this pain is an illusion.
– Tool, "Parabola"

on $\mathbb{R}^n$ ($n \geq 3$). This leads to the system

$$f' = (n-1)\frac{\psi_{rr}}{\psi}, \quad \psi\psi_r f + (n-2)(1-\psi_r^2) = \psi\psi_{rr}.$$

Upon making suitable substitutions, we again are able to obtain a global solution satisfying the required compatibility conditions (see e.g.[6]). When $n \geq 3$, it behaves as

[6] Chow, *Ricci solitons in low dimensions*.

$$\psi \sim \sqrt{r} \text{ as } r \to \infty. \qquad \blacksquare$$

These basic ideas have a vast generalization: recall that a HOMOGENEOUS SPACE may be regarded as a Riemannian manifold (M, g) whose isometry group acts transitively. In short, the manifold "looks the same from any vantage point". This degree of symmetry guarantees that the curvature tensor at any given point is determined *algebraically* by the metric at that point; if we impose the ansatz that the isometry group is preserved, this reduces the Ricci flow to a system of ordinary differential equations. Similar considerations apply to homogeneously fibred solitons (though additional compatibility conditions may be required at any singular fibres). For a much more comprehensive examination of the Ricci flow on homogeneous geometries, see.[7]

[7] Chow and Knopf, *The Ricci flow: an introduction*.

8.4 Uniqueness and (short-time) existence of solutions

We would like to exhibit the Ricci flow equation as an equation or system of equations for which known methods from the theory of partial differential equations may be applied. There is indeed a general short-time existence theory which applies to strictly parabolic second order partial differential equations in vector bundles over compact manifolds. Unfortunately, this cannot be applied to the Ricci flow due to the lack of *strict* parabolicity.

For nonlinear equations, parabolicity is determined by the *linearization*.

Lemma 8.1 (Linearization of the Ricci flow[8]). *Suppose that the two parameter family of metrics g_t^ε, $t \in I$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, forms a one-parameter family of Ricci flows $\{g_t^\varepsilon\}_{t\in I}$ about $g_t \doteqdot g_t^0$. The variation field $h_t \doteqdot \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_t^\varepsilon$ satisfies, in any local coordinate chart,*

[8] See, for example, Andrews and Hopper, *The Ricci flow in Riemannian geometry*, §5.2.

$$\frac{\partial h_{ij}}{\partial t} = g^{k\ell}\left(\nabla_k\nabla_\ell h_{ij} + \nabla_i\nabla_j h_{k\ell} - \nabla_\ell\nabla_j h_{ik} - \nabla_k\nabla_i h_{j\ell}\right). \tag{8.4}$$

After some calculation, the equation (8.4) can be seen to be weakly, but not strictly, parabolic[9]. It turns out that the lack of strict parabolicity is essentially due to the Bianchi identities. Treating the Bianchi identities as a constraint, Hamilton[10] is able to prove short-time ex-

[9] See, for example, ibid.

[10] Richard S. Hamilton, "Three-manifolds with positive Ricci curvature".

istence using direct methods (in particular, the Nash–Moser implicit function theorem). Soon after Hamilton's work, de Turck found a way to relate the Ricci flow to a strictly parabolic equation, to which the standard theory may be more readily applied.

Theorem 8.2 (Short-time existence and uniqueness)**.** *Let M^n be a compact manifold. Given a metric g_0 on M there exists $\delta > 0$ and a Ricci flow $\{g_t\}_{t\in(0,\delta)}$ on M such that g_t converges uniformly to g_0 as $t \to 0$ (in the smooth sense if g_0 is smooth). Moreover, any other Ricci flow starting from g_0 agrees with g_t on their common interval of existence. Finally, the Ricci flow $\{g_t\}_{t\in(0,\delta)}$ depends continuously on g_0 (in the smooth sense if g_0 is smooth).*

Sketch of the de Turck argument[11]. Fix some background metric $\overline{g}$ on M and consider, instead of the Ricci flow, the Ricci-harmonic map flow system

$$\begin{cases} \frac{d}{dt}\Phi_t = \Delta_{g_t,\overline{g}}\Phi_t \\ \frac{d}{dt}g_t = -2\mathrm{Rc}_{g_t}, \end{cases} \tag{8.5}$$

where $\Phi_t : M \to M$ and $\Delta_{g_t,\overline{g}}$ is the map Laplacian with the domain endowed with the metric g_t and the codomain endowed with $\overline{g}$. In fact, don't consider (8.5); consider instead the system

$$\begin{cases} \frac{d}{dt}\Phi_t = \Delta_{\Phi_t^*\tilde{g}_t,\overline{g}}\Phi_t \\ \frac{d}{dt}\tilde{g}_t = -2\mathrm{Rc}_{\tilde{g}_t} - \mathcal{L}_{(\Phi_t^{-1})_*\frac{d}{dt}\Phi_t}\tilde{g}_t, \end{cases} \tag{8.6}$$

which is related to (8.5) by $g_t \doteqdot \Phi_t^*\tilde{g}_t$. The system (8.6) is *strictly* parabolic, and hence admits a (unique) solution $\{(\Phi_t, g_t)\}_{t\in[0,\delta)}$ for a short-time (which depends continuously on g_0), thereby providing the desired Ricci flow $\{g_t \doteqdot \Phi_t^*\tilde{g}_t\}_{t\in[0,\delta)}$. □

[11] For a more in-depth presentation of de Turck's argument, in particular its relation to the Bianchi identities, see, e.g. Andrews and Hopper, *The Ricci flow in Riemannian geometry*.

8.5 *The time-dependent geometric formalism*

A one-parameter family $\{g_t\}_{t\in I}$ of metrics $g_t \in \Gamma(T^*M \otimes T^*M)$ may (perhaps more properly) be viewed as a map $(x,t) \mapsto g_{(x,t)} \doteqdot (g_t)_x \in T^*M \otimes T^*M$ Any such map may be exhibited as a section of a bundle over $M \times I$ whose fibres are those of $T^*M \otimes T^*M$. Indeed, if we introduce the SPATIAL TANGENT BUNDLE[12]

$$\mathfrak{S} \doteqdot \{\xi \in T(M \times I) : dt(\xi) = 0\}$$

of $M \times I$, then any $(x,t) \mapsto g_{(x,t)}$ induces (canonically) a section g of $\mathfrak{S}^* \otimes \mathfrak{S}^*$, which we shall refer to as a TIME-DEPENDENT METRIC. Similarly, the Ricci tensors Rc_t of the metrics g_t induce a section Rc of

[12] Here, $t : M \times I \to \mathbb{R}$ denotes the projection onto the second factor.

Note that the fibres of $\mathfrak{S}$ are canonically identified with those of TM; however, $\mathfrak{S}$ is a bundle over $M^n \times I$ (not M^n), which means that its sections are "time-dependent".

$\mathfrak{S}^* \otimes \mathfrak{S}^*$. From this point of view, the Ricci flow equation becomes

$$\mathcal{L}_{\partial_t} g = -2\mathrm{Rc}, \tag{8.7}$$

where ∂_t is the CANONICAL VECTOR FIELD on I. Indeed, since the flow of ∂_t is $\phi_s(x,t) = (x, t+s)$,

$$\mathcal{L}_{\partial_t} g_{(x,t)} = \left.\frac{d}{ds}\right|_{s=0} (\phi_s^* g)_{(x,t)} = \left.\frac{d}{ds}\right|_{s=0} g_{(x,t+s)} = \frac{d}{dt}(g_t)_x.$$

This may seem like abstract nonsense (and it is), but it does have a more pragmatic purpose: any time-dependent metric g induces a natural (and computationally convenient) "time-dependent" geometric formalism on M^n, which is entirely analogous to the geometric formalism induced by a (time-independent) metric. In particular, it induces a canonical (compatible) connection.

Proposition 8.3 (The time-dependent connection)**.** *Given any metric g on the spatial tangent bundle $\mathfrak{S}$ of $M \times I$ there exists a unique connection*[13] $\nabla : T(M \times I) \times \Gamma(\mathfrak{S}) \to \mathfrak{S}$ *on $\mathfrak{S}$ which is*

[13] called the TIME-DEPENDENT CONNECTION

1. METRIC*: for any $U, V \in \Gamma(\mathfrak{S})$ and $\xi \in T(M \times I)$,*

$$0 = \nabla_\xi g(U,V) \doteqdot \xi g(U,V) - g(\nabla_\xi U, V) - g(U, \nabla_\xi V)$$

2. SPATIALLY SYMMETRIC*: for any $U, V \in \Gamma(\mathfrak{S})$,*

$$\nabla_U V - \nabla_V U = [U,V],$$

and

3. IRROTATIONAL*: the tensor $\mathrm{S} \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S})$ defined by*

$$\mathrm{S}(V) \doteqdot \nabla_t V - [\partial_t, V]$$

is g-self-adjoint.

Proof. Observe that the properties (1)–(3) yield

$$\begin{aligned} 0 &= \partial_t(g(U,V)) - g(\nabla_t U, V) - g(U, \nabla_t V) \\ &= \mathcal{L}_{\partial_t} g(U,V) - g(\mathrm{S}(U), V) - g(U, \mathrm{S}(V)) \\ &= \mathcal{L}_{\partial_t} g(U,V) - 2\,\mathrm{S}(U,V), \end{aligned}$$

and hence

$$\mathrm{S} = \tfrac{1}{2}\mathcal{L}_{\partial_t} g.$$

We shall thus refer to S as the VARIATION TENSOR of g.

In particular, along a Ricci flow,

$$\mathrm{S} = -\mathrm{Rc},$$

and hence, for any time dependent vector field $V \in \Gamma(\mathfrak{S})$, we have the formula

$$\nabla_t V = [\partial_t, V] - \mathrm{Rc}(V). \tag{8.8}$$

Since $T(M \times I) = \mathfrak{S} \oplus \mathbb{R}\partial_t$ and the properties (1) and (2) ensure that $\nabla_\xi V$ satisfies the Levi–Civita formula when $\xi \in \mathfrak{S}$, this completely determines ∇.

Conversely, the Levi–Civita formula combined with (8.8) defines a connection on $\mathfrak{S}$. □

The time-dependent connection provides a natural notion of differentiation in the time direction of TIME-DEPENDENT VECTOR FIELDS (sections of $\mathfrak{S}$). The upshot is that this notion is computationally very convenient, as it is compatible with the time-dependent metric.

In the sequel, when it is clear that we are working in the "time-dependent" setting, we shall conflate $\mathfrak{S}$ with TM and we will often use the data $(M \times I, g)$ (where g is a time-dependent metric satisfying (8.7)) to denote a Ricci flow.

8.6 Exercises

Exercise 8.1. Consider metric on S^2 which takes the form

$$g = dr^2 + \psi^2(r)d\theta^2$$

in spherical polar coordinates.

(a) Show that the sectional curvature of g is given by

$$\mathrm{K} = -\frac{\psi_{rr}}{\psi}.$$

(b) Suppose that g satisfies

$$\mathrm{Rc} = g + \tfrac{1}{2}\mathcal{L}_V g$$

for some radial vector field $V = f(r)\partial_r$. Find f, and hence determine g.

Exercise 8.2. Consider a time-dependent metric g which is locally of the form

$$g_{(x,\theta,t)} = y^2(x,t)\,dx^2 + x^2\,d\theta^2$$

(a) Show that g satisfies

$$\partial_t g = -2\mathrm{Rc} + 2\nabla^2\varphi$$

for some potential function φ if and only if y and φ satisfy the system

$$\begin{cases} y_t = x\left(\dfrac{y_x}{xy^2}\right)_x \\ \varphi_x = \dfrac{y_x}{y^3}. \end{cases} \tag{8.9}$$

(b) Find all time-independent solutions to (8.9).

(c) Which (if any) of these solutions extend to a complete metric?

Exercise 8.3. (a) Show that the Levi–Civita connections ∇^ε of a one-parameter family of metrics g_ε on a manifold M^n satisfy

$$2\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\nabla^\varepsilon\right)(X,Y,Z) = \nabla^\varepsilon_Z h(X,Y) - \nabla^\varepsilon_X h(Y,Z) - \nabla^\varepsilon_Y h(X,Z), \quad (8.10)$$

where $g \doteqdot g_0$, $h \doteqdot \frac{d}{d\varepsilon}\big|_{\varepsilon=0} g_\varepsilon$, and

$$\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\nabla^\varepsilon\right)(X,Y,Z) \doteqdot g\left(\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\nabla^\varepsilon\right)(X,Y),Z\right).$$

(b) Deduce that, under Ricci flow,

$$\frac{d\nabla^{g_t}}{dt}(X,Y,Z) = \nabla^{g_t}_X \operatorname{Rc}_{g_t}(Y,Z) + \nabla^{g_t}_Y \operatorname{Rc}_{g_t}(X,Z) - \nabla^{g_t}_Z \operatorname{Rc}_{g_t}(X,Y). \quad (8.11)$$

Exercise 8.4. Show that the map $V \mapsto \mathrm{S}(V)$ of Proposition 8.3 is indeed linear over the ring of smooth functions and takes values in $\Gamma(\mathfrak{S})$ (and hence induces a genuine tensor field $\mathrm{S} \in \Gamma(\mathfrak{S}^* \otimes \mathfrak{S})$ as claimed).

Exercise 8.5. Equip the time-dependent Riemannian manifold $(M^n \times I, g)$ with its time-dependent connection ∇. Given $(x_0, t_0) \in M^n \times I$ and sufficiently small ε, define the PARALLEL TRANSPORT maps $\tau_\varepsilon : T_{x_0}M^n \to T_{x_0}M^n$ by

$$\tau_\varepsilon(u) \doteqdot U(t_0 + \varepsilon),$$

where, for each $u \in T_{x_0}M^n$, $t \mapsto U(t) \in T_{x_0}M^n$ is the unique solution to

$$\begin{cases} \nabla_t U = 0 \\ U(t_0) = u. \end{cases}$$

(a) Show that τ_ε is an isometry for each ε.

(b) Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\tau_\varepsilon(u) = \mathrm{S}_{(x_0,t_0)}(u)$.

(c) Deduce that the projection of $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\tau_\varepsilon$ onto $\mathfrak{so}(T_x M^n, g_{t_0})$ vanishes.

(This justifies the term "irrotational" to describe the third defining property of the time-dependent connection.)

Exercise 8.6. Show that, on any Riemannian manifold equipped with its Levi–Civita connection,

(a) $\Delta\nabla f = \nabla\Delta f + \operatorname{Rc}(\nabla f)$ for any function f.

Show that, along any Ricci flow equipped with its time-dependent connection,

(c) $\nabla_t \nabla f = \nabla \partial_t f + \mathrm{Rc}(\nabla f)$ and

(d) $\partial_t \Delta f = \Delta \partial_t f + 2g(\mathrm{Rc}, \nabla^2 f)$ for any time-dependent function f.

Exercise 8.7. Let g be a time-dependent metric on $M^n \times I$. Equip $M^n \times I$ with its canonical time-dependent connection ∇. Prove that

$$\mathrm{Rm}(\xi, X, Y, Z) + \mathrm{Rm}(\xi, X, Z, Y) = 0$$

for any $\xi \in T(M \times I)$ and $X, Y, Z \in TM$. *Hint: The argument is the usual one: since the time-dependent connection is metric compatible, so too is the curvature operator:* $\mathrm{Rm}(\partial_t, X)g = 0$ *for any* $X \in TM$.

Exercise 8.8 (Bianchi identities for the time-dependent connection)**.** Let g be a time-dependent metric on $M^n \times I$. Equip $M^n \times I$ with its canonical time-dependent connection ∇. Prove the following identities.

(a) $\mathrm{Rm}(\partial_t, X)Y + \mathrm{Rm}(Y, \partial_t)X = \nabla_X \mathrm{S}(Y) - \nabla_Y \mathrm{S}(X)$.

(b) $\nabla_{\partial_t}(\mathrm{Rm}(X, Y)) = \nabla_X(\mathrm{Rm}(\partial_t, Y)) - \nabla_Y(\mathrm{Rm}(\partial_t, X))$.

Deduce from part (a) that

(c) $\mathrm{Rm}(\partial_t, X, Y, Z) = \nabla_Z \mathrm{S}(X, Y) - \nabla_Y \mathrm{S}(Z, X)$.

9
The groundwork

9.1 The maximum principle

The maximum principle is a fundamental tool in the analysis of partial differential equations of parabolic type, and the Ricci flow is no exception. Indeed, in the context of Ricci flow, the maximum principle exhibits multiple useful manifestations.

9.1.1 Maximum principle for scalars

Proposition 9.1. *Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies*

$$(\partial_t - \Delta - \nabla_b - c)u \leq 0$$

for some time-dependent vector field b and some locally bounded function $c : M^n \times [0,T) \to \mathbb{R}$, where the Laplacian Δ is taken with respect to the time-dependent metric g. If $\max_{M^n \times \{0\}} u \leq 0$, then

$$\max_{M^n \times \{t\}} u \leq 0 \;\; \text{for all} \;\; t \in [0,T]. \tag{9.1}$$

If $c \equiv 0$, then

$$\max_{M^n \times [0,T]} u = \max_{M^n \times \{0\}} u. \tag{9.2}$$

Proof. Given $\sigma \in (0,T)$ and $\varepsilon > 0$, consider $u_{\sigma,\varepsilon}(x,t) \doteqdot u(x,t) - \varepsilon \mathrm{e}^{(C+1)t}$, where $C \doteqdot \max_{M^n \times [0,\sigma]} c$. We claim that $u_{\sigma,\varepsilon} < 0$ in $M^n \times [0,\sigma]$. Suppose, to the contrary, that $u_{\sigma,\varepsilon}(x_0,t_0) \geq 0$ for some point $(x_0,t_0) \in M^n \times [0,\sigma]$. Since $u_{\sigma,\varepsilon}(\cdot,0) < 0$, there exists a positive earliest such time, which we take to be t_0, in which case $u(x_0,t_0) = 0$. At the point (x_0,t_0),

$$\begin{aligned} 0 \leq (\partial_t - \Delta - \nabla_b)u_{\sigma,\varepsilon} &\leq cu - \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &= \varepsilon \mathrm{e}^{(C+1)t} c - \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &\leq -\varepsilon \mathrm{e}^{(C+1)t} < 0, \end{aligned}$$

which is absurd. We conclude that $u_{\sigma,\varepsilon} < 0$ in $M^n \times [0,\sigma]$. But $\sigma \in (0,T)$ and $\varepsilon > 0$ were arbitrary. Taking $\varepsilon \to 0$ and then $\sigma \to T$ yields the claim. □

Of course, the same argument applies with the inequalities reversed, leading to a *minimum principle*.

The following ODE COMPARISON PRINCIPLE is an immediate consequence of the maximum principle.

Proposition 9.2 (ODE comparison principle)**.** *Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T)) \cap C^0(M^n \times [0,T))$ satisfies*

$$(\partial_t - \Delta - \nabla_b)u \leq F(u)\,, \tag{9.3}$$

for some time-dependent vector field b and some locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$, where the Laplacian Δ and covariant derivative ∇ are taken with respect to the time-dependent metric g. If $u \leq \phi_0$ at $t = 0$ for some $\phi_0 \in \mathbb{R}$, then $u(x,t) \leq \phi(t)$ for all $x \in M^n$ and $0 \leq t < T$, where ϕ is the solution to the ODE

$$\begin{cases} \dfrac{d\phi}{dt} = F(\phi) \quad in \quad (0,T)\,, \\ \phi(0) = \phi_0\,. \end{cases} \tag{9.4}$$

Proof. Fix $s \in (0,T)$. Since F is locally Lipschitz, there exists some $L < \infty$ such that

$$\begin{aligned}(\partial_t - \Delta - \nabla_b)(u-\phi) &\leq F(u) - F(\phi) \\ &\leq L|u-\phi| = L\,\mathrm{sign}(u-\phi)(u-\phi)\end{aligned}$$

in $M^n \times (0,s]$, where $\mathrm{sign}(u-\phi)$ is the sign of the expression $u-\phi$. The claim now follows, within $M^n \times [0,s]$, from Theorem 9.1. Taking $s \to T$ completes the proof. □

Again, one can reverse the inequalities to obtain the corresponding ODE comparison from below.

The strong maximum principle also passes to the geometric setting.

Proposition 9.3. *Let $(M^n \times (0,T), g)$ be a Ricci flow on a connected manifold M^n. Suppose that $u \in C^\infty(M^n \times (0,T))$ is nonpositive and satisfies*

$$(\partial_t - \Delta - \nabla_b - c)u \leq 0 \tag{9.5}$$

for some time-dependent vector field b and some function $c : M^n \times (0,T) \to \mathbb{R}$, where the Laplacian Δ and covariant derivative ∇ are taken with respect to the time dependent metric g. If $u(x_0,t_0) = 0$ for some $(x_0,t_0) \in M^n \times (0,T)$, then $u(x,t) = 0$ for all $(x,t) \in M^n \times (0,t_0]$.

Proof. In local coordinates $\{x^i\}_{i=1}^n$ for a connected coordinate patch $U \subset M^n$ about x_0, u satisfies

$$\partial_t u \leq g^{ij}u_{ij} + (b^k + g^{ij}\Gamma_{ij}{}^k)u_k + cu\,.$$

The classical strong maximum principle then implies that $u \equiv 0$ in $U \times (0, t_0]$. Since M^n is connected, the claim follows from a standard 'open-closed' argument. □

9.1.2 A maximum principle for symmetric bilinear forms

Hamilton[1] discovered the following beautiful maximum principle for symmetric bilinear forms.

[1] Richard S. Hamilton, "Three-manifolds with positive Ricci curvature".

Proposition 9.4 (Tensor maximum principle)**.** *Let $(M^n \times [0, T), g)$ be a Ricci flow on a compact manifold M^n. Suppose that $S \in \Gamma(T^*M^n \odot T^*M^n)$ satisfies*

$$(\nabla_t - \Delta - \nabla_b)S_{(x,t)}(v, v) \geq F(x, t, S_{(x,t)})(v, v) \;\; \text{for all} \;\; (x, t, v) \in TM^n$$

for some time-dependent vector field $b \in \Gamma(TM)$ and some TIME-DEPENDENT VERTICAL VECTOR FIELD*—a (time-dependent) section F of $\pi^*(T^*M^n \odot T^*M^n)$—which is Lipschitz in the fibre and satisfies the* NULL EIGENVECTOR CONDITION*:*

$$F(x, t, T_{(x,t)})(v, v) \geq 0 \;\; \text{whenever} \;\; T_{(x,t)} \geq 0 \;\; \text{and} \;\; T_{(x,t)}(v) = 0,$$

where ∇ and Δ are the time-dependent connection and (spatial) Laplacian induced by the time-dependent metric g. If $S_{(x,0)} \geq 0$ for all $x \in M^n$, then $S_{(x,t)} \geq 0$ for all $(x, t) \in M^n \times [0, T)$.

Proof. Fix $\sigma \in (0, T)$ and $\varepsilon > 0$. Setting $C \doteqdot \max_{(x,t)\in M^n \times [0,\sigma]} \operatorname{Lip} F(x, t, \cdot)$, we will show that the function $q_{\sigma,\varepsilon} : TM^n \to \mathbb{R}$ defined by

$$q_{\sigma,\varepsilon}(x, y, t) \doteqdot \left(S_{(x,t)} + \varepsilon e^{(C+1)t} g_{(x,t)}\right)(y, y)$$

is positive in $TM^n|_{M^n \times [0,\sigma]} \setminus \{0\}$. By hypothesis, $q_{\sigma,\varepsilon}(x, 0, y) > 0$ for all $y \in T_x M^n$, $x \in M^n$. So suppose, contrary to the claim, that there exist[2] $(x_0, y_0, t_0) \in TM^n|_{M^n \times (0,\sigma]}$ such that $q_{\sigma,\varepsilon}(x, y, t) > 0$ for each $(x, y, t) \in TM^n|_{M^n \times [0,t_0)}$ but $q_{\sigma,\varepsilon}(x_0, y_0, t_0) = 0$. Without loss of generality, $|y_0| = 1$. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ for $T_{x_0}M^n$ consisting of eigenvectors of $S_{(x_0,t_0)}$, with $y_0 = e_1$, and let $x^i : U \to \mathbb{R}^n$ be the corresponding local normal coordinate chart for M^n and $(x^i, y^j = dx^j) : \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^n$ the induced chart for TM^n. With respect to these coordinates,

[2] This is ensured (despite the noncompactness of $TM \setminus \{0\}$) by the homogeneity of $q_{\sigma,\varepsilon}$ with respect to y.

$$q_{\sigma,\varepsilon} = \left(S_{ij} + \varepsilon e^{(C+1)t} g_{ij}\right) y^i y^j\,.$$

We thereby compute

$$\partial_t q_{\sigma,\varepsilon} = \left(\partial_t S_{ij} + \varepsilon(C+1)\mathrm{e}^{(C+1)t} g_{ij} + \varepsilon \mathrm{e}^{(C+1)t}\partial_t g_{ij}\right) y^i y^j\,,$$

$$\partial_{x^k} q_{\sigma,\varepsilon} = \left(\partial_k S_{ij} + \varepsilon \mathrm{e}^{(C+1)t}\partial_k g_{ij}\right) y^i y^j\,,$$

and

$$\partial_{x^k}\partial_{x^\ell} q_{\sigma,\varepsilon} = \left(\partial_k\partial_\ell S_{ij} + \varepsilon \mathrm{e}^{(C+1)t}\partial_k\partial_\ell g_{ij}\right) y^i y^j\,.$$

Using the normal coordinate conditions, $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$, the vanishing condition $S_{11} = S_{ij}y^i y^j = -\varepsilon \mathrm{e}^{(C+1)t}$, and the gradient condition

$$0 = \left(\partial_k S_{ij} + \varepsilon \mathrm{e}^{(C+1)t}\partial_k g_{ij}\right) y^i y^j + 2\left(S_{kj} + \varepsilon \mathrm{e}^{(C+1)t} g_{kj}\right) y^j = \partial_k S_{11}\,,$$

at the point (x_0, y_0, t_0), we find that

$$\begin{aligned} 0 &\geq \left(\partial_t - \delta^{k\ell}\partial_{x^k}\partial_{x^\ell}\right) q_{\sigma,\varepsilon} \\ &= (\nabla_t - \Delta - \nabla_b) S_{11} + \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &\geq F(x_0, t_0, S_{(x_0,t_0)})_{11} + \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &\geq F(x_0, t_0, S^{\sigma,\varepsilon}_{(x_0,t_0)})_{11} - C\varepsilon \mathrm{e}^{(C+1)t} + \varepsilon(C+1)\mathrm{e}^{(C+1)t} \\ &\geq \varepsilon \mathrm{e}^{(C+1)t} \\ &> 0 \end{aligned}$$

at (x_0, y_0, t_0), which is absurd. We conclude that $q_{\sigma,\varepsilon}$ does indeed remain positive in $M^n \times [0, \sigma]$. The claim now follows by taking $\varepsilon \to 0$ and then $\sigma \to T$. □

9.1.3 A maximum principle for sections of vector bundles

There is even a version of the maximum principle for sections of a vector bundle. As the reader has likely already realized, the above maximum principles do not depend on the condition that the time-dependent metric g satisfies the Ricci flow equation. As such, we state the vector bundle maximum principle without this hypothesis.

Proposition 9.5 (Vector bundle maximum principle)**.** *Let M^n be a compact manifold and $\pi : E \to M^n \times [0, T)$ a time-dependent vector bundle over M^n which is equipped with a metric h and a metric connection ∇, and let $\Omega \subset E$ be a closed subset which is* CONVEX IN THE FIBRE *and* INVARIANT UNDER PARALLEL TRANSPORT*. Given any time-dependent vector field b on M^n and any* TIME-DEPENDENT VERTICAL VECTOR FIELD *$F \in \Gamma(\pi^* E \to E)$ which* POINTS INTO *Ω, any solution $u \in \Gamma(E)$ to*

$$(\nabla_t - \Delta - \nabla_b) u = F(u)$$

satisfying $u(x, 0) \in \Omega$ for all $x \in M$ satisfies $u(x, t) \in \Omega$ for all $x \in M$ and all $t \in [0, T)$.

The conditions of Proposition 9.5 are intuitive enough: a subset Ω of the (time-dependent) vector bundle $\pi : M \times [0,T) \to E$ is CONVEX IN THE FIBRE if its fibres $\Omega_{(x,t)} \doteqdot \Omega \cap E_{(x,t)}$ are convex subsets of $E_{(x,t)}$, and INVARIANT UNDER PARALLEL TRANSPORT if parallel translates of vectors in Ω along curves in $M \times [0,T)$ remain in Ω; a vertical vector field $F \in \Gamma(\pi^* E)$ POINTS INTO Ω AT $(x,t,v) \in \partial\Omega$ if $(x,t,v) + \varepsilon F(x,t,v) \in \Omega$ for all small $\varepsilon > 0$ (where addition is is understood fibrewise).

The proof of Proposition 9.5 uses tools from convex geometry, but is very similar in nature to that of Proposition 9.4 (see e.g.[3]); we omit it for the sake of brevity.

[3] Andrews and Hopper, *The Ricci flow in Riemannian geometry*.

9.2 *Evolution of geometry under Ricci flow*

The Ricci flow equation induces diffusion equations of various types for the various geometric attributes of the evolving metric.

9.2.1 *Distance distortion estimates*

Let $(M \times I, g)$ be a Ricci flow. Given any curve $\gamma : [0,L] \to M$ in M,

$$\begin{aligned}\frac{d}{dt}\operatorname{length}(\gamma) &= \frac{d}{dt}\int_0^L |\gamma'(s)|\,ds\\ &= \frac{d}{dt}\int_0^L \sqrt{g(\gamma'(s),\gamma'(s))}\,ds\\ &= -\int_0^L \operatorname{Rc}\left(\frac{\gamma'(s)}{|\gamma'(s)|},\frac{\gamma'(s)}{|\gamma'(s)|}\right)ds.\end{aligned}$$

Thus, the Ricci curvature determines the rate of change of lengths of curves. Applying this at minimizing geodesics yields the following elementary distance distortion estimates.

Proposition 9.6. *If $\underline{K} g \le \operatorname{Rc} \le \overline{K} g$ along a complete Ricci flow $(M \times [t_1, t_2], g)$, then*

$$\underline{K}\operatorname{dist}(x,y,t) \le -\frac{d}{dt}\operatorname{dist}(x,y,t) \le \overline{K}\operatorname{dist}(x,y,t)$$

in the (forward and backward, respectively) barrier sense, and in the classical sense almost everywhere. Furthermore,

$$\mathrm{e}^{-\overline{K}(t_2-t_1)} \le \frac{\operatorname{dist}(x,y,t_2)}{\operatorname{dist}(x,y,t_1)} \le \mathrm{e}^{-\underline{K}(t_2-t_1)}.$$

Proof. Given any two distinct points $x, y \in M$ and any time $t_0 \in [t_1, t_2]$, we can find a distance minimizing geodesic $\gamma : [0,L] \to M$ with respect to the metric at time t_0 which joins x and y. We may assume that γ is parametrized by arclength, so that the distance $d(x,y,t_0)$ between x

and y with respect to the metric at time t_0 is equal to L. By the above computation and the hypotheses,

$$\underline{K}\,\mathrm{length}(\gamma) \le -\frac{d}{dt}\,\mathrm{length}(\gamma) \le \overline{K}\,\mathrm{length}(\gamma)\,.$$

Since $\mathrm{length}(\gamma) \ge d(x, y, \cdot)$ with equality at time $t = t_0$, we have found a (forward resp. backward) barrier satisfying the inequalities. The a.e. classical inequality then follows because $t \mapsto \mathrm{dist}(x, y, t)$ is Lipschitz (and hence admits a classical derivative at a.e. time, which must be equal to that of the barrier because of the first order contact). We may then integrate to obtain the distance distortion estimates. □

These estimates are quite crude. The following argument (inspired by the proof of the Bonnet–Meyers theorem) provides a much sharper estimate on long geodesics.

Proposition 9.7. *If* $\mathrm{Rc} \le (n-1)Kg$ *for some* $K > 0$ *along a complete Ricci flow* $(M^n \times [t_1, t_2], g)$*, then*

$$-\frac{d}{dt}\,\mathrm{dist}(x, y, t) \le 10K^{\frac{1}{2}}$$

in the barrier sense, and in the classical sense almost everywhere. Thus,

$$\mathrm{dist}(x, y, t_2) \ge \mathrm{dist}(x, y, t_1) - 10K^{\frac{1}{2}}(t_2 - t_1)$$

Proof. Given any two distinct points $x, y \in M$ and any time $t_0 \in [t_1, t_2]$, we can find a distance minimizing geodesic $\gamma : [0, L] \to M$ with respect to the metric at time t_0 which joins x and y. We may assume that γ is parametrized by arclength, so that the distance $d(x, y, t_0)$ between x and y with respect to the metric at time t_0 is equal to L. By the above computation and the hypotheses,

$$-\frac{d}{dt}\,\mathrm{length}(\gamma, t) \le K\,\mathrm{length}(\gamma, t)\,.$$

In case $L \le 2K^{-\frac{1}{2}}$, we have

$$-\frac{d}{dt}\,\mathrm{length}(\gamma, t) \le 2K^{\frac{1}{2}} \le 10K^{\frac{1}{2}}$$

at $t = t_0$. The interesting case is $L \ge 2K^{\frac{1}{2}}$. Choose a parallel orthonormal frame $\{E_i\}_{i=1}^n$ along γ such that $E_1 = \gamma'$ and let $\varphi : [0, L] \to \mathbb{R}$ be a smooth function satisfying

$$0 \le \varphi \le 1\,, \quad \varphi|_{[K^{-\frac{1}{2}}, L-K^{-\frac{1}{2}}]}\,, \quad |\varphi'|^2 \le 4K^{\frac{1}{2}}\,.$$

For $i = 2, \ldots, n$, the second variation formula for length yields

$$\begin{aligned} 0 &\le \int_0^L \left[|\nabla_s(\varphi E_i)|^2 - \mathrm{Rm}(\gamma', \varphi E_i, \gamma', \varphi E_i) \right] ds \\ &= \int_0^L \left[|\varphi'|^2 - \varphi^2 \mathrm{Rm}(\gamma', E_i, \gamma', E_i) \right] ds \end{aligned}$$

at $t = t_0$, since γ is minimizing at $t = t_0$. Tracing, we obtain

$$0 \le \int_0^L \Big[(n-1)|\varphi'|^2 - \varphi^2 \mathrm{Rc}(\gamma'\gamma')\Big]\, ds\,.$$

Thus,

$$\begin{aligned}\int_0^L \mathrm{Rc}(\gamma',\gamma')\, ds &= \int_0^L \Big(\varphi^2\mathrm{Rc}(\gamma',\gamma') + (1-\varphi^2)\mathrm{Rc}(\gamma',\gamma')\Big)\, ds\\ &\le (n-1)\int_0^L \Big(|\varphi'|^2 + (1-\varphi^2)K\Big)\, ds\\ &= (n-1)\int_{[0,K^{-\frac{1}{2}}]\cup[d-K^{-\frac{1}{2}},d]} \Big(|\varphi'|^2 + (1-\varphi^2)K\Big)\, ds\\ &\le 10(n-1)K^{\frac{1}{2}}\,.\end{aligned}$$

The claims now follow as before. □

9.2.2 *The first variation of volume*

Recall that, on any Riemannian manifold (M^n, g), the Riemannian measure of any compact subset $K \subset M$ is defined by

$$\mathrm{volume}(K, g) = \int_K d\mu \doteqdot \sum_\alpha \int_{x_\alpha(U_\alpha)} (x_\alpha^{-1})^*\big(\rho_\alpha \sqrt{\det g_\alpha}\big)\, dx\,,$$

where $\{(U_\alpha, x_\alpha)\}_\alpha$ is any locally finite covering of K, $\{\rho_\alpha\}_\alpha$ is any subordinate partition of unity, dx is the Lebesgue measure on $\mathbb{R}^n$, and g_α is the component matrix of g induced by the α-th chart. If $\{g_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a one-parameter family of metrics on M^n with $g_0 = g$ and $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_\varepsilon = h$, then, with respect to any coordinate chart,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \sqrt{\det g_\varepsilon} = \frac{1}{2}\sqrt{\det g}\, \mathrm{tr}_g\, h\,. \tag{9.6}$$

We thus obtain the FIRST VARIATION FORMULA:

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{volume}(K, g_\varepsilon) = -\frac{1}{2}\int_K \mathrm{tr}_g\, h\, d\mu\,. \tag{9.7}$$

In particular,

Proposition 9.8. *along a Ricci flow $(M \times I, g)$,*

$$\frac{d}{dt}\mathrm{volume}(K,\cdot) = -\int_K \mathrm{R}\, d\mu \tag{9.8}$$

for any compact $K \subset M$, where $\mathrm{volume}(K,t) \doteqdot \mathrm{volume}(K, g_t)$.

9.2.3 *Evolution of the Ricci and scalar curvatures*

Given a Ricci flow $\{g_t\}_{t\in I}$, applying Lemma 8.1 to the one parameter family $\{g_t^\varepsilon \doteqdot g_{t+\varepsilon}\}_{t+\varepsilon\in I}$ of time translated Ricci flows yields

$$\begin{aligned}\frac{d}{dt}\mathrm{Rc}_{ij} &= g^{k\ell}\left(\nabla_k\nabla_\ell\mathrm{Rc}_{ij} + \nabla_i\nabla_j\mathrm{Rc}_{k\ell} - \nabla_\ell\nabla_j\mathrm{Rc}_{ik} - \nabla_k\nabla_i\mathrm{Rc}_{j\ell}\right)\\ &= \Delta\mathrm{Rc}_{ij} + Q_{ij},\end{aligned}$$

where

$$\begin{aligned}Q_{ij} &\doteqdot g^{k\ell}\left(\nabla_i\nabla_j\mathrm{Rc}_{k\ell} - \nabla_k\nabla_j\mathrm{Rc}_{i\ell} - \nabla_k\nabla_i\mathrm{Rc}_{j\ell}\right)\\ &= g^{k\ell}g^{pq}\left(\nabla_i\nabla_j\mathrm{Rm}_{kp\ell q} - \nabla_k\nabla_j\mathrm{Rm}_{ip\ell q} - \nabla_k\nabla_i\mathrm{Rm}_{jp\ell q}\right).\end{aligned}$$

We can write this in a more natural way by applying the Bianchi identities and making use of the time-dependent connection.

Proposition 9.9. *Along a Ricci flow $(M^n\times I, g)$,*

$$(\nabla_t - \Delta)\mathrm{Rc} = Q(\mathrm{Rc}), \tag{9.9}$$

where, with respect to any local basis,

$$Q(\mathrm{Rc})_{ij} \doteqdot 2\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{k\ell}.$$

Proof. With respect to (time-independent) local coordinates about a given point of M^n, the second Bianchi identity and the definition and symmetries of curvature yield

$$\begin{aligned}g^{k\ell}g^{pq}\nabla_i\nabla_j\mathrm{Rm}_{kp\ell q} &= -g^{k\ell}g^{pq}\left(\nabla_i\nabla_k\mathrm{Rm}_{pj\ell q} + \nabla_i\nabla_p\mathrm{Rm}_{jk\ell q}\right)\\ &= -g^{k\ell}g^{pq}\left(\nabla_i\nabla_k\mathrm{Rm}_{pj\ell q} + \nabla_i\nabla_k\mathrm{Rm}_{jpq\ell}\right)\\ &= -2g^{k\ell}g^{pq}\nabla_i\nabla_k\mathrm{Rm}_{pj\ell q}\\ &= -2g^{k\ell}g^{pq}\left(\nabla_k\nabla_i\mathrm{Rm}_{pj\ell q} + (\mathrm{Rm}_{ki}\mathrm{Rm})_{pj\ell q}\right)\\ &= 2g^{k\ell}g^{pq}\left(\nabla_k\nabla_i\mathrm{Rm}_{jp\ell q} + (\mathrm{Rm}_{ik}\mathrm{Rm})_{pj\ell q}\right)\end{aligned}$$

The second and first Bianchi identities then yield

$$\begin{aligned}Q_{ij} &= g^{k\ell}g^{pq}\left(2(\mathrm{Rm}_{ik}\mathrm{Rm})_{pj\ell q} + \nabla_k\nabla_i\mathrm{Rm}_{jp\ell q} - \nabla_k\nabla_j\mathrm{Rm}_{ip\ell q}\right)\\ &= g^{k\ell}g^{pq}\left(2(\mathrm{Rm}_{ik}\mathrm{Rm})_{pj\ell q} - \nabla_k\nabla_p\mathrm{Rm}_{ij\ell q}\right)\\ &= g^{k\ell}g^{pq}\left(2(\mathrm{Rm}_{ik}\mathrm{Rm})_{pj\ell q} + \nabla_k\nabla_p\mathrm{Rm}_{iqj\ell} - \nabla_k\nabla_p\mathrm{Rm}_{i\ell jq}\right)\\ &= -2g^{k\ell}(\mathrm{Rm}_{ik}\mathrm{Rc})_{j\ell} + g^{k\ell}g^{pq}\left(\nabla_k\nabla_p\mathrm{Rm}_{iqj\ell} - \nabla_p\nabla_k\mathrm{Rm}_{iqj\ell}\right)\\ &= 2\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{k\ell} - 2\mathrm{Rc}^2_{ij} + g^{k\ell}g^{pq}(\mathrm{Rm}_{pk}\mathrm{Rm})_{iqj\ell}.\end{aligned}$$

Observe that[4]

[4] There is an easier way to see that this term vanishes: since the terms $Q_{ij} = (\frac{d}{dt} - \Delta)\mathrm{Rc}_{ij}$ and $2(\mathrm{Rm}_{ik}\mathrm{Rm})_{pj\ell q} = 2\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{k\ell} - 2\mathrm{Rc}^2_{ij}$ are symmetric, so must be the remainder, $g^{k\ell}g^{pq}(\nabla_k\nabla_i\mathrm{Rm}_{jp\ell q} - \nabla_k\nabla_j\mathrm{Rm}_{ip\ell q})$. But this term is clearly skew-symmetric.

$$\begin{aligned}
g^{k\ell}g^{pq}(\mathrm{Rm}_{pk}\mathrm{Rm})_{iqj\ell} = &- g^{k\ell}g^{pq}g^{mn}\Big(\mathrm{Rm}_{pkim}\mathrm{Rm}_{nqj\ell}+\mathrm{Rm}_{pkqm}\mathrm{Rm}_{inj\ell}\\
&+\mathrm{Rm}_{pkjm}\mathrm{Rm}_{iqn\ell}+\mathrm{Rm}_{pk\ell m}\mathrm{Rm}_{iqjn}\Big)\\
&= g^{k\ell}g^{pq}g^{mn}\Big(\mathrm{Rm}_{kpim}\mathrm{Rm}_{nqj\ell}-\mathrm{Rm}_{n\ell iq}\mathrm{Rm}_{pkjm}\\
&\mathrm{Rm}_{mqpk}\mathrm{Rm}_{inj\ell}-\mathrm{Rm}_{pk\ell m}\mathrm{Rm}_{iqjn}\Big)\\
&= 0.
\end{aligned}$$

So the claim follows upon applying the identity

$$\nabla_t \mathrm{Rc}_{ij} = \frac{d}{dt}\mathrm{Rc}_{ij} + 2\mathrm{Rc}^2_{ij}. \qquad \square$$

Taking the trace of (9.9), we find that

Corollary 9.10. *along a Ricci flow $(M^n \times I, g)$,*

$$(\partial_t - \Delta)\mathrm{R} = 2|\mathrm{Rc}|^2. \tag{9.10}$$

Applying the maximum principle to these equations yields useful information about the behaviour of curvature under Ricci flow.

Proposition 9.11 (Scalar curvature tends towards positive)**.** *Let $\{g_t\}_{t\in[\alpha,\omega)}$ be a Ricci flow on a compact manifold M.*

1. *If* $\min_{M\times\{\alpha\}} \mathrm{R} = 0$ *then either* $\mathrm{Rc} \equiv 0$ *or* $\mathrm{R} > 0$ *for* $t \in (\alpha, \omega)$.
2. *If* $\min_{M\times\{\alpha\}} \mathrm{R} = n(n-1)r^{-2} > 0$*, then* $\omega \leq \alpha + \frac{n}{2}r^2$ *and*

$$\min_{M\times\{t\}} \mathrm{R} \geq \frac{n(n-1)}{r^2 - 2(n-1)(t-\alpha)}$$

   for $t \in (\alpha, \omega)$.
3. *If* $\min_{M\times\{\alpha\}} \mathrm{R} = -n(n-1)r^{-2} < 0$*, then*

$$\min_{M\times\{t\}} \mathrm{R} \geq -\frac{n(n-1)}{r^2 + 2(n-1)(t-\alpha)}$$

   for $t \in (\alpha, \omega)$.

Proof. In the first case, the maximum principle ensures that R remains nonnegative due to (9.10). The strong maximum principle then guarantees that either $\mathrm{R} > 0$ at interior times, or $\mathrm{R} \equiv 0$. But in the latter case, (9.10) implies $|\mathrm{Rc}| \equiv 0$.

Now, since

$$|\mathrm{Rc}|^2 \geq \tfrac{1}{n}\mathrm{R}^2,$$

we may estimate

$$\partial_t \mathrm{R} \geq \Delta \mathrm{R} + \tfrac{2}{n}\mathrm{R}^2.$$

The ODE comparison principle then yields the remaining claims. $\square$

Proposition 9.11 tells us two important facts. First, a Ricci flow with positive scalar curvature on a compact manifold must become singular in finite time. Second if a Ricci flow on a compact manifold happens to exist on a very large time interval, then the scalar curvature is almost nonnegative at the end time. In particular, if the flow has an infinite past, then the scalar curvature is nonnegative in the present.

Corollary 9.12. *For any* ANCIENT[5] *Ricci flow* $(M^n \times (-\infty, \omega), g)$ *on a compact manifold* M^n*, either* $\mathrm{R} > 0$ *or* $\mathrm{Rc} \equiv 0$*.*

[5] I.e. having an infinite past.

9.2.4 *Evolution of the curvature operator*

It is also possible to derive an evolution equation for the full curvature tensor Rm.

Proposition 9.13. *Along a Ricci flow* $(M^n \times I, g)$*,*

$$(\nabla_t - \Delta)\mathrm{Rm} = 2\big(\mathrm{Rm}^2 + \mathrm{Rm}^\#\big)\,, \tag{9.11}$$

where, as operators on vector fields,

$$\mathrm{Rm}^2(X,Y) \doteqdot \tfrac{1}{2}\,\mathrm{tr}\,\mathrm{Rm}(\mathrm{Rm}(X,Y)\cdot,\cdot)$$

and

$$\mathrm{Rm}^\#(X,Y) \doteqdot \mathrm{tr}\,\big[\mathrm{Rm}(X,\cdot),\mathrm{Rm}(Y,\cdot)\big]\,,$$

or, with respect to a local orthonormal frame,

$$\mathrm{Rm}^2_{ijk\ell} = \tfrac{1}{2}\mathrm{Rm}_{ijpq}\mathrm{Rm}_{k\ell pq}$$

and

$$\mathrm{Rm}^\#_{ijk\ell} = (\mathrm{Rm}_{ipkq}\mathrm{Rm}_{jp\ell q} - \mathrm{Rm}_{ip\ell q}\mathrm{Rm}_{jpkq})\,.$$

Sketch of the proof[6]. On the one hand, the Bianchi identities can be exploited to write the Laplacian of the curvature tensor as

[6] See Exercise 9.3 or, e.g., Andrews and Hopper, *The Ricci flow in Riemannian geometry*; Chow and Knopf, *The Ricci flow: an introduction*; Chow, Lu, and Ni, *Hamilton's Ricci flow*.

$$\begin{aligned}\Delta\mathrm{Rm}(X,Y,Z,W) = {} & \nabla_X\nabla_Z\mathrm{Rc}(Y,W) - \nabla_X\nabla_W\mathrm{Rc}(Y,Z)\\ & - \nabla_Y\nabla_Z\mathrm{Rc}(X,W) + \nabla_Y\nabla_W\mathrm{Rc}(X,Z)\\ & - \mathrm{Rm}(\mathrm{Rc}(X),Y,Z,W) + \mathrm{Rm}(X,\mathrm{Rc}(Y),Z,W)\\ & - 2\big(B(X,Y,Z,W) - B(X,Y,W,Z)\\ & \quad + B(X,Z,Y,W) - B(X,W,Y,Z)\big)\,,\end{aligned}$$

where the tensor B is defined by

$$B(X,Y,Z,W) \doteqdot g(\mathrm{Rm}(X,\cdot,Y,*),\mathrm{Rm}(Z,\cdot,W,*))\,.$$

On the other hand, the "time-dependent Bianchi identities" of Exercise 8.8 can be exploited to write the covariant time-derivative of the curvature tensor as

$$\begin{aligned}\nabla_t \mathrm{Rm}(X,Y,Z,W) = \nabla_X\nabla_Z \mathrm{Rc}(Y,W) - \nabla_X\nabla_W \mathrm{Rc}(Y,Z)\\ - \nabla_Y\nabla_Z \mathrm{Rc}(X,W) + \nabla_Y\nabla_W \mathrm{Rc}(X,Z)\\ - \mathrm{Rm}(\mathrm{Rc}(X),Y,Z,W) + \mathrm{Rm}(X,\mathrm{Rc}(Y),Z,W).\end{aligned}$$

The claim then follows upon recognizing that

$$\begin{aligned}(\mathrm{Rm}^2 + \mathrm{Rm}^\#)(X,Y,Z,W) = B(X,Y,Z,W) - B(X,Y,W,Z)\\ + B(X,Z,Y,W) - B(X,W,Y,Z). \quad \square\end{aligned}$$

The terms on the right hand side of (9.11) have a natural algebraic interpretation. Indeed, the term Rm^2 is at each point the square of Rm as an endomorphism of $\Lambda^2(TM)$, while $\mathrm{Rm}^\#$ is the "Lie algebra square" of Rm (where at each point $\Lambda^2(TM)$ is identified with $\mathfrak{so}(n)$). I.e.

$$\mathrm{Rm}^\# = \mathrm{ad} \circ \mathrm{Rm} \wedge \mathrm{Rm} \circ \mathrm{ad}^*,$$

where

$$\mathrm{ad} : \Lambda^2(\mathfrak{so}(n)) \to \mathfrak{so}(n)$$

is the adjoint representation.

9.3 *Global-in-space Bernstein estimates and long-time existence*

The evolution equation for Rm immediately yields an evolution equation for $|\mathrm{Rm}|^2$:

$$\begin{aligned}(\partial_t - \Delta)|\mathrm{Rm}|^2 &= 2g((\nabla_t - \Delta)\mathrm{Rm}, \mathrm{Rm}) - 2|\nabla \mathrm{Rm}|^2\\ &= 2g(\mathrm{Rm}^2 + \mathrm{Rm}^\#, \mathrm{Rm}) - 2|\nabla \mathrm{Rm}|^2.\end{aligned}$$

The first term is formed from the metric contraction of a linear combination of terms which are cubic tensor products of Rm. In particular, by Young's inequality, we may estimate $2g(\mathrm{Rm}^2 + \mathrm{Rm}^\#, \mathrm{Rm}) \leq C|\mathrm{Rm}|^3$, where the constant C depends only on n.

Let us denote by $S * T$ any tensor which is a linear combination of metric contractions of the tensor product of S and T (of the same type).

Lemma 9.14. *Along a Ricci flow* $(M^n \times I, g)$*,*

$$[\nabla_t - \Delta, \nabla]T = \mathrm{Rm} * \nabla T + \nabla \mathrm{Rm} * T.$$

From this, we find that

$$\begin{aligned}(\partial_t - \Delta)|\nabla \mathrm{Rm}|^2 &= 2g((\nabla_t - \Delta)\nabla \mathrm{Rm}, \nabla \mathrm{Rm}) - 2|\nabla^2 \mathrm{Rm}|^2\\ &= 2g(\nabla(\nabla_t - \Delta)\mathrm{Rm} + \mathrm{Rm}*\nabla \mathrm{Rm} + \nabla \mathrm{Rm}*\mathrm{Rm}, \nabla \mathrm{Rm})\\ &\quad - 2|\nabla^2 \mathrm{Rm}|^2\\ &= \mathrm{Rm} * \nabla \mathrm{Rm} * \nabla \mathrm{Rm} - 2|\nabla^2 \mathrm{Rm}|^2.\end{aligned}$$

If $|\mathrm{Rm}|$ remains bounded on the time interval $[0, T]$, then we can estimate

$$(\partial_t - \Delta)|\nabla \mathrm{Rm}|^2 \leq C|\nabla \mathrm{Rm}|^2,$$

where C depends only on n and the bound for $|\mathrm{Rm}|$. The ODE comparison principle then implies that $|\nabla \mathrm{Rm}|^2$ grows at most exponentially on $[0, T]$:

$$|\nabla \mathrm{Rm}|^2 \leq \max_{t=0} |\nabla \mathrm{Rm}|^2 e^{CT}.$$

This estimate takes a more natural form if we exploit its scale invariance: since $|\mathrm{Rm}|$ scales (under parabolic rescaling of our Ricci flow) like the inverse square of distance, whereas t scales as distance squared, the constant CT will be scale invariant. If we introduce the scale parameter $r = \sqrt{T}$ and assume that $|\mathrm{Rm}| \leq Kr^{-2}$ for $t \in [0, r^2]$ (a scale-invarant assumption), then the estimate becomes

$$|\nabla \mathrm{Rm}|^2 \leq C_1 \max_{t=0} |\nabla \mathrm{Rm}|^2,$$

where C_1 depends only on K and n.

We can also obtain a time-interior version of this estimate: consider, for some to-be-determined constant a, the combination

$$Q \doteqdot 2t|\nabla \mathrm{Rm}|^2 + a|\mathrm{Rm}|^2.$$

Observe that

$$\begin{aligned}(\partial_t - \Delta)Q &= 2|\nabla \mathrm{Rm}|^2 + 2t(\partial_t - \Delta)|\nabla \mathrm{Rm}|^2 + a(\partial_t - \Delta)|\mathrm{Rm}|^2 \\ &\leq 2|\nabla \mathrm{Rm}|^2 + 2tC_1|\mathrm{Rm}||\nabla \mathrm{Rm}|^2 + a(C_0|\mathrm{Rm}|^3 - 2|\nabla \mathrm{Rm}|^2) \\ &= 2(1 + C_1 t|\mathrm{Rm}| - a)|\nabla \mathrm{Rm}|^2 + aC_0|\mathrm{Rm}|^3.\end{aligned}$$

If we know that $|\mathrm{Rm}|$ is bounded by Kr^{-2} on $M \times [0, r^2]$, then

$$(\partial_t - \Delta)Q \leq 2(1 + C_1 K - a)|\nabla \mathrm{Rm}|^2 + aC_0 K^3 r^{-6}$$

Thus, if we choose $a = 1 + C_1 K$, then the ODE comparison principle yields

$$t|\nabla \mathrm{Rm}|^2 \leq Q \leq \max_{t=0} Q + aC_0 K^3 r^{-6} t \leq aK^2(1 + C_0 K)r^{-4}.$$

That is,

$$|\nabla \mathrm{Rm}| \leq \frac{Dr^{-2}}{\sqrt{t}},$$

where $D^2 \doteqdot aK^2(1 + C_0 K)$. This is another manifestation of the diffusive nature of the Ricci flow: even if the curvature is arbitrarily rough at the initial time, it becomes much more regular only a short-time later.

An inductive extension of this argument yields the following estimates.

Theorem 9.15 ((Global-in-space) Bernstein estimates[7]). *For every $n \in \mathbb{N}$, $K < \infty$ and $m \in \mathbb{N}$, there exists $C_m < \infty$ with the following property. Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. If*

$$|\mathrm{Rm}_{(x,t)}| \leq Kr^{-2} \text{ for all } (x,t) \in M^n \times [0, r^2],$$

then

$$|\nabla^m \mathrm{Rm}_{(x,t)}| \leq C_m \max_{M \times \{0\}} |\nabla^m \mathrm{Rm}| \text{ for all } (x,t) \in M^n \times [0, r^2]$$

and

$$|\nabla^m \mathrm{Rm}_{(x,t)}| \leq \frac{C_m r^{-2}}{t^{\frac{m}{2}}} \text{ for all } (x,t) \in M^n \times [0, r^2].$$

[7] Bando, "Real analyticity of solutions of Hamilton's equation".

A fundamental application of the global-in-space Bernstein estimates is the following characterization of finite time singularities.

Theorem 9.16 (Long-time existence). *Let $(M^n \times [0,T), g)$ be a* MAXIMAL[8] *Ricci flow on a compact manifold M^n. If $T < \infty$, then*

$$\limsup_{t \to T} \max_{M \times \{t\}} |\mathrm{Rm}| = \infty.$$

[8] I.e. there is no Ricci flow $(M^n \times [0,T'), g')$ with $T' > T$ such that $g'_{(x,t)} = g_{(x,t)}$ for all $t < T$.

Sketch of the proof. Let $(M^n \times [0,T), g)$ with $T < \infty$ be a maximal Ricci flow on a compact manifold M^n and suppose, contrary to the claim, that

$$|\mathrm{Rm}| \leq K \text{ on } M^n \times [0,T).$$

By the Bernstein estimates, we also have bounds on $M^n \times [0,T)$ for $\nabla^m \mathrm{Rm}$ for all m. These geometric estimates can be converted, by an inductive argument, to estimates in C^k for the metric coefficients in any local coordinate chart. The only subtlety is the $k = 0$ and $k = 1$ cases; to control these terms, we observe that, for any $x \in M^n$ and any $v \in T_x M^n$,

$$\left| \frac{d}{dt} \log \left(g_{(x,t)}(v,v) \right) \right| = \left| \frac{2\mathrm{Rc}_{(x,t)}(v,v)}{g_{(x,t)}(v,v)} \right| \leq C.$$

Integrating, we find that $g_{(x,t)}$ remains uniformly equivalent to $g_{(x,0)}$ under the evolution. (The first derivatives are then bounded due to the interpolation inequality.)

Cover M^n by finitely many compact sets K_α which each lie to the interior of some coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. The Arzelá–Ascoli theorem now implies that, for any sequence of times $t_j \to T$, we can find, for each compact set K_α, a subsequence of times such that the metric coefficients in the chart ϕ_α converge uniformly on K_α in the smooth topology to some limit. Taking appropriate subsequences, we can find limits along the same sequence of times which agree on overlaps. These limits thus define a global smooth metric on M^n, which we now evolve by the Ricci flow using our short-time existence theorem. The so extended family of metrics is smooth at each time and it is also smooth in time across the jump time T since time derivatives of g are related to spatial derivatives by the Ricci flow equation. But this is impossible since our original Ricci flow was assumed to be maximal. □

Proposition 9.17. *Let $(M^n \times [0,T), g)$ be the maximal Ricci flow of a compact Riemannian manifold (M^n, g_0). If $T < \infty$, then*

$$\max_{M^n \times \{t\}} |\mathrm{Rm}| \geq \frac{C}{T-t}$$

where C depends only on n.

Proof. Since $\limsup_{t \nearrow T} \max_{M^n \times \{t\}} |\mathrm{Rm}| = \infty$ and

$$(\partial_t - \Delta)|\mathrm{Rm}|^2 \leq c(n)|\mathrm{Rm}|^3,$$

the claim follows from the ODE comparison principle. □

9.4 *Local-in-space Bernstein estimates and the compactness theorem*

By introducing spatial cutoff functions into the above argument, one may derive the following local-in-space estimates.

Theorem 9.18 (Fully local Bernstein estimates[9]). *For every $n \in \mathbb{N}$, $K < \infty$ and $m \in \mathbb{N}$, there exists $C_m < \infty$ with the following property. Let $(M^n \times I, g)$ be a Ricci flow on a manifold M^n. If $B_r(x,t)$ has compact closure in M^n, $[t - r^2, t] \subset I$ and $\sup_{B_r(x,t) \times [t-r^2,t]} |\mathrm{Rm}| \leq K r^{-2}$, then*

$$|\nabla^m \mathrm{Rm}_{(x,t)}| \leq C_m r^{-m-2}.$$

[9] Shi, "Deforming the metric on complete Riemannian manifolds".

Combining these estimates with the Cheeger–Gromov compactness theorem for Riemannian manifolds with bounded geometry yields the following compactness theorem for Ricci flows under modest geometric assumptions.

Theorem 9.19 (Compactness of the space of Ricci flows with bounded geometry). *Let $\{(M_k \times I_k, o_k, g_k)\}_{k \in \mathbb{N}}$ be a sequence of pointed Ricci flows. Suppose the following conditions hold*

1. *$B_r(o_k, \alpha) \Subset M_k$ and $I \doteqdot [\alpha, \omega] \subset I_k$ for all k.*

2. *$\max_{\overline{B}_r(o_k,\alpha) \times I} |\mathrm{Rm}_{g_k}| \leq C < \infty$ for all k.*

3. *$\mathrm{inj}_{g_k}(o_k, \alpha) \geq \delta > 0$ for all k.*

There exists a pointed Ricci flow $(M \times I, o, g)$ such that, after passing to a subsequence, the Ricci flows $(\overline{B_{\frac{r}{2}}(o_k,\alpha)} \times I_k, o_k, g_k)$ converge uniformly in the smooth sense to the Ricci flow $(\overline{B_{\frac{r}{2}}(o,0)} \times I, o, g)$. That is, there exists a sequence of diffeomorphisms $\phi_k : \overline{B_{\frac{r}{2}}(o,0)} \to M_k$ with $\phi_k(o) = o_k$ such that $\phi_k^* g_k \to g$ uniformly in the smooth topology.

By taking limits along diagonal subsequences, one can obtain a complete limit under global bounds on the curvature and injectivity radius. Note though that the limit can lose or gain topology, and different subsequences can take different limits. Compact limits are better behaved, however (as in this case the convergence is necessarily uniform).

9.5 An estimate for the curvature

By Klingenberg's lemma, lower injectivity radius bounds are equivalent to lower volume bounds under the assumption of bounded curvature.

Proposition 9.20. *Given $\kappa > 0$ and $K < \infty$, there exists $\delta > 0$ with the following property. Let (M^n, g) be a Riemannian manifold. If*

1. $\sup_{B_r(x_0)} |\mathrm{Rm}| \leq K r^{-2}$ *and*
2. $\mathrm{volume}(B_r(x_0)) \geq \kappa r^n$,

then

$$\mathrm{inj}(x_0) \geq \delta r .$$

Proof. See e.g.[10]. □

[10] Chow and Knopf, *The Ricci flow: an introduction*; Chow, Lu, and Ni, *Hamilton's Ricci flow*.

So the lower injectivity radius bound in the compactness theorem may be replaced by lower volume bounds for geodesic balls.

On the other hand, if the volume of a geodesic ball is bounded from below for some time under Ricci flow, then the curvature at the centre is bounded from above.

Theorem 9.21 (Perelman[11]). *For any $n \geq 2$ and any $\kappa > 0$, there exists $C < \infty$ with the following property. Let $(M^n \times I, g)$ be a Ricci flow. Suppose that $B_r(x,t) \times (t - r^2, t] \Subset M^n \times I$. If*

$$\mathrm{Rm} \geq -r^{-2} g \text{ in } B_r(x,t) \times (t-r^2, t] \text{ and } \mathrm{volume}(B_r(x,t), t) \geq \kappa r^n ,$$

then

$$|\mathrm{Rm}_{(x,t)}| \leq C r^{-2} .$$

[11] Perelman, "The entropy formula for the Ricci flow and its geometric applications"

Sketch of the proof. Suppose, to the contrary, that we can find $\kappa > 0$, a sequence $\{(M_j^n \times I_j, g_j)\}_{j \in \mathbb{N}}$ of Ricci flows $(M_j^n \times I_j, g_j)$ containing points $(x_j, t_j) \in M_j^n \times I_j$, and a sequence of scales r_j such that

– $\mathrm{Rm}_j \geq -r_j^{-2} g_j$ in $B_{r_j}(x_j, t_j) \times (t_j - r_j^2, t_j]$ and

– $\mathrm{volume}(B_{r_j}(x_j, t_j), t_j) \geq \kappa r_j^n$, but nonetheless

– $|(\mathrm{Rm}_j)_{(x_j,t_j)}| > j^2 r_j^{-2}$.

Set $Q_j(x,t) \doteqdot |(\mathrm{Rm}_j)_{(x,t)}|$. We claim that points $(\bar{x}_j, \bar{t}_j) \in M_j^n \times I_j$ can be found with the following properties (see Lemma 9.22 below):

1. $(\bar{x}_j, \bar{t}_j) \in B_{\frac{2j}{Q_j(x_j,t_j)}}(x_j, t_j) \times (t_j - \frac{4j^2}{Q_j^2(x_j,t_j)} t_j]$.
2. $Q_j(\bar{x}_j, \bar{t}_j) \geq Q_j(x_j, t_j)$.
3. $Q_j \leq 2Q_j(\bar{x}_j, \bar{t}_j)$ in $B_{\frac{j}{Q_j(\bar{x}_j,\bar{t}_j)}}(\bar{x}_j, \bar{t}_j) \times (\bar{t}_j - \frac{j^2}{Q_j^2(\bar{x}_j,\bar{t}_j)} \bar{t}_j]$.

Set $\bar{r}_j \doteqdot Q_j^{-1}(\bar{x}_j, \bar{t}_j)$. After parabolically rescaling by $\bar{r}_j^{-1}$, we obtain a sequence of pointed Ricci in flows with curvature bounded by two on $B_j(\bar{x}_j, 0) \times (-j^2, 0]$ and $\mathrm{volume}(B_j(\bar{x}_j, 0), 0) \geq \kappa$. By the compactness theorem (Theorem 9.19), a subsequence converges to a complete ancient Ricci flow with nonnegative curvature operator and curvature bounded from above by two, which has *positive asymptotic volume ratio*,

$$\mathcal{V}(M_\infty^n, g_0) \doteqdot \lim_{r\to\infty} \frac{\mathrm{volume}(B_r(x_\infty, 0), 0)}{r^n} > 0.$$

It turns out that this final condition is incompatible with the others. (See Theorem 13.7 in Chapter 13.) □

In the proof of Theorem 9.21, we used the following "point picking" trick.

Lemma 9.22 (Point picking lemma). *Let $(M^n \times I, g)$ be a Ricci flow and $f : M^n \times I \to (0, \infty)$ a continuous function. Given $(x,t) \in M^n \times I$ and any $d > 0$ such that $B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t) \Subset M^n \times I$, there exists $(y,s) \in B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t)$ such that $f(y,s) \geq f(x,t)$ and $f \leq 4f(y,s)$ in $B_{\frac{d}{\sqrt{f(y,s)}}}(y,s) \times (s - \frac{d^2}{f(y,s)}, s)$.*

Proof. Set $(y_0, s_0) \doteqdot (x,t)$. If $(y,s) = (y_0, s_0)$ satisfies the conclusion, we are done. Else there exists $(y_1, s_1) \in B_{\frac{d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{d^2}{f(x,t)}, t)$ such that $f(y_1, s_1) > 4f(y_0, s_0)$. If (y_1, s_1) satisfies the conclusion, we're done. Else, we continue choosing points (y_j, s_j) in the same way. Since the radii form a geometric series, the points (y_j, s_j) never leave the ball $B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t)$. Since f admits some finite bound within $B_{\frac{2d}{\sqrt{f(x,t)}}}(x,t) \times (t - \frac{4d^2}{f(x,t)}, t)$, the process must terminate after finitely many steps. □

While the estimates for derivatives of curvature (under the assumption of bounded curvature) rely entirely on the maximum principle, inspired by a classical argument of Bernstein, the estimate of Theorem 9.21 requires a number of new ideas. We will touch on these ideas in Chapter 12.

9.6 Exercises

Exercise 9.1. Let (M^n, g) be a Riemannian manifold equipped with its Levi-Civita connection ∇. Assuming $f \in C^2(M^n)$ attains a local maximum at $x_0 \in M^n$, show that

$$0 = \nabla f(x_0) \quad \text{and} \quad \nabla^2 f(x_0) \leq 0.$$

Exercise 9.2. Let $\{g_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ be a one-parameter family of metrics on a manifold M^n with $g_0 = g$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} g_\varepsilon = h$.

(a) Show that

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathrm{R}_{g_\varepsilon} = -g(\mathrm{Rc}, h) - \Delta \operatorname{tr} h + \operatorname{div}\operatorname{div} h, \tag{9.12}$$

where the Ricci curvature, trace, Laplacian and divergence are all with respect to the metric g.

(b) Deduce that

$$(\partial_t - \Delta)\mathrm{R} = 2|\mathrm{Rc}|^2$$

under Ricci flow.

Exercise 9.3. Let g be a time-dependent metric on $M^n \times I$ which evolves by Ricci flow.

(a) Using parts (b) and (c) from Exercise 8.8, show that

$$\begin{aligned}
\nabla_t \mathrm{Rm}(X,Y,Z,W) = {}& \nabla_X \nabla_Z \mathrm{Rc}(Y,W) - \nabla_X \nabla_W \mathrm{Rc}(Y,Z) \\
& - \nabla_Y \nabla_Z \mathrm{Rc}(X,W) + \nabla_Y \nabla_W \mathrm{Rc}(X,Z) \\
& - \mathrm{Rm}(\mathrm{Rc}(X),Y,Z,W) + \mathrm{Rm}(X,\mathrm{Rc}(Y),Z,W),
\end{aligned}$$

where

$$B(X,Y,Z,W) \doteqdot g(\mathrm{Rm}(X,\cdot,Y,*), \mathrm{Rm}(Z,\cdot,W,*))$$

(b) Show that

$$\begin{aligned}
\Delta \mathrm{Rm}(X,Y,Z,W) = {}& \nabla_X \nabla_Z \mathrm{Rc}(Y,W) - \nabla_X \nabla_W \mathrm{Rc}(Y,Z) \\
& - \nabla_Y \nabla_Z \mathrm{Rc}(X,W) + \nabla_Y \nabla_W \mathrm{Rc}(X,Z) \\
& - \mathrm{Rm}(\mathrm{Rc}(X),Y,Z,W) + \mathrm{Rm}(X,\mathrm{Rc}(Y),Z,W) \\
& - 2\Big(B(X,Y,Z,W) - B(X,Y,W,Z) \\
& \quad + B(X,Z,Y,W) - B(X,W,Y,Z)\Big).
\end{aligned}$$

(c) Deduce that

$$(\nabla_t - \Delta)\mathrm{Rm} = 2\Big(B(X,Y,Z,W) - B(X,Y,W,Z) \\ + B(X,Z,Y,W) - B(X,W,Y,Z)\Big).$$

Exercise 9.4. Show that any ETERNAL Ricci flow $(M^n \times (-\infty, \infty), g)$ on a compact manifold M^n is Ricci flat.

Exercise 9.5. Show (using the ODE comparison principle) that, along any Ricci flow $(M^n \times [0,T), g)$ on a compact manifold M^n,

$$\max_{M^n \times \{t\}} |\mathrm{Rm}| \leq \frac{1}{\big(\max_{M^n \times \{0\}} |\mathrm{Rm}|\big)^{-1} + c(n)t}, \tag{9.13}$$

where $c(n)$ is a constant which depends only on the dimension n.

10
Pinching and its consequences

We have seen that positivity of scalar curvature is preserved under the Ricci flow, by applying the (scalar) maximum principle to the reaction-diffusion equation for the scalar curvature. The reaction terms in the evolution equation for the Riemann tensor enjoy a far richer algebraic structure. Understanding this structure (in relation to the tensor and vector bundle maximum principles) is a crucial step in understanding the long term behaviour of the Ricci flow. We will explore this paradigm in this chapter.

10.1 Contraction of compact three-manifolds with positive Ricci curvature to round points

In three dimensions, the trace-free part of the Riemann curvature tensor (the Weyl tensor) necessarily vanishes, so the curvature is entirely determined by the Ricci tensor.[1]

Moreover, the inequality $\mathrm{Rc} \geq 0$ implies the inequality $\mathrm{Rm} \geq 0$. The tensor maximum principle guarantees that these inequalities are preserved.

Proposition 10.1. *Let $(M^n \times [0,T), g)$ be Ricci flow on a compact three manifold M^3. If $\mathrm{Rc}|_{t=0} \geq 0$, then either*

1. $\mathrm{Rc} > 0$ *for all* $t > 0$,
2. (M^3, g) *is flat, or*
3. $(M^3 \times I, g)$ *is an isometric quotient of* $(M^2 \times \mathbb{R} \times I, h + dr^2)$ *for some two-dimensional Ricci flow* $(M^2 \times I, h)$.

Proof. Recall that

$$(\nabla_t - \Delta)\mathrm{Rc}_{ij} = \mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{k\ell}.$$

With the tensor maximum principle in mind, consider, for any non-negative definite symmetric two-tensor S, the reaction term[2] $N(S)_{ij} \doteqdot$

[1] Indeed, in general, in dimensions $n \geq 3$, the Riemann curvature tensor admits the decomposition

$$\mathrm{Rm} = \mathrm{Wy} + \mathrm{Sc} \owedge g, \qquad (10.1)$$

where the Schouten tensor, Sc, is defined by

$$\mathrm{Sc} \doteqdot \tfrac{1}{n-2}\left(\mathrm{Rc} - \tfrac{1}{2(n-1)}\,\mathrm{R}\,g\right),$$

the Kulkarni–Nomizu product, $\owedge$, of two symmetric $(0,2)$ tensors S and T is defined by

$$\begin{aligned}(S \owedge T)(u,v,w,z) &\doteqdot S(u,w)T(v,z) - S(v,w)T(u,z)\\ &\quad + S(v,z)T(u,w) - S(u,z)T(v,w),\end{aligned}$$

and the Weyl tensor, Wy, which may be taken to be defined by the formula (10.1), is totally trace-free.

[2] We may write $\mathrm{Rm}(S)$ explicitly using the formula

$$\mathrm{Rm}(S) \doteqdot \left(S - \tfrac{1}{4}\,\mathrm{tr}(S)g\right) \owedge g.$$

But it will suffice to know that $\mathrm{Rm}(S)$ has the (algebraic) symmetries of the Riemann tensor and its trace is S.

$\mathrm{Rm}(S)_{ikj\ell}S^{k\ell}$. Given any null eigenvector v of S, we claim that

$$N(S)(v,v) = \mathrm{Rm}(S)_{ikj\ell}S^{k\ell}v^iv^j \geq 0.$$

To see this, let $\{e_1 = v/|v|, e_2, e_3\}$ be an orthonormal frame of eigenvectors, with corresponding eigenvalues $0 = \rho_1 \leq \rho_2 \leq \rho_3$. Since

$$\begin{bmatrix}\rho_1\\ \rho_2\\ \rho_3\end{bmatrix} = \begin{bmatrix}\sigma_{12}+\sigma_{13}\\ \sigma_{12}+\sigma_{23}\\ \sigma_{13}+\sigma_{23}\end{bmatrix},$$

where $\sigma_{ij} \doteqdot \mathrm{Rm}(S)_{ijij}$, we may express

$$\begin{bmatrix}\sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\rho_2+\rho_3-\rho_1\\ \rho_1+\rho_3-\rho_2\\ \rho_1+\rho_2-\rho_3\end{bmatrix},$$

and hence

$$\begin{aligned}N(S)(v,v) &= 2\mathrm{Rm}(S)_{1k1k}\rho_k\\ &= 2\sigma_{12}\rho_2 + 2\sigma_{13}\rho_3\\ &= (\rho_2-\rho_3)\rho_2 + (\rho_3-\rho_2)\rho_3\\ &= (\rho_3-\rho_2)^2\\ &\geq 0.\end{aligned}$$

So the tensor maximum principle indeed implies that $\mathrm{Rc} \geq 0$.

In fact, the strong maximum principle implies that either $\mathrm{Rc} > 0$ or

$$\min_{|v|=1} \mathrm{Rc}(v,v) \equiv 0,$$

and hence Rc admits a null eigenvector v (at every point). Now, starting at some point (x,t), parallel transport v along radial geodesics to form a vector field, and then extend this vector field in time to form a time dependent vector field V by solving $\nabla_t V = 0$. Note that this vectorfield will satisfy $\nabla V = 0$ and $\Delta V = 0$ at the point (x,t). Thus, since $\mathrm{Rc}(V,V) \geq 0$ with equality at (x,t), we find at (x,t) that

$$\nabla\mathrm{Rc}(V,V) = \nabla(\mathrm{Rc}(V,V)) = 0,$$

$$\Delta\mathrm{Rc}(V,V) = \Delta(\mathrm{Rc}(V,V)) \geq 0,$$

and

$$0 = \partial_t(\mathrm{Rc}(V,V)) = \nabla_t\mathrm{Rc}(V,V) = \Delta\mathrm{Rc}(V,V) + Q(\mathrm{Rc}),$$

and hence both terms on the right (being each nonnegative) must vanish. In an eigenframe $\{e_1, e_2, e_3\}$ for Rc at (x,t) with $e_1 = v/|v|$ and corresponding eigenvalues $0 = \rho_1 \leq \rho_2 \leq \rho_3$,

$$0 = Q(\mathrm{Rc}) = 2\sec(e_1 \wedge e_j)\rho_j = (\rho_3-\rho_2)^2,$$

and hence $\rho_2 = \rho_3$. So Rc has eigenvalues $\{0, \rho(x,t), \rho(x,t)\}$ at each point (x,t). If ρ vanishes at some (x,t), then so does R, and the strong maximum principle implies that $2\rho = \mathrm{R} \equiv 0$, and hence $\mathrm{Rc} \equiv 0$. So we may assume that $\rho > 0$ everywhere. This guarantees that there is a smooth null eigenvector field, U. Computing as above, we find that

$$0 \geq \mathrm{Rc}(\nabla U, \nabla U) = \rho^2 |\nabla U|^2$$

and hence U is parallel in space. It then follows that

$$\Delta \mathrm{Rc}(U) \equiv 0$$

and hence

$$\mathrm{Rc}([\partial_t, U]) = \mathrm{Rc}(\nabla_t U) = \nabla_t(\mathrm{Rc}(U)) - \nabla_t \mathrm{Rc}(U) \equiv 0.$$

The claim now follows from the Frobenius theorem (consider the distribution $\mathcal{U} \doteqdot \ker \mathrm{Rc}$). □

When $\mathrm{Rc} > 0$, we may estimate $|\mathrm{Rc}|^2 \leq \mathrm{R}^2$, so the ODE comparison principle give the following blow-up estimates for the scalar curvature.

Proposition 10.2. *Let $(M^3 \times [0,T), g)$ be Ricci flow on a compact three manifold M^3 with positive Ricci curvature.*

$$\min_{M \times \{t\}} \tfrac{1}{3}\mathrm{R} \leq \frac{1}{2(T-t)} \leq \max_{M \times \{t\}} \mathrm{R}.$$

Proof. Since $\mathrm{Rc} > 0$, we may estimate $|\mathrm{Rc}|^2 \leq \mathrm{R}^2$, and hence

$$\tfrac{2}{3}\mathrm{R}^2 \leq (\partial_t - \Delta)\mathrm{R} \leq 2\mathrm{R}^2.$$

Since $\limsup_{t \to T} \max_{M^3 \times \{t\}} \mathrm{R} = \infty$, the ODE comparison principle yields the claims. □

The tensor maximum principle can also be used to show that any uniform positive pinching of the Ricci tensor is preserved.

Proposition 10.3 (Pinching is preserved)**.** *Let $(M^3 \times [0,T), g)$ be a Ricci flow on a compact manifold M^3 such that* $\mathrm{Rc} > 0$ *at the initial time. There exists $\alpha > 0$ such that*

$$\mathrm{Rc} \geq \alpha \mathrm{R} g > 0$$

at all times.

Proof. Since M^3 is compact and $\mathrm{Rc} > 0$, a constant $\alpha > 0$ may be found such that the inequality holds at the initial time. Given such a constant, consider the tensor $S \doteqdot \mathrm{Rc} - \alpha \mathrm{R} g$. Observe that

$$\begin{aligned}(\nabla_t - \Delta)S_{ij} &= (\nabla_t - \Delta)\mathrm{Rc}_{ij} - \alpha(\partial_t - \Delta)\mathrm{R} g_{ij} \\ &= 2\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{k\ell} - 2\alpha|\mathrm{Rc}|^2 g_{ij}.\end{aligned}$$

If v is a null eigenvector of S, then v is an eigenvector of Rc with eigenvalue $\rho_1 = \alpha\mathrm{R}$. Consider an orthonormal basis $\{e_1 = v, e_2, e_3\}$ which diagonalizes Rc. With respect to this basis,

$$\begin{aligned}\left(\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{k\ell} - \alpha|\mathrm{Rc}|^2 g_{ij}\right) v_i v_j &= \left(\mathrm{Rm}_{1k1\ell} - \alpha\mathrm{Rc}_{kl}\right)\mathrm{Rc}^{k\ell} \\ &= \left(\sigma_{1k} - \alpha\rho_k\right)\rho_k \\ &= -\alpha\rho_1^2 + (\sigma_{12} - \alpha\rho_2)\rho_2 + (\sigma_{13} - \alpha\rho_3)\rho_3,\end{aligned}$$

where $\sigma_{ij} = \sec(e_i \wedge e_j)$ $(= \rho_i + \rho_j - \frac{1}{2}\mathrm{R})$. Since

$$\sigma_{12} - \alpha\rho_2 + \sigma_{13} - \alpha\rho_3 = \rho_1 - \alpha(\rho_2 + \rho_3) = \alpha(\mathrm{R} - \rho_2 - \rho_3) = \alpha\rho_1 > 0,$$

we have

$$\max\{\sigma_{12} - \alpha\rho_2, \sigma_{13} - \alpha\rho_3\} > 0$$

and hence

$$\begin{aligned}(\sigma_{12} - \alpha\rho_2)\rho_2 + (\sigma_{13} - \alpha\rho_3)\rho_3 &\geq \left((\sigma_{12} - \alpha\rho_2) + (\sigma_{13} - \alpha\rho_3)\right)\min\{\rho_2, \rho_3\} \\ &= \alpha\rho_1 \min\{\rho_2, \rho_3\} \\ &\geq \alpha\rho_1^2.\end{aligned}$$

So the claim follows from the tensor maximum principle.[3] □

[3] In fact, our argument is not quite rigorous. It may be easily made so by expressing Rm, Rc and R algebraically in terms of the tensor S.

Consider now the ratio $|\mathring{\mathrm{R}}\mathrm{c}|^2 / \mathrm{R}^2$, where $\mathring{\mathrm{R}}\mathrm{c} = \mathrm{Rc} - \frac{1}{3}\mathrm{R}g$ denotes the trace-free part of Rc. Since $\mathring{\mathrm{R}}\mathrm{c}$ vanishes precisely at Einstein points, this ratio is a scale invariant pointwise measure of the "roundness" of our hypersurface. We will show that this measure of roundness becomes optimal in regions of very large curvature. First, observe that it does not decay.

Proposition 10.4 (Roundness is preserved)**.** *Let $(M^3 \times [0, T), g)$ be a Ricci flow on a compact manifold M^3 such that* $\mathrm{Rc} > 0$ *at the initial time.*

$$\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2} \leq \max_{M^3 \times \{0\}} \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}.$$

Proof. Since

$$\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2} = \frac{|\mathrm{Rc}|^2}{\mathrm{R}^2} - \frac{1}{3},$$

we find that

$$
\begin{aligned}
(\partial_t - \Delta)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} &= (\partial_t - \Delta)\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2} \\
&= 2g\left((\nabla_t - \Delta)\frac{\mathrm{Rc}}{\mathrm{R}}, \frac{\mathrm{Rc}}{\mathrm{R}}\right) - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 \\
&= 2g\left(\frac{(\nabla_t - \Delta)\mathrm{Rc}}{\mathrm{R}} - (\partial_t - \Delta)\mathrm{R}\frac{\mathrm{Rc}}{\mathrm{R}^2} + 2\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\frac{\mathrm{Rc}}{\mathrm{R}}, \frac{\mathrm{Rc}}{\mathrm{R}}\right) \\
&\quad - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 \\
&= 4\frac{\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{ij}\mathrm{Rc}^{k\ell}}{\mathrm{R}^2} - 4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3} + 2\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left|\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 \\
&= 4\frac{\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{ij}\mathrm{Rc}^{k\ell}}{\mathrm{R}^2} - 4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3} + 2\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left|\frac{\mathring{\mathrm{Rc}}}{\mathrm{R}}\right|^2 - 2\left|\nabla\frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 .
\end{aligned}
$$

Observe that, with respect to an eigenframe for Rc,

$$
\begin{aligned}
\mathrm{R}\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{ij}\mathrm{Rc}^{k\ell} - |\mathrm{Rc}|^4 &= \sum_{i,k}\mathrm{R}\sec(e_i \wedge e_k)\rho_i\rho_k - \left(\sum_i \rho_i^2\right)^2 \\
&= \sum_{i \neq k}\mathrm{R}(\rho_i + \rho_k - \tfrac{1}{2}\mathrm{R})\rho_i\rho_k - \left(\sum_i \rho_i^2\right)^2 .
\end{aligned}
$$

Since, by Exercise 10.1, this is nonpositive, we find that

$$
(\partial_t - \Delta)\frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2} \leq 2\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left|\frac{\mathring{\mathrm{Rc}}}{\mathrm{R}}\right|^2 .
$$

So the claim follows from the maximum principle. □

By taking a little more care in estimating the reaction terms, we are able to show that roundness improves at the onset of a singularity.

Proposition 10.5 (Roundness improves). *Let $(M^3 \times [0, T), g)$ be a Ricci flow on a compact manifold M^3 such that $\mathrm{Rc} > 0$ at the initial time. For every $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ (which depends only on ε and the initial data) such that*

$$
|\mathring{\mathrm{Rc}}|^2 \leq \varepsilon \mathrm{R}^2 + C_\varepsilon . \tag{10.2}
$$

Proof. Given σ, consider the function $\mathrm{R}^\sigma \frac{|\mathring{\mathrm{Rc}}|^2}{\mathrm{R}^2}$. We aim to show, using the maximum principle, that an initial upper bound for this function is preserved, for some $\sigma > 0$ (which will depend on the preserved pinching constant α). The claim then follows from Young's inequality.

Observe that

$$
\begin{aligned}
&(\partial_t - \Delta)\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) \\
&\qquad = \mathrm{R}^\sigma (\partial_t - \Delta)\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2} + \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}(\partial_t - \Delta)\mathrm{R}^\sigma - 2g\left(\nabla \mathrm{R}^\sigma, \nabla \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) \\
&\qquad = \mathrm{R}^\sigma (\partial_t - \Delta)\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2} + \sigma \mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\left[\frac{(\partial_t - \Delta)\mathrm{R}}{\mathrm{R}} + (1-\sigma)\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\right] \\
&\qquad\quad - 2\sigma \mathrm{R}^\sigma g\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}, \nabla \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) \\
&\qquad = \mathrm{R}^\sigma \left[4\frac{\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{ij}\mathrm{Rc}^{k\ell}}{\mathrm{R}^2} + 2\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\frac{|\mathrm{Rc}|^2}{\mathrm{R}} - 4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3}\right. \\
&\qquad\quad \left. + 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2} - 2\left|\nabla \frac{\mathrm{Rc}}{\mathrm{R}}\right|^2 + \sigma(1-\sigma)\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\right].
\end{aligned}
$$

Since

$$
\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) = \sigma \mathrm{R}^\sigma \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2} + \mathrm{R}^\sigma \nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2},
$$

we arrive at

$$
\begin{aligned}
&(\partial_t - \Delta)\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) \\
&\qquad = \mathrm{R}^\sigma \left[4\frac{\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{ij}\mathrm{Rc}^{k\ell}}{\mathrm{R}^2} + 2\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\frac{|\mathrm{Rc}|^2}{\mathrm{R}} - 4\frac{|\mathrm{Rc}|^4}{\mathrm{R}^3} - 2\left|\nabla \frac{\mathrm{Rc}}{\mathrm{R}}\right|^2\right. \\
&\qquad\quad \left. -\sigma(1-\sigma)\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\right] + 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) \\
&\qquad \leq 4\mathrm{R}^{1+\sigma}\left(Z + \frac{\sigma}{2}\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2}\frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) + 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right),
\end{aligned}
$$

where

$$
Z \doteqdot \frac{\mathrm{Rm}_{ikj\ell}\mathrm{Rc}^{ij}\mathrm{Rc}^{k\ell}}{\mathrm{R}^3} - \frac{|\mathrm{Rc}|^4}{\mathrm{R}^4}.
$$

By Exercise 10.1, this expression is nonpositive, with equality only if at least one of the eigenvalues is zero. It follows that the homogeneous expression Z (as an algebraic function of the Ricci eigenvalues) takes a negative maximum, $-\zeta_\alpha$, on the cone described by the condition $\mathrm{Rc} \geq \alpha \mathrm{R}$. Since $|\mathring{\mathrm{R}}\mathrm{c}| \leq |\mathrm{Rc}| \leq \mathrm{R}$ on the positive cone, we may take σ to be twice ζ_α to obtain

$$
(\partial_t - \Delta)\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right) \leq 2(1-\sigma)\nabla_{\frac{\nabla \mathrm{R}}{\mathrm{R}}}\left(\mathrm{R}^\sigma \frac{|\mathring{\mathrm{R}}\mathrm{c}|^2}{\mathrm{R}^2}\right),
$$

at which point the claim follows from the maximum principle. □

Proposition 10.5 ensures that the metric is becoming round at any point where the curvature is becoming large, in the sense that the scale invariant ratio $|\mathring{\mathrm{Rc}}|/\mathrm{R}$ is becoming small. We already know that $\max \mathrm{R} \geq \frac{1}{2(T-t)}$ is blowing up at the final time. We thus need to show that $\min \mathrm{R}$ blows up at the same rate. So we should try to control the *gradient* of Rc. In order to do this, we need to compare $|\nabla \mathrm{Rc}|^2$ to some function (of curvature) whose evolution equation can overcome the bad reaction terms $\mathrm{Rc} * \nabla \mathrm{Rc} * \nabla \mathrm{Rc}$ in the evolution equation for $|\nabla \mathrm{Rc}|^2$. We can exploit the estimate (10.2) in this regard.

Proposition 10.6. *Let $(M^3 \times [0,T), g)$ be a Ricci flow on a compact manifold M^3 such that $\mathrm{Rc} \geq \alpha \mathrm{R}$ at the initial time. For any $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ such that*

$$|\nabla \mathrm{Rc}|^2 \leq \varepsilon \mathrm{R}^3 + C_\varepsilon$$

at all times.

Proof. Recall that

$$(\partial_t - \Delta)|\nabla \mathrm{Rc}|^2 \leq c|\mathrm{Rc}||\nabla \mathrm{Rc}|^2 - 2|\nabla^2 \mathrm{Rc}|^2.$$

Given $\varepsilon > 0$, choose C_ε (as permitted by Proposition 10.5) so that

$$|\mathring{\mathrm{Rc}}|^2 \leq \varepsilon \mathrm{R}^2 + C_\varepsilon$$

and consider, for suitable $C_\varepsilon < \infty$, the function

$$G_\varepsilon \doteqdot 2C_\varepsilon + \varepsilon \mathrm{R}^2 - |\mathring{\mathrm{Rc}}|^2 \geq C_\varepsilon > 0.$$

Estimating $Z \geq 0$, $|\mathrm{Rc}| \leq \mathrm{R}$, and (see Exercise 10.2) $|\nabla \mathrm{Rc}|^2 \geq \frac{7}{20}|\nabla \mathrm{R}|^2$, we find that

$$\begin{aligned}(\partial_t - \Delta)G_\varepsilon &= 4\left(\left(\tfrac{1}{3}+\varepsilon\right)\mathrm{R}|\mathrm{Rc}|^2 - \mathrm{Rm}(\mathrm{Rc},\mathrm{Rc})\right)\\ &\quad + 2\left(|\nabla \mathrm{Rc}|^2 - \left(\tfrac{1}{3}+\varepsilon\right)|\nabla \mathrm{R}|^2\right)\\ &\geq 4\frac{|\mathrm{Rc}|^2}{\mathrm{R}}(G_\varepsilon - 2C_\varepsilon) + \kappa|\nabla \mathrm{Rc}|^2\\ &\geq -4|\mathrm{Rc}|G_\varepsilon + \kappa|\nabla \mathrm{Rc}|^2,\end{aligned}$$

where $\kappa \doteqdot \frac{1}{21}$, say, so long as $\varepsilon \leq \frac{1}{120}$.

We aim to preserve upper bounds for the function $\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}$. So consider

$$\begin{aligned}&(\partial_t - \Delta)\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\\ &\quad = \frac{(\partial_t - \Delta)|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} - \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left(\frac{(\partial_t-\Delta)\mathrm{R}}{\mathrm{R}} - \frac{(\partial_t - \Delta)G_\varepsilon}{G_\varepsilon}\right)\\ &\qquad + 2g\left(\nabla\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}, \nabla \log(\mathrm{R}G_\varepsilon)\right) + 2\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} g\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}, \frac{\nabla G_\varepsilon}{G_\varepsilon}\right).\end{aligned}$$

We estimate the terms on the first line as above. To control the terms on the second line, observe that, at a new local maximum of $\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}$,

$$0 = \nabla_k \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} = 2\frac{g(\nabla_k \nabla \mathrm{Rc}, \nabla \mathrm{Rc})}{\mathrm{R}G_\varepsilon} - \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left(\frac{\nabla_k \mathrm{R}}{\mathrm{R}} + \frac{\nabla_k G_\varepsilon}{G_\varepsilon}\right)$$

and hence

$$4\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} g\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}, \frac{\nabla G_\varepsilon}{G_\varepsilon}\right) \leq \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\left|\frac{\nabla \mathrm{R}}{\mathrm{R}} + \frac{\nabla G_\varepsilon}{G_\varepsilon}\right|^2 \leq 4\frac{|\nabla^2 \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}.$$

Thus, at such a point,

$$0 \leq (\partial_t - \Delta)\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} \leq \frac{|\nabla \mathrm{Rc}|^2}{G_\varepsilon}\left((c+4)\frac{|\mathrm{Rc}|}{\mathrm{R}} + 2\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2} - \kappa\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\right)$$

and hence

$$\kappa\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} \leq (c+4)\frac{|\mathrm{Rc}|}{\mathrm{R}} + 2\frac{|\mathrm{Rc}|^2}{\mathrm{R}^2} \leq c+6.$$

We conclude that

$$\frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon} \leq C \doteqdot \max\left\{\frac{c+6}{\kappa}, \max_{M^3\times\{0\}} \frac{|\nabla \mathrm{Rc}|^2}{\mathrm{R}G_\varepsilon}\right\}.$$

The claim now follows from Young's inequality. □

Proposition 10.7. *Let $(M^3 \times [0,T), g)$ be the maximal Ricci flow of a compact Riemannian three-manifold (M^3, g) with positive Ricci curvature.*

$$\frac{\mathrm{R}_{\max}(t)}{\mathrm{R}_{\min}(t)} \to 1 \;\; \textit{and} \;\; \operatorname{diam}(M^3, g_{(\cdot,t)}) \to 0 \;\; \textit{as} \;\; t \to T, \tag{10.3}$$

where $\mathrm{R}_{\max} \doteqdot \max_{M^3} \mathrm{R}$ *and* $\mathrm{R}_{\min} \doteqdot \min_{M^3} \mathrm{R}$.

Proof. By the gradient estimate (Proposition 10.6), for every $\eta > 0$ there is a constant $C_\eta < \infty$ such that

$$|\nabla \mathrm{R}| \leq \tfrac{1}{2}\eta^2 \mathrm{R}^{\frac{3}{2}} + C_\eta.$$

Since $\mathrm{R}_{\max}(t) \to \infty$ as $t \to T$, there is, for every $\eta > 0$, some point $(x_\eta, t_\eta) \in M^3 \times [0,T)$ such that

$$\mathrm{R}_\eta^{\frac{3}{2}} \doteqdot \mathrm{R}^{\frac{3}{2}}(x_\eta, t_\eta) = \mathrm{R}_{\max}^{\frac{3}{2}}(t_\eta) \geq 8C_\eta/\eta^2$$

and hence

$$|\nabla \mathrm{R}|(x, t_\eta) \leq \eta^2 \mathrm{R}^{\frac{3}{2}}(x_\eta, t_\eta)$$

for all $x \in M$. Now let γ be a unit speed $g_{(\cdot,t_\eta)}$-geodesic through $\gamma(0) = x_\eta$. For each $s \leq L \doteqdot \eta^{-1}\mathrm{R}_\eta^{-\frac{1}{2}}$, the mean value theorem provides some $s_0 \in (0,s)$ such that

$$\mathrm{R}(\gamma(s), t_\eta) = \mathrm{R}_\eta + s\nabla_{\gamma'(s_0)}\mathrm{R}(\gamma(s_0), t_\eta) \geq \mathrm{R}_\eta(1-\eta). \tag{10.4}$$

Applying the preserved pinching estimate $\mathrm{Rc} \geq \alpha \mathrm{R} g$, we may estimate

$$\mathrm{Rc}(\gamma', \gamma') \geq \alpha \mathrm{R} \geq \alpha \mathrm{R}_\eta (1 - \eta)$$

for $s \leq L$. If $\eta < \frac{1}{2}$, then

$$\mathrm{Rc}(\gamma', \gamma') \geq 2Kg,$$

where $K \doteqdot \frac{\alpha}{4}\mathrm{R}_\eta$. Choosing further $\eta \leq \frac{\alpha}{4\pi}$, we obtain $L \geq \pi K^{-1}$. Myers' theorem then implies that every point of M^3 is reached by a $g_{(\cdot, t_\eta)}$-geodesic of length at most L and we conclude from (3.15) that

$$\mathrm{R}_{\min}(t_\eta) \geq (1 - \eta)\mathrm{R}_{\max}(t_\eta).$$

Since $\mathrm{R}_{\min}$ is nondecreasing, we then have

$$\mathrm{R}^2_{\max}(t) \geq (1 - \eta)^2 \mathrm{R}^2_{\max}(t_\eta) \geq \frac{1}{4}\mathrm{R}^2_\eta \quad \text{for all} \quad t \geq t_\eta,$$

so that the above arguments hold for all $t \geq t_\eta$. We now conclude that, given any $\eta \leq \min\{\frac{\alpha}{4\pi}, \frac{1}{2}\}$, there is some time $t_\eta \in [0, T)$ such that

$$\mathrm{diam}(M, g_{(\cdot, t)}) \leq \frac{1}{\eta \mathrm{R}_{\max}(t)} \quad \text{and} \quad \mathrm{R}_{\min}(t) \geq (1 - \eta)\mathrm{R}_{\max}(t)$$

for all $t > t_\eta$. The proposition follows since $\mathrm{R}_{\max}(t) \geq \frac{1}{2(T-t)}$. □

It follows that the diameter of the rescaled metrics $\frac{1}{2(n-1)(T-t)} g_{(\cdot, t)}$ remains bounded, and their scalar curvature converges uniformly to a constant as $t \to T$. Bootstrapping arguments then yield smooth convergence to a round metric.

Theorem 10.8 (Hamilton[4]). *Let (M^3, g_0) be a compact Riemannian three manifold with positive Ricci curvature. The maximal Ricci flow $(M^3 \times [0, T), g)$ of (M^3, g_0) satisfies*

$$\frac{1}{4(T - t)} g_{(\cdot, t)} \to \bar{g}$$

uniformly in the smooth topology as $t \to T$, where $\overline{g}$ is a metric of constant sectional curvature one. In particular, M^3 is a quotient of S^3.

[4] Richard S. Hamilton, "Three-manifolds with positive Ricci curvature"

10.2 Manifolds with positive curvature operator

In higher dimensions, Böhm and Wilking[5] were able to exploit the algebraic structure of the reaction terms in the evolution equation for the curvature tensor to prove (using the vector bundle maximum principle) that pinching of the curvature operator is preserved and improves under Ricci flow in all dimensions. As a result, they obtained the following higher dimensional analogue of Hamilton's theorem.

[5] Böhm and Wilking, "Manifolds with positive curvature operators are space forms".

Theorem 10.9. *Let (M^n, g_0) be a compact Riemannian manifold with positive curvature operator. The maximal Ricci flow $(M^n \times [0,T), g)$ of (M^n, g_0) satisfies*

$$\frac{1}{2(n-1)(T-t)} g_{(\cdot,t)} \to \bar{g}$$

uniformly in the smooth topology as $t \to T$, where $\overline{g}$ is a metric of constant sectional curvature one. In particular, M^n is a quotient of S^n.

10.3 Positive isotropic curvature and the quarter-pinched differentiable sphere theorem

The quarter pinched sphere theorem of Rauch–Klingenberg–Berger states that a simply connected, complete Riemannian manifold whose sectional curvature satisfies $\frac{1}{4} < \sec \leq 1$ must be homeomorphic to a sphere.[6]

Micallef and Moore[7] later showed that every manifold of POSITIVE ISOTROPIC CURVATURE is homeomorphic to a sphere. This condition states that the curvature operator takes only positive values when acting on totally isotropic two planes. Since, by Berger's lemma, any manifold whose curvature is pointwise quarter pinched has positive isotropic curvature, this generalizes the quarter pinched sphere theorem.

It is natural to expect that these results also hold within the smooth category (i.e. such a manifold should be *diffeomorphic* to the sphere) but attempts to prove this failed for almost fifty years, with the problem becoming known as the QUARTER PINCHED DIFFERENTIABLE SPHERE CONJECTURE. The conjecture was finally resolved in 2009 by Brendle and Schoen[8] using the Ricci flow.

[6] The hypothesis is optimal since the sectional curvatures of the Fubini–Study metric on $\mathbb{C}P^n$ take values between $1/4$ and 1 inclusive.

[7] Micallef and Moore, "Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes".

[8] Brendle and Schoen, "Manifolds with 1/4-pinched curvature are space forms".

Theorem 10.10. *Let (M^n, g_0) be a compact Riemannian manifold. If g_0 has positive isotropic curvature, then the unique Ricci flow starting from g_0 deforms g_0 through a family of metrics $\{g_t\}_{t\in[0,T)}$, $T < \infty$, each having positive isotropic curvature. Moreover,*

$$\frac{1}{2(n-1)(T-t)} g_t \to \overline{g}$$

uniformly in the smooth topology as $t \to T$, where $\overline{g}$ is a metric of constant curvature one. In particular, M^n is diffeomorphic to S^n.

Corollary 10.11 (Quarter pinched differentiable sphere theorem)**.** *Let (M^n, g) be a compact Riemannian manifold. If $\frac{1}{4} < \sec \leq 1$, then M^n is diffeomorphic to S^n.*

A key ingredient is the observation (following Böhm and Wilking) that nonnegative isotropic curvature is preserved by Ricci flow (established independently by Nguyen[9]).

[9] Huy T. Nguyen, "Isotropic curvature and the Ricci flow".

10.4 Pinched manifolds are compact

By establishing local versions of Hamilton's arguments, it becomes possible to apply them in the *noncompact* setting.

Theorem 10.12 (Ricci pinched three-manifolds are compact[10]). *If a complete Riemannian three-manifold (M^3, g) satisfies $\mathrm{R} > 0$ and*

$$\mathrm{Rc} \geq \alpha \mathrm{R} g$$

for some $\alpha > 0$, then M^3 is compact (indeed, $M^3 \cong S^3/\Gamma$).

[10] Deruelle, Schulze, and M. Simon, "Initial stability estimates for Ricci flow and three dimensional Ricci-pinched manifolds."; Lee and P. Topping, "Three-manifolds with non-negatively pinched Ricci curvature"; Lott, "On 3-manifolds with pointwise pinched nonnegative Ricci curvature"

The idea is to flow the metric by Ricci flow, preserving and improving the pinching condition until it converges to a round point in finite time. Note, though, that this is much harder to achieve in the (*a priori*) absence of compactness!

Theorem 10.12 should be compared with the Bonnet–Myers theorem (which guarantees that a complete Riemannian manifold with uniformly positive Ricci curvature, $\mathrm{Rc} \geq \alpha g > 0$, is compact).

There are higher dimensional versions of Theorem 10.12 which hold under stronger conditions.[11]

[11] B.-L. Chen and Zhu, "Complete Riemannian manifolds with pointwise pinched curvature"; Lee and P. Topping, "Manifolds with PIC1 pinched curvature"; Ni and Wu, "Complete manifolds with nonnegative curvature operator".

10.5 Exercises

Exercise 10.1. Given nonnegative numbers ρ_1, ρ_2 and ρ_3, show that

$$\sum_{i \neq k} \mathrm{R}(\rho_i + \rho_k - \tfrac{1}{2}\mathrm{R})\rho_i\rho_k - \left(\sum_i \rho_i^2\right)^2 \leq 0$$

with equality only if at least one of the numbers ρ_i vanish, where $\mathrm{R} \doteqdot \rho_1 + \rho_2 + \rho_3$.

Exercise 10.2. Show that, on any Riemannian three-manifold (M^3, g),

$$|\nabla \mathrm{Rc}|^2 \geq \tfrac{7}{20}|\nabla \mathrm{R}|^2.$$

HINT: *Split* $\nabla \mathrm{Rc}$ *into its trace and trace-free components.*

11
Conformal flow of surfaces by curvature

A key step in the proof of Hamilton's theorem on the convergence of three-manifolds of positive Ricci curvature (and its higher dimensional analogues) was the improvement of pinching of the eigenvalues of the Ricci curvature (or curvature operator). No such estimate is possible in the two-dimensional setting as, in that case, the curvature operator has only one component! Fortunately, in two-dimensions, the Ricci flow enjoys some additional structure, which actually allows us to prove something far stronger.

11.1 *Special properties of the Ricci flow in two space dimensions*

Since in two dimensions the Ricci tensor is in proportion to the metric[1], the Ricci flow takes the form

$$\partial_t g = -2\mathrm{K} g, \tag{11.1}$$

[1] This is a straightforward consequence of the algebraic symmetry properties of the Riemann curvature tensor.

where the scalar of proportion, K, is called the GAUSS CURVATURE. This equation is also the two-dimensional special case of a number of other higher dimensional flows (e.g. the Kähler Ricci flow, the Yamabe flow, and conformal flows by functions of the Schouten tensor). With this in mind, it is perhaps not surprising that (11.1) displays properties of these higher dimensional flows that are not necessarily shared by the Ricci flow in higher dimensions.

11.1.1 *The logarithmic fast diffusion equation and conformal invariance*

Two dimensional Ricci flow $(M^2 \times I, g)$ of a compact manifold M^2 is actually a CONFORMAL FLOW; that is, we can find a function $u \in C^\infty(M^2 \times I)$ such that

$$g_{(x,t)} = \mathrm{e}^{-2u(x,t)} g_{(x,0)} . \tag{11.2}$$

To prove this, observe that a time-dependent metric of the form (11.2) satisfies Ricci flow if and only if

$$\partial_t u g = -\frac{1}{2}\mathcal{L}_{\partial_t} g = \mathrm{Rc} = \mathrm{K} g.$$

That is,

$$\partial_t u = \mathrm{K}.$$

By Exercise 11.1,

$$\mathrm{K}(x,t) = \mathrm{e}^{2u(x,t)}(\Delta_0 u(x,t) + \mathrm{K}_0(x)),$$

where Δ_0 and K_0 are the Laplace–Beltrami operator and sectional curvature of g_0, so we conclude that $\mathrm{e}^{-2u} g_0$ satisfies Ricci flow if and only if

$$\partial_t u = \mathrm{e}^{2u}(\Delta_0 u + \mathrm{K}_0). \tag{11.3}$$

But this is a parabolic equation, and hence admits a (unique) solution u for a short-time, given the initial condition $u_0 = 0$. By uniqueness of solutions to Ricci flow on compact manifolds, $g = \mathrm{e}^{-2u} g_0$ must be the unique Ricci flow starting from g_0.

We note that (11.3) is equivalent to the LOGARITHMIC FAST DIFFUSION EQUATION

$$\partial_t v = \Delta_0 \log v - 2\mathrm{K}_0 \tag{11.4}$$

on (M^2, g_0) for the conformal factor $v = \mathrm{e}^{-2u}$.

11.1.2 *Preservation of negative curvature*

Since $\mathrm{Rc} = \mathrm{K} g$, the Gauss curvature (which is half the scalar curvature) evolves according to

$$(\partial_t - \Delta)\mathrm{K} = 2\mathrm{K}^2. \tag{11.5}$$

This means that negativity of curvature is preserved in two dimensions (recall that positivity of the scalar curvature is preserved in all dimensions). We also obtain an analogue of Proposition 9.11:

Proposition 11.1. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a compact two-manifold M^2.*

1. *If $\max_{M^2\times\{\alpha\}} \mathrm{K} = 0$ then either $\mathrm{K} \equiv 0$ or $\mathrm{K} < 0$ for $t \in (\alpha, \omega)$.*
2. *If $\max_{M^2\times\{\alpha\}} \mathrm{K} = -r^{-2} < 0$, then*

$$\max_{M^2\times\{t\}} \mathrm{K} \le -\frac{1}{r^2 + 2(t-\alpha)}$$

for $t \in (\alpha, \omega)$.

3. *If* $\max_{M^2\times\{\alpha\}} \mathrm{K} = r^{-2} > 0$, *then*

$$\max_{M^2\times\{t\}} \mathrm{K} \le \frac{1}{r^2 - 2(t-\alpha)}$$

for $t \in (\alpha, \omega)$.

In fact, we can do better by making use of the Gauss–Bonnet theorem.

11.1.3 *A simple formula for the area*

By the Gauss–Bonnet theorem and the first variation of area, the area of a two-dimensional Ricci flow changes at a precise rate:

$$\frac{d}{dt}\,\mathrm{area}(t) = -2\int_{M^2} \mathrm{K}\, d\mu = -4\pi\chi(M^2), \tag{11.6}$$

where $\chi(M^2)$ is the Euler characteristic of M^2. Integrating yields

$$\mathrm{area}(M^2, t) = \mathrm{area}(M^2, 0) - 4\pi\chi(M^2)t, \tag{11.7}$$

a remarkably simple—and useful—formula. Indeed, consider the average Gauss curvature

$$\kappa(t) \doteqdot \frac{\int_{M^2} \mathrm{K} d\mu}{\int_{M^2} d\mu} = \frac{2\pi\chi(M^2)}{\mathrm{area}(M^2,t)} = \frac{2\pi\chi(M^2)}{\mathrm{area}(M^2,0) - 4\pi\chi(M^2)t}.$$

By (11.6) (or (11.7)),

$$\frac{d}{dt}\kappa = -\frac{2\pi\chi(M^2)}{\mathrm{area}^2(M^2,t)}\frac{d}{dt}\,\mathrm{area}(M^2,t) = 2\kappa^2.$$

Recalling (11.5), we thus find that

$$(\partial_t - \Delta)(\mathrm{K}-\kappa) = 2(\mathrm{K}-\kappa)\left(\mathrm{K}-\kappa + \frac{4\pi\chi(M^2)}{\mathrm{area}(M^2,0) - 4\pi\chi(M^2)t}\right)$$

and hence, if we normalize so that $\mathrm{area}(M^2, 0) = 4\pi$,

$$\min_{M^2\times\{t\}} \mathrm{K} \ge \kappa + \phi \tag{11.8}$$

for $t \in [0, T)$, where ϕ is the solution to the problem

$$\begin{cases} \dfrac{d\phi}{dt} = 2\phi\left(\phi + \dfrac{\chi(M^2)}{1-\chi(M^2)t}\right) \\ \phi(0) = \phi_0 \doteqdot \min\limits_{M^2\times\{0\}}(\mathrm{K}-\kappa)\,; \end{cases}$$

that is (note that $\phi_0 \le 0$),

$$\phi(t) = \frac{\phi_0}{(1-\chi(M^2)t)(1-\chi(M^2)t - 2\phi_0 t)}$$
$$\sim \begin{cases} -\frac{1}{t^2} \text{ as } t \to \infty \text{ if } \chi(M^2) < 0 \\ -\frac{1}{t} \text{ as } t \to \infty \text{ if } \chi(M^2) = 0 \\ \frac{-1}{1-\chi(M^2)t} \text{ as } t \to \frac{1}{\chi(M^2)} \text{ if } \chi(M^2) > 0. \end{cases}$$

In particular, $T \le \frac{1}{\chi(M^2)}$ if $\chi(M^2) > 0$.

An analogous argument may be carried out to establish an upper bound for $\max_{M^2\times\{t\}} \mathrm{K}$, but that estimate will prove of little utility. We will obtain a congruous estimate from above by a different argument, which is strongly informed by the behaviour of solitons.

11.1.4 *The Chow–Hamilton entropy*

The Chow–Hamilton entropy[2] of a Riemannian surface $(M^2, g)$of positive curvature is defined to be

$$\mathscr{E}(M^2, g) \doteqdot \frac{\operatorname{area}(M^2, g)}{\chi(M^2)} \exp\left(\frac{1}{\chi(M^2)} \int_{M^2} \mathrm{K} \log \mathrm{K}\, d\mu\right). \tag{11.9}$$

[2] Compare this to the Nash entropy, $-\int u \log u$, of a positive function u, introduced by Nash, "Continuity of solutions of parabolic and elliptic equations".

Proposition 11.2 (Monotonicity of the Chow–Hamilton entropy[3])**.** *Along any Ricci flow with positive curvature $(M^2 \times I, g)$ on a compact surface M^2,*

$$\frac{d}{dt} \mathscr{E}(M^2, g_t) \le 0$$

at all times, with strict inequality unless $\partial_t \log \mathrm{K} - |\nabla \log \mathrm{K}|^2$ is constant in space.

[3] The stated result was established by Richard S. Hamilton, "The Ricci flow on surfaces". A modified version which allows the curvature to change sign was established by Chow, "The Ricci flow on the 2-sphere".

Proof. To make the calculations slightly simpler, we assume that $M^2 \cong S^2$ but the general case is the same. Using the evolution equations for curvature (11.5) and area (9.8), we find that

$$\frac{d}{dt} \int_{M^2} \mathrm{K} \log \mathrm{K}\, d\mu = \int_{M^2} \left(\frac{\partial_t \mathrm{K}}{\mathrm{K}} - \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}^2}\right) \mathrm{K}\, d\mu.$$

Set

$$Q \doteqdot \partial_t \log \mathrm{K} - |\nabla \log \mathrm{K}|^2.$$

Using the formulae

$$[\nabla, \Delta] f = -\mathrm{K}\, \nabla f, \quad \nabla_t \nabla f = \nabla \partial_t f + \mathrm{K}\, \nabla f \text{ and } [\partial_t, \Delta] f = 2\, \mathrm{K}\, \Delta$$

(and a little elbow grease) we find that

$$(\partial_t - \Delta) Q = 2g(\nabla \log \mathrm{K}, \nabla Q) + 2\left|\nabla^2 \log \mathrm{K} + \mathrm{K}\, g\right|^2. \tag{11.10}$$

We thus find that

$$\begin{aligned}\frac{d^2}{dt^2}\int_{M^2} \mathrm{K}\log\mathrm{K}\, d\mu &= \frac{d}{dt}\int_{M^2} Q\mathrm{K}\, d\mu \\ &= \int_{M^2} (\mathrm{K}\,\partial_t Q + Q\Delta\,\mathrm{K})\, d\mu \\ &= \int_{M^2} \left(\Delta(\mathrm{K}Q) + 2\,\mathrm{K}\left|\nabla^2\log\mathrm{K} + \mathrm{K}\,g\right|^2\right) d\mu \\ &= 2\int_{M^2} \left|\nabla^2\log\mathrm{K} + \mathrm{K}\,g\right|^2 \mathrm{K}\, d\mu.\end{aligned}$$

Estimating

$$\begin{aligned}\left|\nabla^2\log\mathrm{K} + \mathrm{K}\,g\right|^2 &= \left|\nabla^2\log\mathrm{K} - \tfrac{1}{2}\Delta\log\mathrm{K}\,g + \tfrac{1}{2}\left(\Delta\log\mathrm{K} + 2\mathrm{K}\right)g\right|^2 \\ &= \left|\nabla^2\log\mathrm{K} - \tfrac{1}{2}\Delta\log\mathrm{K}\,g\right|^2 + \tfrac{1}{2}\left(\Delta\log\mathrm{K} + 2\mathrm{K}\right)^2 \\ &\geq \tfrac{1}{2}\left(\Delta\log\mathrm{K} + 2\mathrm{K}\right)^2 \\ &= \tfrac{1}{2}Q^2, \qquad (11.11)\end{aligned}$$

this becomes

$$\frac{d^2}{dt^2}\int_{M^2} \mathrm{K}\log\mathrm{K}\, d\mu \geq \int_{M^2} Q^2\,\mathrm{K}\, d\mu.$$

Applying Hölder's inequality and the Gauss–Bonnet theorem, we arrive at

$$\frac{d^2}{dt^2}\int_{\Gamma_t} \mathrm{K}\log\mathrm{K}\, ds \geq \frac{1}{4\pi}\left(\frac{d}{dt}\int_{\Gamma_t} \mathrm{K}\log\mathrm{K}\, ds\right)^2. \qquad (11.12)$$

On the other hand, recalling (4.2), we see that the function

$$\phi(t) \doteqdot \frac{32\pi^2}{\mathrm{area}(M^2, g_t)} = \frac{4\pi}{\frac{\mathrm{area}(M^2,g_0)}{8\pi} - t}$$

satisfies the corresponding ODE

$$\frac{d\phi}{dt} = \tfrac{1}{4\pi}\phi^2.$$

Moreover, by Perelman's curvature estimate (Theorem 9.21), the flow may be continued until the area tends to zero[4]; i.e. (by (9.8)) until time $T \doteqdot \frac{\mathrm{area}(M^2,g)}{8\pi}$. This means that

[4] We shall present an alternative argument for this below.

$$\phi(t) \to \infty \;\text{ as }\; t \to T,$$

and we may thereby deduce, by ODE comparison, that

$$\begin{aligned}\frac{d}{dt}\int_{M^2} \mathrm{K}\log\mathrm{K}\, d\mu &\leq \frac{32\pi^2}{\mathrm{area}(M^2, g_t)} \\ &= -4\pi\frac{d}{dt}\log\mathrm{area}(M^2, g_t).\end{aligned}$$

Rearranging, we conclude that

$$\frac{d}{dt}\log \mathscr{E}(\Gamma_t) \leq 0.$$

Now, if the inequality is saturated at some time t_0, then we may deduce from (11.12) that is saturated for all $t \leq t_0$. But this guarantees that the Hölder inequality is saturated, which ensures that Q is constant in space for $t \leq t_0$. □

11.2 *Self-similar solutions*

Recall that a metric g on a two-manifold M^2 generates a self-similarly expanding, steady or shrinking Ricci flow if there are a constant $\lambda \in \mathbb{R}$ and a vector field V such that

$$\mathrm{Rc} = \lambda g - \tfrac{1}{2}\mathcal{L}_V g. \tag{11.13a}$$

An important special class of solutions are those with $V = \operatorname{grad} f$ for some POTENTIAL FUNCTION f, in which case,

$$\mathrm{Rc} = \lambda g - \nabla^2 f. \tag{11.13b}$$

Theorem 11.3. *Every compact, two-dimensional gradient Ricci soliton has constant curvature.*

Proof. Let (M^2, g, f) be a gradient Ricci soliton on a compact two-manifold. By Exercise 11.2, the vector field $K \doteqdot \mathrm{J}(\nabla f)$ is Killing. Since M^2 is compact, there must be some $o \in M^2$ such that $\nabla f(o) = 0$ and hence $K(o) = 0$. It follows that K generates rotations, and hence we can find coordinates $(r, \theta) \in (0, R) \times \mathbb{R}/2\pi\mathbb{Z}$ such that $g = dr^2 + \psi^2(r)d\theta^2$. The claim now follows from the result of Exercise 8.1. □

Essentially the same argument yields the following.

Theorem 11.4. *The cigar is the only steady two-dimensional gradient Ricci soliton with positive curvature.*

Sketch of the proof. By Theorem 11.3, M^2 cannot be compact. It follows from Theorem 9.21 (though indirectly; see Theorem 13.2 below) that $\mathrm{K} \to 0$ as the distance to any fixed point x of M^2 goes to infinity. But then K attains a (positive) maximum at some point, at which $\nabla f = \nabla \mathrm{K}/\mathrm{K} = 0$. The claim now follows as in the previous theorem and Example 17. □

By manipulating the (gradient) soliton equation, we shall establish a suite of identities for two-dimensional (gradient) Ricci solitons.

First observe that taking the trace of (11.13a) yields (note that, for any vector field V, $\frac{1}{2}\mathcal{L}_V g$ is equal to the symmetric part of ∇V)

$$\mathrm{K} = \lambda - \tfrac{1}{2}\operatorname{div} V, \tag{11.14a}$$

or, in the gradient case,

$$-\Delta f = 2(\mathrm{K} - \kappa). \tag{11.14b}$$

From this, we see that (11.13a) is equivalent to

$$\mathcal{L}_V g - \operatorname{div} V g = 0. \tag{11.15a}$$

or, in the gradient case,

$$\nabla^2 f - \tfrac{1}{2}\Delta f g = 0. \tag{11.15b}$$

Moreover, in case M^2 is compact,

$$0 = -\int_{M^2} \operatorname{div} V \, d\mu = 2\int_{M^2} (\mathrm{K} - \lambda)\, d\mu$$

and hence

$$\lambda = \kappa \doteqdot \frac{\int_{M^2} \mathrm{K}\, d\mu}{\int_{M^2} d\mu}.$$

Taking the divergence of (11.15a), we find that

$$\Delta V + \operatorname{Rc}(V) = 0 \tag{11.16a}$$

which, on a gradient Ricci soliton becomes

$$\nabla \mathrm{K} - \mathrm{K}\nabla f = 0. \tag{11.16b}$$

Next observe that taking the divergence of (11.16b) yields, in the gradient case,

$$\Delta \mathrm{K} - \nabla_{\nabla f}\mathrm{K} + 2\mathrm{K}(\mathrm{K} - \lambda) = 0. \tag{11.17}$$

We may also rewrite (11.16b), using (11.15b), as

$$\begin{aligned} 0 &= \nabla \mathrm{K} - (\mathrm{K} - \lambda)\nabla f - \lambda \nabla f \\ &= \nabla \mathrm{K} + \tfrac{1}{2}\Delta f \nabla f - \lambda \nabla f \\ &= \nabla \mathrm{K} + \nabla_{\nabla f}\nabla f - \lambda \nabla f \\ &= \nabla\left(\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \lambda f\right). \end{aligned} \tag{11.18}$$

Thus, in the gradient case,

$$\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \lambda f = C$$

for some constant $C \in \mathbb{R}$. Equivalently (by (11.14b)),

$$-\Delta f + \tfrac{1}{2}|\nabla f|^2 - \mathrm{K} - \lambda f = C - 2\lambda.$$

Remarkably, this is the Euler–Lagrange equation for a certain constrained energy functional.

Proposition 11.5. *Given any compact Riemannian surface (M^2, g) and any $\lambda \in \mathbb{R}$, define, for any smooth function f,*

$$F(f) \doteqdot \int_{M^2} \left(\tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right) \mathrm{e}^{-f}\, d\mu. \qquad (11.19)$$

If $\{f_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ is a smooth variation of $f = f_0$ which satisfies the weighted volume constraint

$$\frac{d}{d\varepsilon} \int_{M^2} \mathrm{e}^{-f_\varepsilon}\, d\mu \equiv 0,$$

then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(f_\varepsilon) = -\int_{M^2} \left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right) h\, \mathrm{e}^{-f}\, d\mu,$$

where $h \doteqdot \frac{d}{d\varepsilon}|_{\varepsilon=0} f_\varepsilon$. In particular, if f is a stationary point of the action with respect to constrained variations, then $-\Delta f + \frac{1}{2}|\nabla f|^2 - \mathrm{K} - \lambda f$ is constant.

Proof. Since the weighted volume constraint guarantees that

$$0 = -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{M^2} \mathrm{e}^{-f_\varepsilon}\, d\mu = \int_{M^2} h \mathrm{e}^{-f}\, d\mu,$$

we find that

$$\begin{aligned} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(f_\varepsilon) &= \int_{M^2} \left(g(\nabla f, \nabla h) + \lambda h - h\left[\tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right]\right) \mathrm{e}^{-f}\, d\mu \\ &= -\int_{M^2} \left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda f\right) h \mathrm{e}^{-f}\, d\mu. \end{aligned}$$

This is the first claim. The second claim follows since any function h which is $L^2(\mathrm{e}^{-f}\, d\mu)$-orthogonal to the constant functions gives rise to an admissible variation. □

Theorem 11.6. *All compact, two-dimensional shrinking Ricci solitons are gradient.*

Sketch of the proof. We will prove this statement in all dimensions in §12.2. The idea is to find a minimizer f for the functional F (using classical methods from the calculus of variations). If $\lambda g - \mathrm{Rc} = \frac{1}{2}\mathcal{L}_V g$ for some vectorfield V (the Ricci soliton equation), then this minimizer will satisfy $\lambda g - \mathrm{Rc} = \nabla^2 f$ (the *gradient* Ricci soliton equation). (Note that this does not necessarily mean that the original soliton vector field V is given by $V = \nabla f$—the two could differ by a Killing vector field). □

Combined with Theorem 11.3, we find that

Corollary 11.7. *every compact, two-dimensional shrinking Ricci soliton has constant curvature.*

Consider now, for some gradient Ricci soliton (M^2, g, f), the corresponding self-similar Ricci flow $\phi^* g$, ϕ being the flow of ∇f. This Ricci flow will satisfy the soliton equation with

$$\lambda(t) = \begin{cases} \frac{1}{-2t} & \text{for } t \in (-\infty, 0) \text{ (shrinking case)} \\ 0 & \text{for } t \in (-\infty, \infty) \text{ (steady case)} \\ \frac{1}{2t} & \text{for } t \in (0, \infty) \text{ (expanding case).} \end{cases}$$

Thus, by (11.18),

$$\begin{aligned} \partial_t f &= \nabla_{\nabla f} f \\ &= |\nabla f|^2 \\ &= -2\,\mathrm{K} + 2\lambda f + C \\ &= -2(\mathrm{K} - \lambda) - 2\lambda + 2\lambda f + C \\ &= \Delta f + 2\lambda f + C - 2\lambda. \end{aligned}$$

Since we are free to modify the potential function, at each time, by addition of a constant, some choice of potential function will satisfy the heat equation

$$(\partial_t - \Delta) f = 2\lambda f. \tag{11.20}$$

Alternatively, since $-\Delta f = 2(\mathrm{K} - \lambda)$, we may exhibit f as a solution to the *backwards* heat equation

$$\begin{aligned} (\partial_t + \Delta) f + 2\,\mathrm{K} &= 2\Delta f + 2\lambda f + C + 2(\mathrm{K} - \lambda) \\ &= -2(\mathrm{K} - \lambda) + 2\lambda f + C \\ &= |\nabla f|^2 + 2\lambda. \end{aligned} \tag{11.21}$$

Remarkably, this means that the function $h \doteqdot \lambda e^{-f}$ satisfies the CONJUGATE HEAT EQUATION:

$$-(\partial_t + \Delta - 2\mathrm{K}) h = 0.$$

The name comes from the fact that, along any two-dimensional Ricci flow $(M^2 \times I, g)$,

$$\begin{aligned} \frac{d}{dt} \int_{M^2} u\varphi \, d\mu &= \int_{M^2} \big(\partial_t u \varphi + u \partial_t \varphi - 2\mathrm{K} u\varphi\big) \, d\mu \\ &= \int_{M^2} \big((\partial_t - \Delta) u\varphi + u(\partial_t + \Delta - 2\mathrm{K})\varphi\big) \, d\mu. \end{aligned}$$

so long as $\varphi(\cdot, t)$ is compactly supported. In particular, a smooth function $u : M^2 \times (a, b) \to \mathbb{R}$ satisfies the heat equation if and only if every smooth function $\varphi : M^2 \times (a, b) \to \mathbb{R}$ which is compactly supported in $M^2 \times (a, b)$ satisfies

$$\int_a^b \int_{M^2} u(\partial_t - \Delta)^* \varphi \, d\mu \, dt = 0.$$

where

$$(\partial_t - \Delta)^* = -(\partial_t + \Delta - 2\mathrm{K})$$

is the CONJUGATE HEAT OPERATOR.

11.3 *The differential Harnack inequality*

The classical heat equation exhibits a remarkable property, known as the (matrix) DIFFERENTIAL HARNACK INEQUALITY, which states that any positive solution $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ must satisfy

$$\nabla^2 \log u + \frac{\mathrm{I}}{2t} \geq 0. \tag{11.22}$$

In fact, the inequality must be strict, unless u is a constant multiple of the (self-similar) fundamental solution, $\rho(x,t) \doteqdot (4\pi t)^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-x_0|^2}{4t}}$ for some x_0. Integrating the trace of (11.22) along spacetime curves of the form $t \mapsto (\gamma(t), t)$, with γ a geodesic joining points x_1 and x_2, yields the classical HARNACK INEQUALITY:

$$(4\pi t_2)^{\frac{n}{2}}\, u(x_2, t_2) \geq (4\pi t_1)^{\frac{n}{2}}\, u(x_1, t_1) \exp\left(-\frac{|x_2 - x_1|^2}{4(t_2 - t_1)} \right), \tag{11.23}$$

for any x_2, x_1 and any $t_2 > t_1$.

For an ANCIENT[5] solution $u : \mathbb{R}^n \times (-\infty, \infty) \to \mathbb{R}$, performing a series of time-translations yields the stronger inequality

[5] I.e. having an infinite past.

$$\nabla^2 \log u \geq 0.$$

Again, we have strict inequality, except in the exceptional circumstance that $\nabla^2 \log u = 0$; that is, u is a constant multiple of the travelling wave solution, $u(x,t) = \mathrm{e}^{(x+tv)\cdot v}$ for some $v \in \mathbb{R}^n$.

Observe that, by (11.16b) and (11.17), a two-dimensional expanding gradient self-similar Ricci flow must satisfy

$$\partial_t \mathrm{K} = \Delta \mathrm{K} + 2\mathrm{K}^2 = \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}} - \frac{\mathrm{K}}{t},$$

while a two-dimensional steady gradient self-similar Ricci flow must satisfy

$$\partial_t \mathrm{K} = \Delta \mathrm{K} + 2\mathrm{K}^2 = \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}}.$$

Theorem 11.8 (Differential Harnack inequality[6]). *Along any Ricci flow $(M^2 \times [0,T), g)$ with positive curvature on a compact two-manifold,*

[6] Richard S. Hamilton, "The Ricci flow on surfaces". Cf. Chow, "The Ricci flow on the 2-sphere"

$$\frac{\partial_t \mathrm{K}}{\mathrm{K}} - \frac{|\nabla \mathrm{K}|^2}{\mathrm{K}^2} + \frac{1}{t} \geq 0 \;\; \textit{for} \;\; t \in (0,T). \tag{11.24}$$

Moreover, if (11.24) *holds along a Ricci flow $(M^2 \times (0,T), g)$ on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless $(M^2 \times (0,T), g)$ is an expanding self-similar solution.*

On any non-flat ancient two-dimensional Ricci flow $(M^2 \times (-\infty, T), g)$,

$$\frac{\partial_t K}{K} - \frac{|\nabla K|^2}{K^2} \geq 0. \tag{11.25}$$

Moreover, if (11.25) *holds along Ricci flow* $(M^2 \times (-\infty, T), g)$ *on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless* $(M^2 \times (-\infty, T), g)$ *is a steady self-similar solution.*

Proof. Consider the functions

$$Q \doteqdot \partial_t \log K - |\nabla \log K|^2$$

and

$$P \doteqdot t(\partial_t \log K - |\nabla \log K|^2) + 1.$$

Note that $P \equiv 0$ if and only if $(M^n \times I, g)$ is an expanding self-similar solution and $Q \equiv 0$ if and only if $(M^n \times I, g)$ is a steady self-similar solution.

Recalling (11.10), we have

$$(\partial_t - \Delta)Q = 2g(\nabla \log K, \nabla Q) + 2\left|\nabla^2 \log K + K g\right|^2.$$

Applying (11.11), we thus find that

$$(\partial_t - \Delta)P \geq 2g(\nabla \log K, \nabla P) + QP.$$

Since $P|_{t=0} = 1 > 0$, the maximum principle implies that $P \geq 0$ for positive times, and either $P > 0$ or $P \equiv 0$. The claims follow. □

Note that, by continuity, smooth limits of Ricci flows on compact surfaces satisfy the differential Harnack inequality (and hence also the rigidity case by the strong maximum principle).

Corollary 11.9 ((Integral) Harnack inequality)**.** *Along any Ricci flow* $(M^2 \times [0, T), g)$ *with positive curvature on a compact two-manifold,*

$$\frac{K(x_2, t_2)}{K(x_1, t_1)} \geq \left[\frac{t_2}{t_1} \exp\left(\frac{d^2(x_1, x_2, t_1)}{4(t_2 - t_1)}\right)\right]^{-1}$$

for any $x_1, x_2 \in M^2$ *and any* $0 < t_1 < t_2 < T$*, with strict inequality unless* $(M^2 \times [0, T), g)$ *is an expanding self-similar solution.*

Proof. Integrate the differential Harnack inequality along curves of the form $t \mapsto (t, \gamma(t))$. □

In fact, Theorem 11.8 is the trace version of the following more general "matrix Harnack inequality".

Theorem 11.10 (Matrix differential Harnack inequality[7]). *Along any Ricci flow* $(M^2 \times [0,T), g)$ *with positive curvature on a compact two-manifold* M^2,

$$\left(\partial_t \mathrm{K} - \mathrm{K}^2 + \frac{1}{t}\mathrm{K}\right)|W|^2 - \nabla_W \nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}|U|^2 \geq 0 \tag{11.26}$$

for every time-dependent vector field W *and two-form* U*. Moreover, if* (11.26) *holds along a Ricci flow* $(M^2 \times (0,T), g)$ *on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless* $(M^2 \times (0,T), g)$ *is an expanding self-similar solution.*

Along any ancient Ricci flow $(M^2 \times (-\infty, T), g)$ *with positive curvature on a compact two-manifold* M^2,

$$\left(\partial_t \mathrm{K} - \mathrm{K}^2\right)|W|^2 - \nabla_W \nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K}|U|^2 \geq 0 \tag{11.27}$$

for every time-dependent vector field W *and two-form* U*. Moreover, if* (11.27) *holds along a Ricci flow* $(M^2 \times (-\infty,T), g)$ *on a (not necessarily compact) connected two manifold, then it holds with strict inequality, unless* $(M^2 \times (-\infty,T), g)$ *is a steady self-similar solution.*

[7] Richard S. Hamilton, "The Harnack estimate for the Ricci flow"

Proof. Motivated by various identities which hold on expanding (and steady) solitons, one considers the forms

$$Q(U,W) \doteqdot \left(\partial_t \mathrm{K} - \mathrm{K}^2\right) g(W,W) - \nabla_W \nabla_W \mathrm{K} + 2g(\nabla \mathrm{K} \wedge W, U) + \mathrm{K} g(U,U)$$

and

$$P(U,W) \doteqdot tQ(U,W) + \mathrm{K} g(W,W).$$

After some arduous computations (motivated by various identities which hold on solitons), it is possible to obtain a suitable differential inequality for P. ☐

11.4 The monotonicity formula for Perelman's functional

Given a Ricci flow $(M^2 \times I, g)$ on a compact surface M^2, define, for any $f : M^2 \times I \to \mathbb{R}$ and $\tau : I \to \mathbb{R}$, the functional

$$\mathscr{P}(f,g,\tau) \doteqdot \int_{M^2} \left[\tau\left(|\nabla f|^2 + 2\mathrm{K}\right) + f - 2\right](4\pi\tau)^{-1} \mathrm{e}^{-f} d\mu. \tag{11.28}$$

Observe that, when τ is identified with backwards time, Perelman's functional $\mathscr{P}$ is just a multiple of the functional F of (11.19) in the shrinking case, $\lambda > 0$ (with f replaced by $f - 2$).

Now, on a self-similarly shrinking Ricci flow $(M^2 \times (-\infty, 0), g)$ with potential function f and τ taken to be negative time,

$$\begin{aligned}\mathscr{P}(f, g, \tau) &= \int_{M^2} \Big[\tau\left(|\nabla f|^2 + 2\mathrm{K}\right) + f - 2\Big](4\pi\tau)^{-1}\mathrm{e}^{-f} d\mu \\ &= \frac{1}{2\pi}\int_{M^2} \left[\tfrac{1}{2}|\nabla f|^2 + \mathrm{K} + \lambda(f-2)\right) \mathrm{e}^{-f} d\mu \\ &= \frac{1}{\pi}\int_{M^2} \lambda f \mathrm{e}^{-f} d\mu\end{aligned}$$

due to (11.18) (and the choice of normalization of f). Since (by (11.21)) $\lambda\mathrm{e}^{-f}$ satisfies the conjugate heat equation, we find that

$$\begin{aligned}\frac{d}{dt}\mathscr{P}(f, g, \tau) &= \frac{1}{\pi}\int_{M^2} (\partial_t - \Delta) f \lambda \mathrm{e}^{-f}\, d\mu \\ &= \frac{1}{\pi}\int_{M^2} \left(\partial_t f - |\nabla f|^2\right) \lambda \mathrm{e}^{-f}\, d\mu \\ &= 0.\end{aligned}$$

The following remarkable *inequality* holds along a general Ricci flow on a compact surface.[8]

Theorem 11.11 (Perelman's monotonicity formula[9])**.** *Let $(M^2 \times I, g)$ be a Ricci flow on a compact surface M^2. If f and τ satisfy*

$$\begin{cases}(\partial_t + \Delta + 2\mathrm{K}) f = |\nabla f|^2 + \frac{1}{\tau}, \\ \dfrac{d\tau}{dt} = -1,\end{cases}$$

then

$$\frac{d}{dt}\mathscr{P}(f, g, \tau) = 2\tau \int_{M^2} \left|\mathrm{Rc} + \nabla^2 f - \tfrac{1}{2\tau} g\right|^2 \mathrm{e}^{-f}\, d\mu \qquad (11.29)$$

so long as $\tau > 0$. In particular, the Perelman entropy

$$\mu(M^2, g_t, t_0 - t) \doteqdot \inf\left\{\mathscr{P}(g_t, f, t_0 - t) : \frac{1}{4\pi(t_0 - t)}\int_{M^2} \mathrm{e}^{-f}\, d\mu_t = 1\right\}$$

is nondecreasing for $t < t_0$ (strictly, unless (M^2, g_{t_0+t}) is a gradient shrinking soliton with potential $f(\cdot, t_0 + t)$).

[8] We omit the proof, as we will establish a generalization of the formula to all dimensions in §12.4.

[9] Perelman, "The entropy formula for the Ricci flow and its geometric applications"

11.5 Noncollapsing

Roughly speaking, a sequence of Riemannian surfaces (M_j^2, g_j) is said to collapse if some sequence of neighbourhoods $U_j \subset M_j^2$ and scales λ_j can be found such that $(U_j, \lambda_j g_j)$ resemble a one-dimensional manifold as $j \to \infty$. One precise way to quantify this is to ask for a sequence of points $p_j \in M_j$ such that

$$\mathrm{inj}_{g_j}(p_j) \sup_{B_{j\,\mathrm{inj}_{g_j}(p_j)}(p_j)} |\mathrm{K}|^{\frac{1}{2}} \le j^{-1}, \qquad (11.30)$$

where $\mathrm{inj}_g(p)$ denotes the INJECTIVITY RADIUS of (M^2, g) at p—the radius of the largest ball in (T_pM^2, g_p) on which the exponential map is a diffeomorphism.

Note that $\mathrm{inj}_g |\mathrm{K}|^{\frac{1}{2}}$ is scale invariant. Thus, if (11.30) holds, then, at the scale of the *curvature*, the injectivity radius degenerates to zero. On the other hand, at the scale of the *injectivity radius*, the curvature is tending towards zero in arbitrarily large regions, and at this scale the regions converge to a flat surface.

Example 19. Consider the constant sequence $(M_j^2, g_j) = (\mathbb{R}^2, g_{\mathrm{cigar}})$, where, in polar coordinates

$$g_{\mathrm{cigar}} = dr^2 + \tanh^2 r\, d\theta^2$$

is the cigar metric. If p_j are a sequence of points with $r_j \to \infty$, then, on the one hand, $\mathrm{inj}_j(p_j) \to \pi$ as $j \to \infty$. On the other hand, since $r_j \to \infty$ as $j \to \infty$, we may arrange, by passing to a subsequence, that $r_j - j\pi \to \infty$, and hence (recalling that $\mathrm{K} = 2\,\mathrm{sech}^2\, r$)

$$\begin{aligned}\sup_{B_{j\,\mathrm{inj}_j(p_j)}(p_j)} \mathrm{K} &\le \sup_{B_{j\pi}(p_j)} \mathrm{K}\\ &\le \sup_{r_j - j\pi \le r \le r_j + j\pi} \mathrm{K}\\ &= \mathrm{K}(r_j + j\pi))\\ &= 2\,\mathrm{sech}^2(r_j - j\pi)\\ &= o(1) \ \text{ as } \ j \to \infty.\end{aligned}$$

So the sequence is collapsing. ■

11.5.1 *The isoperimetric estimate*

The RELATIVE ISOPERIMETRIC CONSTANT of a Riemannian two-sphere $(M^2 \cong S^2, g)$ is defined to be

$$\mathcal{I}(M^2, g) \doteqdot \inf_{\Gamma} \mathrm{relength}(\Gamma, g),$$

where the infimum is taken over all SEPARATING CURVES—regular Jordan curves $\Gamma \subset M^2$ which[10] separate M^2 into two topological disks, Ω_1 and Ω_2—and the RELATIVE LENGTH of a separating curve is defined by

$$\mathrm{relength}(\Gamma, g) \doteqdot \frac{\mathrm{length}(\Gamma, g)}{\mathrm{length}(\overline{\Gamma}, \overline{g})},$$

where the COMPARISON ARC $\overline{\Gamma}$ is the (unique up to isometry) shortest Jordan curve which separates the round sphere $(S^2, \overline{g})$ of the same area as (M^2, g) into regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ of the same areas as Ω_1 and Ω_2, respectively.

[10] Necessarily, by the Schoenflies theorem.

Obviously, the relative isoperimetric constant of a round sphere is one. Moreover, since

$$\mathrm{relength}(\Gamma, g) \to 1 \ \text{ as } \ \mathrm{length}(\Gamma, g) \to 0,$$

the relative isoperimetric constant cannot exceed one on any two-sphere (M^2, g). In fact, $\mathcal{I}(M^2, g) < 1$ unless (M^2, g) is isometric to a round sphere.

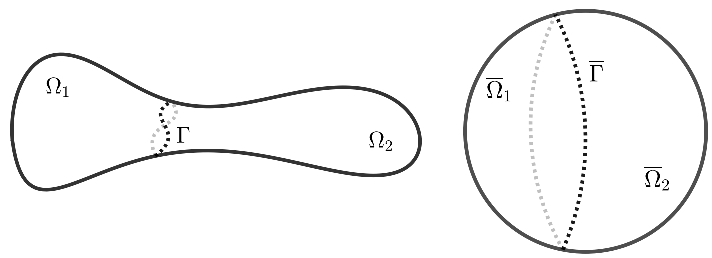


Figure 11.1: Given a curve, Γ, separating a surface $(M^2 \cong S^2, g)$, into regions Ω_1 and Ω_2, the comparison curve, $\overline{\Gamma}$, is the shortest curve separating M^2 into regions $\overline{\Omega}_1$ and $\overline{\Omega}_2$ which when measured in the round geometry on M^2 of the same area as g have the same areas as Ω_1 and Ω_2, respectively, as measured in the original geometry.

Hamilton proved that the isoperimetric constant of a Riemannian sphere does not decrease under Ricci flow.

Proposition 11.12. *Let $(M^2 \times [0, T), g)$ be a Ricci flow on a surface $M^2 \cong S^2$.*

$$\frac{d}{dt}\mathcal{I}(M^2, g_t) \geq 0$$

in the viscosity sense[11] *whenever $\mathcal{I}(M^2, g_t) < 1$. In particular,*

$$\mathcal{I}(M^2, g_t) \geq \mathcal{I}(M^2, g_0).$$

[11] This is a weak formulation of the differential inequality $\frac{du}{dt} \geq 0$ which applies to any continuous function. It asserts, for every $t_0 \in (0, T)$, that every smooth function $\varphi : [0, T) \to \mathbb{R}$ which touches u from below at t_0, in the sense that $u \leq \varphi$ for t in a backward neighbourhood $(t_0 - \delta, t_0]$ of t_0 with equality at t_0, satisfies $\frac{d\varphi}{dt}(t_0) \geq 0$.

Sketch of the proof. First note that, given any separating curve Γ for a surface (M^2, g), the first variation formula for the length of a separating curve in the comparison surface $(M^2, \overline{g})$, subject to the area constraint, guarantees that any comparison curve $\overline{\Gamma}$ has constant curvature. For such curves, we have the formula

$$\frac{4\pi}{\mathrm{length}^2(\overline{\Gamma}, \overline{g})} = \frac{1}{\mathrm{area}(\overline{\Omega}_1, \overline{g})} + \frac{1}{\mathrm{area}(\overline{\Omega}_2, \overline{g})},$$

where $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are the two regions bounded by Γ, which gives the formula

$$\mathrm{relength}(\Gamma, g) = \frac{\mathrm{length}(\Gamma, g)}{\sqrt{4\pi}} \left(\frac{1}{\mathrm{area}(\Omega_1, g)} + \frac{1}{\mathrm{area}(\Omega_2, g)} \right)^{\frac{1}{2}},$$

where Ω_1 and Ω_2 are the regions bounded by Γ.

Recall now that the length and area of a variation $\{\Gamma_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ of $\Gamma = \Gamma_0$ vary according to

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{length}(\Gamma_\varepsilon) = -\int_\Gamma g(\vec{\kappa}, V)\, ds,$$

and

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \operatorname{area}(\Omega_\varepsilon) = \int_\Gamma g(\mathsf{N}, V)\, ds\,,$$

where V is the variation field and N is the outward unit normal corresponding to the choice of bounded region, Ω. It follows that

$$\frac{\int_\Gamma \kappa g(\mathsf{N}, V)\, ds}{\int_\Gamma g(\mathsf{N}, V)\, ds} = \frac{1}{2}\operatorname{length}(\Gamma, g)\left(\frac{1}{\operatorname{area}(\Omega, g)} - \frac{1}{\operatorname{area}(M^2 \setminus \Omega, g)}\right)$$

at a minimizer of the relative length for any (nontrivial) variation V. From this we may conclude that a miminizer has constant curvature,

$$\kappa \equiv \frac{1}{2}\operatorname{length}(\Gamma, g)\left(\frac{1}{\operatorname{area}(\Omega, g)} - \frac{1}{\operatorname{area}(M^2 \setminus \Omega, g)}\right). \tag{11.31}$$

Consider now the constant distance variation, $\Gamma_\varepsilon = \{\exp_p \varepsilon\, \mathsf{N}_p : p \in \Gamma\}$. The second variation identities, along this variation, are given by

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \operatorname{length}(\Gamma_\varepsilon) = -\int_\Gamma \mathrm{K}\, ds$$

and

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \operatorname{area}(\Omega_\varepsilon) = \int_\Gamma \kappa\, ds\,.$$

(The first of these is established by differentiating the Gauss–Bonnet formula, $\int_{\Omega_\varepsilon} \mathrm{K}\, d\mu + \int_{\Gamma_\varepsilon} \kappa_\varepsilon\, ds = 2\pi$ while the second follows from the identity $\frac{d}{d\varepsilon}\operatorname{area}(\Omega_\varepsilon, g) = \operatorname{length}(\Gamma_\varepsilon, g)$.) Combining these and recalling (11.31), we conclude that

$$\int_\Gamma \mathrm{K}\, ds \leq \frac{\operatorname{length}^3(\Gamma, g)}{\operatorname{area}(\Omega, g)\operatorname{area}(M^2 \setminus \Omega, g)} \tag{11.32}$$

at a minimizer Γ of the relative length.

Now, if $\mathcal{I}(M^2, g) < 1$, then (since $\operatorname{relength}(\Gamma, g)$ approaches 1 as $\operatorname{length}(\Gamma, g)$ approaches 0) a minimizing sequence[12] of separating curves Γ_j will have lengths bounded uniformly from below. It is then possible to extract a suitable *weak* limit curve, Γ. Though this limiting curve may not be smooth *a priori*, the vanishing of the first variation of the relative length at Γ ensures that Γ has constant curvature in the corresponding weak sense, which guarantees that it is smooth (and connected, else a better constant is given by one of the components).

[12] I.e. $\operatorname{relength}(\Gamma_j, g) \to \mathcal{I}(M^2, g)$ as $j \to \infty$.

Now suppose that the metrics $\{g_t\}_{t\in[0,T)}$ on M^2 evolve by Ricci flow. Given $t_0 \in (0, T)$, if $\mathcal{I}(M^2, g_{t_0}) < 1$, then we can find some minimizing curve, Γ_{t_0}, as above. Given any variation $\{\Gamma_t\}_{t\in(t_0-\delta,t_0]}$ of Γ_{t_0}, the inequality $\operatorname{relength}(\Gamma_t, g_t) \geq \mathcal{I}(M^2, g_t)$ holds for $t \in (t_0 - \delta, t_0]$, with equality at time t_0. Thus, if φ is a lower support for $\mathcal{I}(M^2, g_t)$ at time t_0, then $\varphi(t) \leq \operatorname{relength}(\Gamma_t, g_t)$ with equality at time t_0, and hence, at time t_0,

$$\frac{d}{dt}\varphi \geq \frac{d}{dt}\operatorname{relength}(\Gamma_t, g_t)\,.$$

If we take $\{\Gamma_t\}_{t\in(t_0-\delta,t_0]}$ to be the constant distance variation (in the outwards direction with respect to a choice of bounded domain Ω), then

$$\frac{d}{dt}\,\mathrm{length}(\Gamma_t, g_t) = \int_{\Gamma_t}(\kappa - \mathrm{K})\,ds,$$

$$\begin{aligned}\frac{d}{dt}\,\mathrm{area}(\Omega_t, g_t) &= \mathrm{length}(\Gamma_t, g_t) - 2\int_{\Omega_t}\mathrm{K}\,d\mu\\ &= \mathrm{length}(\Gamma_t, g_t) + 2\int_{\Gamma_t}\kappa\,d\mu - 4\pi\end{aligned}$$

and

$$\frac{d}{dt}\,\mathrm{area}(M^2\setminus\Omega_t, g_t) = -\,\mathrm{length}(\Gamma_t, g_t) - 2\int_{\Gamma_t}\kappa\,d\mu - 4\pi,$$

where $\Gamma_t = \partial\Omega_t$ is either choice of orientation. Thus,

$$\begin{aligned}&\frac{\frac{d}{dt}\,\mathrm{relength}(\Gamma_t, g_t)}{\mathrm{relength}(\Gamma, g_t)}\\ &= \frac{\int_{\Gamma_t}(\kappa-\mathrm{K})\,ds}{\mathrm{length}(\Gamma, g_t)}\\ &\quad+\frac{2\pi\left(\mathrm{area}^2(\Omega_t, g_t)+\mathrm{area}^2(M^2\setminus\Omega_t, g_t)\right)}{\mathrm{area}(\Omega_t, g_t)\,\mathrm{area}(M^2\setminus\Omega_t, g_t)\,\left(\mathrm{area}(\Omega_t, g_t)+\mathrm{area}(M^2\setminus\Omega_t, g_t)\right)}\\ &\quad-\frac{1}{2}\left(\mathrm{length}(\Gamma_t, g_t)+2\int_{\Gamma_t}\kappa\,d\mu\right)\left(\frac{1}{\mathrm{area}(\Omega_t, g_t)}-\frac{1}{\mathrm{area}(M^2\setminus\Omega_t, g_t)}\right).\end{aligned}$$

Recalling (11.31) and (11.32), we find, at time $t = t_0$, that

$$\begin{aligned}\frac{d}{dt}\ln\varphi &\geq \frac{d}{dt}\ln\mathrm{relength}(\Gamma_t, g_t)\\ &\geq \frac{2\pi\left[\mathrm{area}^2(\Omega_t, g_t)+\mathrm{area}^2(M^2\setminus\Omega_t, g_t)\right](1-\mathrm{relength}^2(\Gamma_t, g_t))}{\mathrm{area}(\Omega_t, g_t)\,\mathrm{area}(M^2\setminus\Omega_t, g_t)\left[\mathrm{area}(\Omega_t, g_t)+\mathrm{area}(M^2\setminus\Omega_t, g_t)\right]}.\end{aligned}$$

The first claim follows.

To prove the second claim, it suffices to establish that

$$\mathcal{I}(\Gamma_t) - \mathcal{I}(\Gamma_0) + \varepsilon(1+t) \geq 0$$

for all $t \in [0, T)$ for any $\varepsilon > 0$. Note that the inequality holds strictly at time $t = 0$ for any positive ε. Suppose then that some $\varepsilon > 0$ and $t_0 \in (0, T)$ can be found such that

$$\mathcal{I}(\Gamma_t) - \mathcal{I}(\Gamma_0) + \varepsilon(1+t) \geq 0$$

for all $t \leq t_0$, but with equality at time $t = t_0$. But then the function

$$\varphi(t) \doteqdot \mathcal{I}(\Gamma_0) - \varepsilon(1+t)$$

is a lower support for $\mathcal{I}$ at time $t = t_0$, and hence

$$0 \leq \frac{d}{dt}\varphi = -\varepsilon < 0,$$

which is absurd. □

Combining this with Klingenberg's lemma yields the following lower bound for the injectivity radius.

Corollary 11.13. *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a surface $M^2 \cong S^2$.*

$$\mathrm{inj}^2(M^2, g_t) \geq \frac{\pi}{4} \frac{\mathcal{I}(M^2, g_0)}{\mathrm{K}_{\max}(t)} . \tag{11.33}$$

11.5.2 *A lower bound for area at the scale of the curvature*

Recall, from Theorem 11.11, that, for any choice of backwards time $\tau(t) = t_0 - t$, Perelman's entropy

$$\mu(M^2, g_t, \tau(t)) = \inf \left\{ \mathscr{P}(g_t, f, \tau(t)) : \frac{1}{4\pi\tau(t)} \int_{M^2} \mathrm{e}^{-f}\, d\mu_t = 1 \right\}$$

is nondecreasing along a Ricci flow $(M^2 \times [0,T), g)$ whilever $\tau(t) > 0$. Given $t_0 \in [0,T)$, set $\tau = t_0 + r^2 - t$ and consider the test function $u(\cdot, t_0) = (4\pi r^2)^{-\frac{n}{2}} \mathrm{e}^{-f(\cdot,t_0)}$ with $\mathrm{e}^{-f(\cdot,t_0)} = A\chi_{B_r(x_0,t_0)}$. Observe that, in order to satisfy the constraint

$$\int_{M^2} u(\cdot, t_0) d\mu_{t_0} = 1 ,$$

we should take $A \sim \frac{\mathrm{area}(B_r(x_0,t_0),t_0)}{r^2}$. Monotonicity of the entropy then implies

$$\begin{aligned} \mu(M^2, g_0, t_0 + r^2) &\leq \mu(M^2, g_{t_0}, r^2) \\ &\leq \mathscr{P}(g, f(\cdot, t_0), r^2) \\ &= \int_{M^2} \left[r^2 \left(|\nabla f|^2 + 2\mathrm{K} \right) + f - 2 \right] (4\pi r^2)^{-1} \mathrm{e}^{-f} d\mu \\ &\lesssim r^2 \max_{B_r(x_0,t_0)} \mathrm{K}(\cdot, t_0) + \ln \frac{\mathrm{area}(B_r(x_0, t_0), t_0)}{r^2} . \end{aligned}$$

Thus, if $\mathrm{K}(\cdot, t_0) \lesssim r^{-2}$, then we obtain the lower area bound

$$\frac{\mathrm{area}(B_r(x_0, t_0), t_0)}{r^n} \geq \kappa(M^2, g_0, T) .$$

I.e. areas are bounded uniformly from below at the scale of the curvature. By Proposition 9.20, this yields a uniform lower bound for the injectivity radius at the scale of the curvature, so the flow is noncollapsing.

Note though, that this argument is not quite rigorous, as the test function is not smooth (we took the gradient term to be zero), but it can easily be made so by introducing a cut-off function.[13]

[13] We omit the proof as we shall carry it out in general dimensions in §12.4.

Theorem 11.14. [14] *Let $(M^2 \times [0,T), g)$ be a Ricci flow on a compact surface M^2. Given $(x,t) \in M^2 \times [0,T)$, if $|\mathrm{K}| \leq r^{-2}$ on $B_r(x,t)$, $r \leq 1$, then*

$$\mathrm{area}(B_r(x,t), t) \geq \kappa r^2 ,$$

where $\kappa = \kappa(M^2, g_0, T)$.

[14] Perelman, "The entropy formula for the Ricci flow and its geometric applications".

11.6 Uniformization of surfaces by Ricci flow

Recall the lower curvature bound

$$\mathrm{K}-\kappa \gtrsim \begin{cases} -\dfrac{1}{t^2} \ \text{ as } \ t\to\infty \ \text{ if } \ \chi(M^2)<0 \\ -\dfrac{1}{t} \ \text{ as } \ t\to\infty \ \text{ if } \ \chi(M^2)=0 \\ -\dfrac{1}{1-\chi(M^2)t} \ \text{ as } \ t\to\dfrac{1}{\chi(M^2)} \ \text{ if } \ \chi(M^2)>0. \end{cases}$$

from (11.8). We shall obtain a complimentary upper bound by seeking an estimate which is saturated by soliton solutions. Recall that, on a gradient Ricci soliton, the potential function f satisfies

$$-\Delta f = 2(\mathrm{K}-\kappa). \tag{11.34}$$

On the other hand, since its right hand side has zero average, the equation (11.34) admits a solution f on *any* compact two-dimensional Ricci flow. Moreover, by the maximum principle, the solution f is unique up to the addition of a function of time.

Lemma 11.15. *Every Ricci flow $(M^2\times[0,T),g)$ on a compact two-manifold M^2 admits a curvature potential function satisfying*

$$(\partial_t-\Delta)f = 2\kappa f$$

and hence, assuming $\mathrm{area}(M^2,0)=4\pi$,

$$\frac{\min_{M^2\times\{0\}} f}{1-\chi(M^2)t} \le f \le \frac{\max_{M^2\times\{0\}} f}{1-\chi(M^2)t}. \tag{11.35}$$

Proof. Since, for any function u,

$$\partial_t\Delta u = \Delta\partial_t u + 2\mathrm{K}\Delta u,$$

we find that

$$\begin{aligned}\Delta\partial_t f &= \partial_t\Delta f - 2\mathrm{K}\Delta f\\ &= -2\partial_t(\mathrm{K}-\kappa)+4\mathrm{K}(\mathrm{K}-\kappa)\\ &= -2\Delta(\mathrm{K}-\kappa)-4(\mathrm{K}^2-\kappa^2)+4\mathrm{K}(\mathrm{K}-\kappa)\\ &= \Delta\Delta f + 2\kappa\Delta f\\ &= \Delta(\Delta f+2\kappa f).\end{aligned}$$

That is,

$$\Delta(\partial_t f-\Delta f-2\kappa f)=0.$$

So $\partial_t f-\Delta f-2\kappa f$ is a function of t only. By exploiting the freedom to add a function of t to f, we can easily guarantee that

$$(\partial_t-\Delta)f-2\kappa f=0$$

as claimed. The second claim then follows from the maximum principle, since, under the area normalization, $\kappa=\frac{\chi(M^2)}{1-\chi(M^2)t}$ □

Recall from (11.18) that, on a two-dimensional Ricci soliton,

$$0 = \nabla \left(\mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \kappa f \right) .$$

That is, $\mathrm{K} + \frac{1}{2}|\nabla f|^2 - \kappa f$ is a function of time only. Consider then, on a general (compact) two dimensional Ricci flow, the function

$$F \doteqdot \mathrm{K} + \tfrac{1}{2}|\nabla f|^2 - \kappa f$$

where f is a curvature potential satisfying Lemma 11.15.

Proposition 11.16. *The function F satisfies*

$$(\partial_t - \Delta)F = 2\kappa F - 2\left|\nabla^2 f - \tfrac{1}{2}\Delta f g\right|^2 \tag{11.36}$$

and hence

$$F \leq \frac{\max_{M^2\times\{0\}} F}{1 - \chi(M^2)t} \tag{11.37}$$

with strict inequality unless $(M^2 \times I, g)$ is a soliton.

Proof. We leave the verification of (11.36) as an exercise. The inequality (11.37) follows from the maximum principle, with strict inequality unless it holds identically. But in that case (11.36) implies that $\nabla^2 f - \frac{1}{2}\Delta f g = 0$. The final claim follows. □

This is an extremely useful estimate. For instance, we immediately obtain precise control on the maximal time of existence.

Corollary 11.17. *Let $(M^2 \times [0, T), g)$ be the maximal Ricci flow of a compact Riemannian surface (M^2, g_0). If $\chi(M^2) \leq 0$, then $T = \infty$. If $\chi(M^2) > 0$, then $T = \frac{1}{\chi(M^2)}$.*

Proof. By (11.35) and (11.37), there is a constant $C < \infty$ such that

$$\mathrm{K} \leq \frac{C}{1 - \chi(M^2)t}\left(1 - \frac{\chi(M^2)}{1 - \chi(M^2)t}\right) . \tag{11.38}$$

So the claim follows from the long-time existence theorem (Theorem 9.16). □

In fact, the estimate (11.37) in conjunction with the lower bound (11.8) will be sufficient to establish infinite time existence and convergence of the flow in case $\chi(M^2) \leq 0$. The case $\chi(M^2) > 0$ is somewhat trickier due to the finite time singularity. In that case, we analyze the singularity by rescaling and applying Theorem 9.19. The rescaling normalizes the curvature, but we still need to establish lower bounds for the injectivity radius. Note that, in the elliptic case, $\chi(M^2) > 0$, the universal cover is S^2 (which is compact); so it suffices to work on S^2, in which case Corollary 11.13 yields the desired bound.

Theorem 11.18 (Chow and Hamilton[15]). *Given a compact Riemannian surface (M^2, g_0), let $(M \times [0,T), g)$ be the maximal Ricci flow starting at (M^2, g_0).*

- *If $\chi(M^2) > 0$, then $T < \infty$ and $\frac{1}{2(T-t)} g_t$ converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = +1$ as $t \to T$.*
- *If $\chi(M^2) = 0$, then $T = \infty$ and g_t converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = 0$ as $t \to T$.*[16]
- *If $\chi(M^2) < 0$, then $T = \infty$ and $\frac{1}{2t} g_t$ converges uniformly in the smooth topology to a metric of constant curvature $\mathrm{K} = -1$ as $t \to \infty$.*

[15] Chow, "The Ricci flow on the 2-sphere"; Richard S. Hamilton, "The Ricci flow on surfaces"

[16] Observe that, in contrast to the proof of Hamilton's theorem (Theorem 10.8), the argument presented here does not provide a *rate* of convergence of the rescaled curves to the shrinking sphere. This may be remedied by a *stability argument* (cf. Exercise 4.8).

Sketch of the proof. Consider first the case $\chi(M^2) = 0$. In this case, $\kappa = 0$, and (11.8) becomes

$$\mathrm{K} \geq -\frac{1}{2t}.$$

The uniform upper bound for K of (11.38) then implies a uniform bound for $\nabla \mathrm{K}$ via the Bernstein estimates. Since the average of K is zero, we are then able to conclude that $\mathrm{K} \to 0$ as $t \to \infty$. Convergence of g_t to a limit metric then follows from the Ricci flow equation via the identity

$$-\frac{d}{dt} \log g_{(x,t)}(v, v) = 2\mathrm{K}(x,t) \tag{11.39}$$

for any $x \in M^2$ and any $v \in T_x M^2$. The limit metric is flat and the convergence is smooth, since the higher order Bernstein estimates and the interpolation inequality yield $\mathrm{K} \to 0$ to *all* orders as $t \to \infty$.

The hyperbolic case, $\chi(M^2) < 0$, may be treated similarly as the flat case, $\chi(M^2) = 0$. We omit the details.

The elliptic case, $\chi(M^2) > 0$, is more difficult. But at least we may work on the universal cover, S^2 (since it is compact). The lower bound (11.33) for the injectivity radius allows us to blow-up at the final time, $T < \infty$, to obtain an ancient limit Ricci flow. Note that (by the ODE comparison principle) $\max_{M^2 \times \{t\}} \mathrm{K} \geq \frac{1}{2(T-t)}$. Assume first that $\max_{M^2 \times \{t\}} \mathrm{K} \leq C(T-t)^{-1}$ (the expected rate of blow-up). Choose any sequence of times $t_j \nearrow T$ and points $x_j \in M^2$ such that

$$r_j^{-2} \doteqdot \max_{M^2 \times [0, t_j]} \mathrm{K} = \mathrm{K}_{(x_j, t_j)}$$

and consider the pointed rescaled Ricci flows $(M^2 \times I_j, x_j, g_j)$, where $I_j \doteqdot [-r_j^{-2} t_j, r_j^{-2}(T - t_j))$ and $(g_j)_{(x,t)} \doteqdot r_j^{-2} g(x, r_j^2 t + t_j)$. Observe that the curvature K_j of the rescaled Ricci flow satisfies

$$\mathrm{K}_j(x,t) = r_j^2 \mathrm{K}(x, r_j^2 t + t_j) \leq \frac{C r_j^2}{T - t_j - r_j^2 t} = \frac{C}{r_j^{-2}(T - t_j) - t} \leq \frac{2C}{1 - 2t}.$$

Since, by Corollary 11.13,

$$\operatorname{inj}(M^2,(g_j)_t) \geq \frac{\sqrt{\pi\,\mathcal{I}(M^2,g_0)}}{2},$$

some subsequence of the pointed rescaled Ricci flows $(M^2 \times I_j, x_j, g_j)$ converges locally uniformly in the smooth sense to a limit ancient Ricci flow $(M^2_\infty \times (-\infty,1), g_\infty)$. Since $\operatorname{area}(M^2, g_t) \to 0$ as $t \to T$, Proposition 11.12 implies that $\operatorname{diam}(M^2, g_t) \to 0$ as well. Proposition 9.7 then implies that

$$\begin{aligned}\operatorname{diam}(M^2,(g_j)_t) &= r_j^{-1}\operatorname{diam}(M^2, g_{r_j^2 t+t_j})\\ &\leq 10 r_j^{-2}(T-t_j-r_j^2 t)\\ &\leq C(1-2t).\end{aligned}$$

So the limit is compact, and hence $M^2_\infty = M^2 \cong S^2$.

Next, we claim that $\max_{M^2\times\{t\}} F/\kappa$ is constant on the limit flow. Recall that $\max_{M^2\times\{t\}} F/\kappa$ is nonincreasing on the original flow since

$$(\partial_t - \Delta)\frac{F}{\kappa} = -2\left|\nabla^2 f - \tfrac{1}{2}\Delta f g\right|^2.$$

In particular, $\max_{M^2\times\{t\}} F/\kappa$ takes a limit as $t \to T$. Now, since both numerator and denominator scale like curvature, we have, for any $a < b \in (-\infty, 1)$,

$$\max_{M^2\times\{b\}}\frac{F_j}{\kappa_j} - \max_{M^2\times\{a\}}\frac{F_j}{\kappa_j} = \max_{M^2\times\{r_j^2 b+t_j\}}\frac{F}{\kappa} - \max_{M^2\times\{r_j^2 a+t_j\}}\frac{F}{\kappa}$$

for all j sufficiently large. But both $r_j^2 a + t_j$ and $r_j^2 a + t_j$ tend to T, so the right hand side tends to zero. So $\max_{M^2\times\{t\}} F/\kappa$ is indeed constant on the limit flow. But then $\frac{F}{\kappa}$ must be constant, due to the strong maximum principle. We conclude that

$$\nabla^2 f - \tfrac{1}{2}\Delta f g \equiv 0$$

on the limit flow, which must therefore be a gradient Ricci soliton[17], and hence the shrinking sphere by Theorem 11.3. The theorem now follows from bootstrapping arguments.

[17] Alternatively, we could have invoked the Chow–Hamilton entropy and Proposition 11.2 here, since non-flat compact ancient Ricci flows on surfaces have positive curvature (by Corollary 9.12).

It remains to prove that $\mathrm{K}(T-t)$ remains bounded. Suppose then that, to the contrary,

$$\limsup_{t\nearrow T}\max_{M^2\times\{t\}}\mathrm{K}(T-t) = \infty.$$

For each j, choose $(x_j, t_j) \in M^2 \times [0, T)$ so that

$$(T-j^{-1}-t_j)\mathrm{K}(x_j,t_j) = \max_{M^2\times[0,T-j^{-1}]}(T-j^{-1}-t)\mathrm{K}$$

and set $r_j^{-2} \doteqdot \mathrm{K}(x_j, t_j)$. Consider the pointed rescaled Ricci flows $(M^2 \times [\alpha_j, \omega_j), x_j, g_j)$, where $\alpha_j \doteqdot -r_j^{-2} t_j$, $\omega_j \doteqdot r_j^{-2}(T - j^{-1} - t_j)$ and $(g_j)_{(x,t)} \doteqdot r_j^{-2} g_{(x, r_j^2 t + t_j)}$. Observe in this case that

$$\alpha_j \to -\infty, \quad \omega_j \to \infty,$$

and

$$\mathrm{K}_j(x,t) = r_j^2 \mathrm{K}(x, r_j^2 t + t_j) \le \frac{T - j^{-1} - t_j}{T - j^{-1} - r_j^2 t + t_j} = \frac{\omega_j}{\omega_j - t},$$

which is uniformly bounded on any compact time interval for j sufficiently large. Since, by Proposition 11.13, the injectivity radii remain uniformly bounded from below after rescaling, some subsequence of the pointed, rescaled Ricci flows $(M^2 \times [\alpha_j, \omega_j), x_j, g_j)$ must converge to an eternal limit pointed Ricci flow $(M^2_\infty \times (-\infty, \infty), x_\infty, g_\infty)$. Since this Ricci flow is the limit of compact Ricci flows, it satisfies the differential Harnack inequality. But, by construction,

$$\mathrm{K} \le \limsup_{j \to \infty} \frac{\omega_j}{\omega_j - t} = 1 = \mathrm{K}(x_\infty, 0).$$

Thus, at $(x_\infty, 0)$, $\partial_t \mathrm{K} = 0$ and $\nabla \mathrm{K} = 0$, and hence the rigidity case of the differential Harnack inequality implies that $(M^2_\infty \times (-\infty, \infty), g_\infty)$ is a steady soliton, which must be a cigar by Theorem 11.4 and the curvature normalization at $(x_\infty, 0)$. But the cigar violates the (scale invariant) lower bound for the isoperimetric constant (which passes to the limit as it is scale invariant and lower semi-continuous under local uniform convergence). This completes the proof. □

The original argument of Hamilton and Chow made use of the Kazdan–Warner identity—which relies on the uniformization theorem—to establish Theorem 11.3. The argument presented here for Theorem 11.3 (which does not require the uniformization theorem) was pointed out by Chen–Lu–Tian.[18]

A different proof of Theorem 11.18 was later found by Andrews–Bryan[19] and Bryan[20] (following Hamilton[21]). They were able to obtain a very sharp estimate for the *isoperimetric profile* under Ricci flow, sharp enough indeed to obtain sharp control on the curvature (which appears in the second variation of the isoperimetric profile), and thereby obtain convergence directly.

[18] X. Chen, Lu, and Tian, "A note on uniformization of Riemann surfaces by Ricci flow".

[19] Andrews and Bryan, "Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere".

[20] Bryan, "Curvature bounds via an isoperimetric comparison for Ricci flow on surfaces".

[21] Richard S. Hamilton, "An isoperimetric estimate for the Ricci flow on the two-sphere".

11.7 *Exercises*

Exercise 11.1. Suppose that the two metrics g and g_0 on a surface M^2 are related by $g = \mathrm{e}^{-2u} g_0$ for some function u. Show that the respective sectional curvatures K and K_0 are related by

$$K = e^{2u}(\Delta_0 u + K_0),$$

where Δ_0 is the Laplace–Beltrami operator induced by g_0.

Exercise 11.2. Let (M^2, g, f) be a two-dimensional gradient Ricci soliton. Show that

$$K \doteqdot J(\nabla f)$$

is a Killing vector field, where $J : TM^2 \to TM^2$ denotes counterclockwise rotation in the fibres through 90 degrees. Hint: *first show that* J *is parallel*.

Exercise 11.3. Show that a solution to the heat equation $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ satisfies

$$\nabla^2 \log u + \frac{n}{2t} = 0.$$

if and only if it is a fundamental solution.

Exercise 11.4. Prove that

$$\Delta \log u + \frac{1}{2t} \geq 0$$

for any positive periodic solution $u : T^n \times [0, \infty) \to \mathbb{R}$ to the heat equation. Hint: Consider the function $P \doteqdot 2t\Delta \log u + 1$.

Exercise 11.5. Prove that

$$\nabla^2 \log u + \frac{I}{2t} \geq 0$$

for any positive periodic solution $u : T^n \times [0, \infty) \to \mathbb{R}$ to the heat equation, where I is the Euclidean inner product. Hint: Consider the function $P \doteqdot 2t\nabla_V \nabla_V \log u + I$ for any fixed vector $V \in S^n$.

Exercise 11.6. Set $U = V \wedge W$ in (11.26) and trace with respect to W, and then optimize with respect to V to obtain (11.24).

Exercise 11.7. Andrews' inequality[22] states that

$$\tfrac{n}{n-1} \int_{M^n} \varphi^2 \, d\mu \leq \int_{M^n} |F|^2 + \int_{M^n} \mathrm{Rc}^{-1} \left(\nabla\varphi - \operatorname{div} F, \nabla\varphi - \operatorname{div} F \right) d\mu$$

on any compact Riemannian manifold (M^n, g) with positive Ricci curvature for every zero-average smooth function φ and every trace-free, symmetric, smooth two-tensor F, with equality only if

$$-\tfrac{n-1}{n} F = \nabla^2 f - \tfrac{1}{n} \Delta f \, g \quad \text{and} \quad \tfrac{n-1}{n} \operatorname{div} F = \tfrac{n-1}{n} \nabla\varphi - \mathrm{Rc}(\nabla f)$$

for some (any) potential function f for φ (solution to $-\Delta f = \varphi$).

[22] See Chow, Lu, and Ni, *Hamilton's Ricci flow*, Theorem B.18 for a proof.

(a) Using Andrews' inequality, show that, on any compact Riemannian manifold (M^n, g) with positive Ricci curvature,

$$\frac{n}{n-1}\int_{M^n}(\mathrm{R}-\rho)^2 d\mu \leq \alpha^2 \int_{M^n} |\mathring{\mathrm{Rc}}|^2 d\mu + \Big(1-\alpha\big(\tfrac{1}{2}-\tfrac{1}{n}\big)\Big)^2 \int_{M^n} \mathrm{Rc}^{-1}(\nabla \mathrm{R}, \nabla \mathrm{R})\, d\mu$$

for any $\alpha \in \mathbb{R}$, with equality only if

$$-\tfrac{n-1}{n}\alpha\mathring{\mathrm{Rc}} = \nabla^2 f - \tfrac{1}{n}\Delta f g \ \text{ and } \ (\tfrac{1}{2}+\tfrac{1}{n})\nabla \mathrm{R} = \tfrac{n}{n-1}\mathrm{Rc}(\nabla f)$$

for some (any) scalar curvature potential function f (solution to $-\Delta f = \mathrm{R}-\rho$), where ρ denotes the average scalar curvature.

(b) (Hamilton's inequality) Deduce (or otherwise prove) that, on any compact Riemannian surface (M^2, g) with positive curvature,

$$\int_{M^2}(\mathrm{K}-\kappa)^2 d\mu \leq \frac{1}{2}\int_{M^2}\frac{|\nabla \mathrm{K}|^2}{\mathrm{K}}d\mu \tag{11.40}$$

with equality only if

$$\nabla^2 f - \tfrac{1}{n}\Delta f g = 0 \ \text{ and } \ \nabla \mathrm{K} = \mathrm{K}\nabla f$$

for some (any) curvature potential function f (solution to $-\Delta f = 2(\mathrm{K}-\kappa)$).

Exercise 11.8. Use Hamilton's inequality to establish the monotonicity formula for the Chow–Hamilton entropy (Proposition 11.2).

12

Singularities and their analysis

There's only you
Nothing before you
Only you
Nothing beyond you
So now I'm without you
– Puscifer, "A Singularity"

We have seen that finite time singularities will necessarily occur under Ricci flow if the scalar curvature is initially positive. Under Ricci flow on the two-sphere, or in higher dimensions when the Ricci curvature is positive, we were able to deal with finite time singularities by "blowing up" and classifying the possible blow-up limits. As a result, we saw that the Ricci flow "uniformizes" compact surfaces, and positively curved manifolds in any dimension. One could therefore be forgiven for hoping that Ricci flow might deform *any* (not necessarily positively curved) Riemannian sphere to a round metric. This turns out to be too optimistic, however.

Example 20 (A "neckpinch" singularity[1])**.** Consider a Riemannian three-sphere which looks like two large, round three-spheres which are far apart but smoothly connected by a long, thin "neck" (as in Figure 12.1, say). If the neck is sufficiently thin compared to the spheres, then its curvature will be much larger, and it will contract much more quickly under the flow, and it seems likely that it should "pinch off" while the spherical components remain large. Configurations in which this behaviour is indeed exhibited were rigorously constructed by Angenent and Knopf.[2] ■

[1] Such examples appear to have first been described by Hamilton.

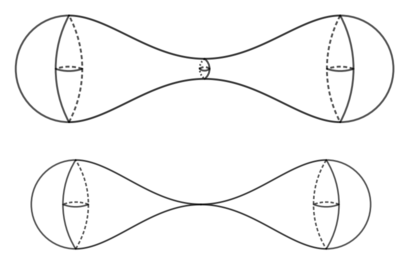


Figure 12.1: A "barbell" configuration. If the "bar" is sufficiently thin compared to the "bells", then it will "pinch off" before the bells disappear.

[2] S. Angenent and Knopf, "An example of neckpinching for Ricci flow on S^{n+1}".

Example 21 (A "degenerate neckpinch" singularity)**.** In the above example, we could imagine a continuous deformation of the initial configuration which shrinks one of the spherical components down to a radius comparable to the neck radius (as in Figure 12.2, say). In this configuration, we expect that the small spherical component of the initial manifold is able to contract quickly enough to slip through the neck before it pinches, the solution thereafter becoming positively curved and shrinking to a round point according to Hamilton's theorem. But then there must be a critical stage in the deformation such that for smaller deformations a neckpinch forms, while for larger deformations there is no neckpinch. Somehow, at the critical deformation, the smaller spherical component attempts a run through the neck,

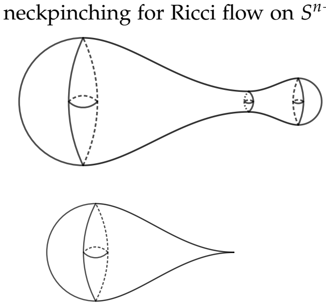


Figure 12.2: An asymmetric barbell configuration. If one of the bells is sufficiently small, it will "pass through" the bar before it pinches off. There is a critical configuration at which the bar pinches off just as the smaller bell is passing through it.

but gets caught just as it is about to emerge from the other side. Configurations in which this behaviour is indeed exhibited were rigorously constructed by Angenent, Isenberg and Knopf.[3] ■

[3] Sigurd B. Angenent, Isenberg, and Knopf, "Degenerate neckpinches in Ricci flow".

Example 22 (A "doubly degenerate neckpinch" singularity)**.** Imagine now performing this deformation in a symmetric manner, so that *both* bells get caught in the neck as it collapses. In this configuration, the metric does indeed shrink to a point at the singular time, T, but its asymptotic shape cannot be that of a round sphere: for at each time $t < T$, the Ricci tensor is not positive at the neck, so the pinching ratio $\inf \rho_1/\rho_n$ can be no better than zero at the singular time. ■

These examples demonstrate that singularities can potentially be quite complicated in dimensions $n \geq 3$, even in the absence of topology. On the other hand, at a neckpinch singularity, most of the manifold remains "non-singular" and the flow appears to be performing the opposite of a connected sum. This begs the question, "Can the flow be continued after a singularity, while keeping track of any topological changes at singular times?" Rather than attempting a comprehensive answer to this (very difficult) question, we shall merely present some basic results and tools which suggest that singularities are indeed somewhat "tamable", at least in certain special settings.

We begin by noting the following immediate corollary of Theorem 9.19, which demonstrates the importance of *ancient* Ricci flows in the analysis of singularities.

Lemma 12.1. *Let $(M^n \times [0,T), g)$, $T < \infty$, be a Ricci flow and $\{(x_k, t_k)\}_{k\in\mathbb{N}}$ a sequence of spacetime points $(x_k, t_k) \in M^n \times [0,T)$ with $t_k \to T$. Suppose that*

1. $r_k^{-2} \doteqdot |\mathrm{Rm}_{(x_k,t_k)}| \to \infty$ *as* $k \to \infty$*;*
2. *for every $A < \infty$ some $C < \infty$ can be found such that $B_{Ar_k}(x_k, t_k) \times (t_k - A^2 r_k^2, t_k] \Subset M^n \times [0,T)$ and*

$$\sup_{B_{Ar_k}(x_k,t_k)\times(t_k - A^2 r_k^2, t_k]} |\mathrm{Rm}| \leq C r_k^{-2}$$

   for every k; and

3. *there exists $\kappa > 0$ such that*

$$\mathrm{volume}(B_{Ar_k}(x_k, t_k), t_k) \geq \kappa r_k^n$$

   for every k.

For each k, define the rescaled Ricci flow $(M^n \times [-r_k^{-2} t_k, r_k^{-2}(T - t_k)), g_k)$ by

$$(g_k)_{(x,t)} \doteqdot r_k^{-2} g_{(x, r_k^2 t + t_k)}\,.$$

There exists a complete pointed ancient Ricci flow $(M^n \times (-\infty, \omega), x_\infty, g_\infty)$ *such that, after passing to a subsequence, the pointed rescaled Ricci flows* $(M^n \times (-r_k^2 t_k, 0], x_k, g_k)$ *converge locally uniformly in the smooth topology to* $(M^n \times (-\infty, 0], x_\infty, g_\infty)$*. That is, there exists an exhaustion* $\{U_k\}_{k\in\mathbb{N}}$ *of* M_∞ *by precompact open sets* U_k *satisfying* $\overline{U}_k \subset U_{k+1}$ *and a sequence of diffeomorphisms* $\phi_k : \overline{U}_k \to M$ *with* $\phi_k(x_\infty) = x_k$ *such that* $\phi_k^* g \to g_\infty$ *uniformly in the smooth topology on any compact subset of* $M_\infty \times (-\infty, 0]$.

12.1 Curvature pinching improves

Recall that the Ricci flow forces scalar curvature towards the positive[4]: if $\min_{M^n \times \{0\}} \mathrm{R} \geq n(n-1)r^{-2}$, then

$$\min_{M^n \times \{t\}} \mathrm{R} \geq -\frac{n(n-1)}{r^2 + 2(n-1)t}.$$

[4] See Proposition 9.11.

For three dimensional Ricci flow, a similar phenomenon holds for the full curvature operator.

Theorem 12.2 (Ellipticity improves[5]). *Let* $(M^3 \times [0, T), g)$ *be a Ricci flow on a compact three-manifold* M^3*. Denote by* $\lambda_1 \leq \lambda_2 \leq \lambda_3$ *the eigenvalues of the curvature operator. If* $\lambda_1(\cdot, 0) \geq -r^{-2}$*, then*

$$-2\lambda_1(\log(-\lambda_1) + \log(r^2 + 4t) - 3) \leq \mathrm{R} \tag{12.1}$$

for all $t \in [0, T)$ *wherever* $\lambda_1 < 0$.

[5] Richard S. Hamilton, "Non-singular solutions of the Ricci flow on three-manifolds"; Ivey, "Ricci solitons on compact three-manifolds"

Sketch of the proof. Note that the hypotheses ensure that the claim is true at the initial time, since

$$\begin{aligned} \tfrac{1}{2}\mathrm{R} + \lambda_1(\log(-r^2\lambda_1) - 3) &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_1(\log(-r^2\lambda_1) - 3) \\ &\geq \lambda_1 \log(-r^2\lambda_1), \end{aligned}$$

which is nonnegative wherever $-r^{-2} \leq \lambda_1 \leq 0$. We will establish the claim using the vector bundle maximum principle. After passing to the orientation cover in case M^3 is not orientable, we can make life easier by identifying the bundle of two-planes with the tangent bundle via the Hodge star operator, $* : TM^3 \mapsto TM^3 \wedge TM^3$, which is characterized by

$$*e_1 = e_2 \wedge e_3, \quad *e_2 = e_3 \wedge e_3, \quad *e_3 = e_1 \wedge e_2$$

on any orthonormal frame, $\{e_j\}_{j=1}^3$. This identifies Rm with a self-adjoint endomorphism/symmetric bilinear form on TM^3 via

$$\mathrm{Rm}(u, v) \doteqdot \mathrm{Rm}(*u, *v).$$

Under this identification, the Lie algebra square becomes the usual cofactor operator[6]. In particular, in an orthonormal frame that diagonal-

[6] See, e.g., Andrews and Hopper, *The Ricci flow in Riemannian geometry*, Claim 12.15

izes an algebraic curvature operator $S = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, the reaction term in the evolution equation for Rm becomes[7]

$$Q(S) = 2(S^2 + S^\#) = 2\,\mathrm{diag}(\lambda_1^2 + \lambda_2\lambda_3, \lambda_2^2 + \lambda_3\lambda_1, \lambda_3^2 + \lambda_1\lambda_2)\,.$$

So set

$$\psi(\lambda_1, \lambda_2, \lambda_3, t) \doteqdot \lambda_1 + \lambda_2 + \lambda_3 + \lambda_1(\log(-\lambda_1) + \log(r^2 + 4t) - 3)$$

and consider the subset K of algebraic curvature operators defined by imposing either the inequality

1. $-\frac{1}{r^2+4t} \le \lambda_1 < 0$,

or the inequalities

2. (a) $\lambda_1 + \lambda_2 + \lambda_3 \ge -\frac{3}{r^2+4t}$ and
   (b) $\psi(\lambda_1, \lambda_2, \lambda_3, t) \ge 0$.

This set is clearly invariant under parallel translation (a direct consequence of the fact that an orthonormal frame remains orthonormal under parallel translation). We shall omit the proof that K is convex in the fibre.[8]

Observe now that the second inequality is preserved under the flow, while the claim automatically holds under first (which implies the second). This reduces our task to showing that the algebraic curvature operator $\mathrm{diag}(\lambda_1^2 + \lambda_2\lambda_3, \lambda_2^2 + \lambda_3\lambda_1, \lambda_3^2 + \lambda_1\lambda_2)$ points into K at a boundary point $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \in \partial K_{(x,t)}$ on the $\{\psi = 0\} \cap \{\lambda_1 \le -\frac{1}{r^2+4t}\}$ component. Now, at such a point,

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_1(\log(-\lambda_1) + \log(\tfrac{1}{2}r^2 + 2t) - 3) = 0,$$

so that the boundary gradient takes the form

$$\begin{aligned} D\psi|_{(\lambda_1,\lambda_2,\lambda_3,\cdot)} &= \Big(2 + \log(-\lambda_1) + \log(r^2 + 4t) - 3, 1, 1\Big) \\ &= \left(2 + \frac{\lambda_1 + \lambda_2 + \lambda_3}{-\lambda_1}, 1, 1\right) \\ &= \left(1 + \frac{\lambda_2 + \lambda_3}{-\lambda_1}, 1, 1\right). \end{aligned}$$

We thus find that

$$\begin{aligned} &D\psi|_{(\lambda_1,\lambda_2,\lambda_3,\cdot)} \cdot (\lambda_1^2 + \lambda_2\lambda_3, \lambda_2^2 + \lambda_3\lambda_1, \lambda_3^2 + \lambda_1\lambda_2) \\ &\qquad = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3 + \frac{\lambda_2\lambda_3(\lambda_2 + \lambda_3)}{-\lambda_1}. \end{aligned}$$

This is clearly nonnegative if $\lambda_2 \ge 0$. To see that it is nonnegative when $\lambda_2 < 0$, we rewrite

$$\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3 + \frac{\lambda_2\lambda_3(\lambda_2 + \lambda_3)}{-\lambda_1} = \frac{\lambda_2^3}{\lambda_1} + \frac{\lambda_2 - \lambda_1}{-\lambda_1}(\lambda_2^2 + \lambda_3^2 + \lambda_2\lambda_3),$$

and estimate $\lambda_2\lambda_3 \ge -\frac{1}{2}(\lambda_2^2 + \lambda_3^2)$. □

[7] Ignoring the diffusion term in the evolution equation for Rm then yields the system

$$\frac{d}{dt}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = 2\begin{bmatrix}\lambda_1^2 + \lambda_2\lambda_3\\ \lambda_2^2 + \lambda_3\lambda_1\\ \lambda_3^2 + \lambda_1\lambda_2\end{bmatrix}.$$

In the rotationally symmetric setting (as in the dumbell example described above), $\lambda_1 \equiv \lambda_2$, which reduces this system to

$$\frac{d}{dt}\begin{bmatrix}\lambda_1\\ \lambda_3\end{bmatrix} = 2\begin{bmatrix}\lambda_1^2 + \lambda_1\lambda_3\\ \lambda_3^2 + \lambda_1^2\end{bmatrix}.$$

This system may be reformulated as

$$\frac{d\lambda_1}{d\lambda_3} = \frac{\lambda_1(\lambda_1 + \lambda_3)}{\lambda_1^2 + \lambda_3^2},$$

which admits the implicit solution

$$\log(-\lambda_1) = \frac{\lambda_2}{-\lambda_1} + 2\log\left(\frac{-\lambda_1}{\lambda_3 - \lambda_1}\right) + C$$

in the region $\lambda_1 < 0$. This solution exhibits the behaviour

$$\tfrac{1}{2}\mathrm{R} \sim \lambda_3 > -\lambda_1\log(-\lambda_1) \text{ as } \lambda_1 \to -\infty,$$

which provides some inspiration for the curvature terms in the inequality (12.1). See e.g. Chow, Lu, and Ni, *Hamilton's Ricci flow*, §6.5.

[8] See, e.g., Chow and Knopf, *The Ricci flow: an introduction*, Lemma 9.5.

Since $\mathrm{R} \geq 0$ on any compact ancient Ricci flow, replacing t by $t - \alpha$ and taking $\alpha \to -\infty$, it follows immediately from Theorem 12.2 that

Corollary 12.3. *any ancient Ricci flow $(M^3 \times (-\infty, \omega), g)$ on a compact three-manifold has nonnegative curvature operator.*

Observe, moreover, that any sequence of eigenvalues $\lambda_1^j \leq \lambda_3^j \leq \lambda_3^j$ such that

- $\lambda_1^j + \lambda_2^j + \lambda_3^j \geq -3r^{-2}$,
- $\lambda_1^j \to -\infty$, and
- $-\lambda_1^j(\log(-\lambda_1^j) + \log(r^2 + 4T) - 3) \leq \lambda_1^j + \lambda_2^j + \lambda_3^j$

satisfies

$$-\frac{\lambda_1^j}{\lambda_3^j} \leq \frac{3}{\log(-\lambda_1^j) + \log(r^2 + 4T) - 3} \to 0 \;\text{ as }\; j \to \infty.$$

It therefore follows from Theorem 12.2 that any (not necessarily compact) blow-up limit (i.e. an ancient Ricci flow obtained as in Lemma 12.1) about a finite time singularity of a Ricci flow on a compact three-manifold has nonnegative curvature.

12.2 *Self-similar solutions*

Recall that a triple (M^n, g, V) is a Ricci soliton if

$$\mathrm{Rc} = \lambda g - \tfrac{1}{2}\mathcal{L}_V g \tag{12.2a}$$

for some $\lambda \in \mathbb{R}$. When $V = \nabla f$, the triple (M^n, g, f) is a gradient Ricci soliton, and

$$\mathrm{Rc} = \lambda g - \nabla^2 f. \tag{12.2b}$$

As in the two dimensional case (gradient) Ricci solitons satisfy a number of informative identities. Indeed, tracing the soliton equation yields

$$\mathrm{R} = n\lambda - \operatorname{div} V \tag{12.3a}$$

which for a gradient soliton becomes

$$\mathrm{R} = n\lambda - \Delta f. \tag{12.3b}$$

Taking the divergence of the soliton equation and applying the contracted second Bianchi identity then yields

$$\Delta V + \mathrm{Rc}(V) = 0, \tag{12.4a}$$

which for a gradient soliton becomes

$$\tfrac{1}{2}\nabla \mathrm{R} - \mathrm{Rc}(\nabla f) = 0 \tag{12.4b}$$

Contracting the gradient soliton equation with ∇f and applying (12.4b) yields

$$\mathrm{R} + |\nabla f|^2 - 2\lambda f = C, \tag{12.5a}$$

where C is constant. Applying (12.3b) then yields

$$-\Delta f + |\nabla f|^2 + n\lambda - 2\lambda f = C \tag{12.5b}$$

Taking the difference between (12.5b) and half of (12.5a) yields

$$-\Delta f + \tfrac{1}{2}|\nabla f|^2 - \tfrac{1}{2}\mathrm{R} - \lambda f = C - n\lambda. \tag{12.5c}$$

As in the two-dimensional setting, (12.5c) is the Euler–Lagrange equation for a certain constrained energy functional. (The proof is the same as that of Proposition 11.5).

Proposition 12.4. *Given any compact Riemannian manifold (M^n, g) and any $\lambda \in \mathbb{R}$, define, for any smooth function f,*

$$F(f) \doteqdot \int_{M^n} \left(\tfrac{1}{2}\left[|\nabla f|^2 + \mathrm{R}\,\right] + \lambda f\right) \mathrm{e}^{-f}\, d\mu. \tag{12.6}$$

If $\{f_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a smooth variation of $f = f_0$ which satisfies the weighted volume constraint

$$\frac{d}{d\varepsilon}\int_{M^n} \mathrm{e}^{-f_\varepsilon}\, d\mu \equiv 0,$$

then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(f_\varepsilon) = -\int_{M^n} \left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\,\mathrm{R} + \lambda f\right) h\, \mathrm{e}^{-f}\, d\mu,$$

where $h \doteqdot \frac{d}{d\varepsilon}|_{\varepsilon=0} f_\varepsilon$. In particular, if f is a stationary point of the action with respect to constrained variations, then $-\Delta f + \frac{1}{2}|\nabla f|^2 - \frac{1}{2}\,\mathrm{R} - \lambda f$ is constant.

Theorem 12.5. *All compact shrinking Ricci solitons are gradient.*

Proof. Let (M^n, g, V) be a compact shrinking Ricci soliton. We seek a solution f to the equation[9]

$$\tfrac{1}{2}\mathcal{L}_V g = \nabla^2 f.$$

Equivalently, we seek a function f such that the tensor

$$S \doteqdot \mathrm{Rc} + \nabla^2 f - \lambda g$$

vanishes identically. Observe that

$$\operatorname{div} S = \tfrac{1}{2}\nabla \mathrm{R} + \nabla\Delta f + \mathrm{Rc}(\nabla f)$$

and

$$S(\nabla f) = \mathrm{Rc}(\nabla f) + \tfrac{1}{2}\nabla|\nabla f|^2 - \lambda\nabla f$$

[9] Rather than the equation $\nabla f = V$, which may not be possible since $(M^n, g, V + K)$ is also a shrinking Ricci soliton for any Killing vector field K.

and hence

$$\nabla \left(\tfrac{1}{2}\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f \right) = \operatorname{div} S - S(\nabla f),$$

which we may rewrite as

$$\nabla \left(\tfrac{1}{2}\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f \right) \mathrm{e}^{-f} = \operatorname{div}(\mathrm{e}^{-f} S).$$

Thus,

$$\begin{aligned}
&\int_{M^n} |S|^2 \mathrm{e}^{-f}\, d\mu \\
&\qquad = \int_{M^n} g(\nabla(\nabla f - V), \mathrm{e}^{-f} S)\, d\mu \\
&\qquad = -\int_{M^n} g(\nabla f - V, \operatorname{div}(\mathrm{e}^{-f} S))\, d\mu \\
&\qquad = -\int_{M^n} g\Big(\nabla f - V, \nabla \left(\tfrac{1}{2}\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f \right) \Big)\mathrm{e}^{-f}\, d\mu.
\end{aligned}$$

So it suffices to find a constant C and a function f satisfying

$$\tfrac{1}{2}\mathrm{R} + \Delta f - \tfrac{1}{2}|\nabla f|^2 + \lambda f = C \tag{12.7a}$$

or equivalently, a function $h = \mathrm{e}^{-\frac{f}{2}}$ satisfying

$$\Delta h - \tfrac{1}{4}\mathrm{R}h + \lambda h \log h = -\tfrac{1}{2}Ch. \tag{12.7b}$$

The equations (12.7a) and (12.7b) are the Euler–Lagrange equations for the constrained functionals

$$\begin{cases} F(f) \doteqdot \displaystyle\int_{M^n} \left(\tfrac{1}{2}\left[|\nabla f|^2 + \mathrm{R}\right] + \lambda f \right) \mathrm{e}^{-f}\, d\mu \\ \text{subject to } \displaystyle\int_{M^n} \mathrm{e}^{-f}\, d\mu = \text{const.} \end{cases} \tag{12.8a}$$

and

$$\begin{cases} G(h) \doteqdot 2\displaystyle\int_{M^n} \left(|\nabla h|^2 + \tfrac{1}{4}\mathrm{R}h^2 - \lambda h^2 \log h \right)\, d\mu \\ \text{subject to } \displaystyle\int_{M^n} h^2\, d\mu = \text{const.} \end{cases} \tag{12.8b}$$

respectively. We have thus reduced the problem to finding a minimizer for (12.8b). This is fairly classical: first observe that, by Jensen's inequality, interpolation and the Poincaré–Sobolev inequality, we may estimate, for any $\varepsilon > 0$,

$$\int_{M^n} h^2 \log h\, d\mu \le \varepsilon \left(\int_{M^n} |\nabla h|^2\, d\mu \right)^{\frac{1}{2}} + C_\varepsilon \left(\int_{M^n} h^2\, d\mu \right)^{\frac{1}{2}}.$$

Choosing ε sufficiently small, we find that

$$G(h) \ge \int_{M^n} |\nabla h|^2\, d\mu - C.$$

From this we deduce two things: first that G is bounded from below, and second that the H^1 norm is uniformly bounded along any infimizing sequence $\{h_j\}_{j\in\mathbb{N}}$. Since $h \mapsto h^2 \log h$ is continuous in H^1, it follows that a minimizer exists in H^1. Smoothness of the minimizer may be established using the de Giorgi–Nash–Moser and Schauder estimates.

Alternatively, we may exploit the gradient flow

$$\partial_t f = \operatorname{div}\left(\mathrm{e}^{-f}\left(\nabla f - V\right)\right) = \Delta f + \nabla_V f - |\nabla f|^2 + \mathrm{R} - n\lambda .$$

Indeed, under this equation,

$$\frac{d}{dt}F(f) = -\int_{M^n} |S|^2 \mathrm{e}^{-f}\, d\mu \text{ and } \frac{d}{dt}\int_{M^n} \mathrm{e}^{-f}\, d\mu = 0.$$

So the energy decreases (strictly unless u is a stationary point of E) and the constraint is maintained. Since the equation is parabolic, we obtain short-time existence from any smooth (say) initial condition. Longtime existence and smooth convergence to a stationary point of F (a minimizer if the initial energy is sufficiently close to the minimum) may be obtained by exploiting estimates for the (divergence form) *linear* equation

$$\partial_t u = \Delta u + \nabla_V u + (\mathrm{R} - n\lambda)u$$

satisfied by $u \doteqdot \mathrm{e}^{-f}$. □

Observe now that, on the self-similarly shrinking Ricci flow $(M^n \times (-\infty, 0), \phi^* g)$, $\frac{d\phi}{dt} = \phi^* \nabla f$, corresponding to a gradient shrinking Ricci soliton (M^n, g, f),

$$\begin{aligned}\partial_t f &= \nabla_{\nabla f} f \\ &= |\nabla f|^2 \\ &= -\Delta f + |\nabla f|^2 - \mathrm{R} + \frac{n}{-2t}\end{aligned}$$

due to (12.3b). Writing $h \doteqdot (-2t)^{-\frac{n}{2}} \mathrm{e}^{-f}$, we find that

$$-(\partial_t + \Delta - \mathrm{R})h = 0.$$

Just as in the two-dimensional setting, this is the CONJUGATE HEAT EQUATION (and $(\partial_t - \Delta)^* \doteqdot -(\partial_t + \Delta - \mathrm{R})$ is the CONJUGATE HEAT OPERATOR).

12.3 *The differential Harnack inequality*

The differential Harnack inequalities for two-dimensional Ricci flow (Theorems 11.8 and 11.10) have the following higher dimensional generalization.

Theorem 12.6 (Matrix differential Harnack inequality[10]). *Along any Ricci flow $(M^n \times [0,T), g)$ with positive curvature operator on a compact manifold M^n,*

$$M_{ij}W_iW_j + 2P_{ijk}U_{ij}W_k + \mathrm{Rm}_{ikjl}U_{ik}U_{jl} + \frac{1}{2t}\mathrm{Rc}_{ij}W_iW_j \geq 0 \tag{12.9}$$

for every time-dependent vector field W and two-form U, where

$$M_{ij} \doteqdot \Delta\mathrm{Rc}_{ij} + 2\mathrm{Rm}_{ikjl}\mathrm{Rc}_{kl} - \tfrac{1}{2}\left(\nabla_i\nabla_j\mathrm{R} + 2\mathrm{Rc}^2_{ij}\right)$$

and

$$P_{ijk} \doteqdot \nabla_i\mathrm{Rc}_{jk} - \nabla_j\mathrm{Rc}_{ik}.$$

The inequality (12.9) *is strict unless $(M^n \times [0,T), g)$ is an expanding soliton.*

Along any ancient Ricci flow $(M^n \times (-\infty, 0), g)$ with positive curvature operator on a compact manifold M^n,

$$M_{ij}W_iW_j + 2P_{ijk}U_{ij}W_k + \mathrm{Rm}_{ikjl}U_{ik}U_{jl} \geq 0 \tag{12.10}$$

for every time-dependent vector field W and two-form U, with strict inequality unless $(M^n \times [0,T), g)$ is a steady soliton.

Sketch of the proof. Motivated by various identities which hold on expanding (and steady) solitons, one considers the forms

$$Q(U,W) \doteqdot M(W,W) + 2P(U,W) + \mathrm{Rm}(U,U)$$

and

$$P(U,W) \doteqdot 2tQ(U,W) + \mathrm{Rc}(W,W).$$

After many arduous computations (motivated by various identities which hold on solitons), it is possible to obtain a suitable differential inequality for P. □

Theorem 12.7 (Trace differential Harnack inequality). *Along any Ricci flow $(M^n \times [0,T), g)$ with positive curvature operator on a compact manifold M^n,*

$$\partial_t\mathrm{R} + 2\nabla_V\mathrm{R} + 2\mathrm{Rc}(V,V) + \frac{1}{t}\mathrm{R} \geq 0 \tag{12.11}$$

for every time-dependent vector field V, with strict inequality unless $(M^n \times [0,T), g)$ is an expanding soliton.

Along any ancient Ricci flow $(M^n \times (-\infty, 0), g)$ with positive curvature operator on a compact manifold M^n,

$$\partial_t\mathrm{R} + 2\nabla_V\mathrm{R} + 2\mathrm{Rc}(V,V) \geq 0 \tag{12.12}$$

for every time-dependent vector field V, with strict inequality unless $(M^n \times [0,T), g)$ is a steady soliton.

[10] Richard S. Hamilton, "The Harnack estimate for the Ricci flow"

Proof. Take the trace of (12.9) and (12.10). □

Note that, by continuity, smooth limits of Ricci flows on compact manifolds satisfy the differential Harnack inequality (and hence also the rigidity case by the strong maximum principle).

Corollary 12.8 ((Integral) Harnack inequality)**.** *Along any Ricci flow* $(M^n \times [0,T), g)$ *with positive curvature on a compact two-manifold,*

$$\frac{\mathrm{R}(x_2,t_2)}{\mathrm{R}(x_1,t_1)} \geq \left[\frac{t_2}{t_1}\exp\left(\frac{d^2(x_1,x_2,t_1)}{4(t_2-t_1)}\right)\right]^{-1}$$

for any $x_1, x_2 \in M^n$ *and any* $0 < t_1 < t_2 < T$*, with strict inequality unless* $(M^n \times [0,T), g)$ *is an expanding self-similar solution.*

Proof. Integrate the trace differential Harnack inequality along curves of the form $t \mapsto (t, \gamma(t))$. □

When $\mathrm{Rc} > 0$, the differential Harnack inequality (12.12) is optimized by the vector field $V = -\frac{1}{2}\mathrm{Rc}^{-1}(\nabla \mathrm{R})$, giving

$$\partial_t \mathrm{R} \geq \tfrac{1}{2}\mathrm{Rc}^{-1}(\nabla \mathrm{R}, \nabla \mathrm{R}). \tag{12.13}$$

Equivalently, $\mathrm{R}(\phi^\tau(\cdot,t), t+\tau)$ is pointwise monotone nondecreasing with respect to t for each $\tau < 0$, where ϕ^τ is the solution to

$$\begin{cases} \dfrac{d\phi^\tau}{dt}(x,t) = V(\phi^\tau(x,t), \tau + t) \\ \phi^\tau(x,0) = x. \end{cases}$$

Thus, the scalar curvature R^τ of the reparametrized flow $\{g^\tau_t\}_{t\in(-\infty,-\tau)}$, where $g^\tau_t \doteqdot \phi^\tau(\cdot,t)^* g_{t+\tau}$, is uniformly bounded on any time interval of the form $(-\infty, T]$, and hence, in any (pointed) limit as $\tau_j \to -\infty$, we obtain a Ricci flow (plus Lie derivative term) for which R is constant in t—a steady soliton!

Corollary 12.9 (Ancient solutions decompose into steady solutions)**.** *Let* $(M^n \times (-\infty, 0), g)$ *be an ancient Ricci flow on a compact manifold* M^n*. Given any point* $o \in M^n$ *and any sequence of times* $t_j \to -\infty$*, some subsequence of the pointed Ricci flows* $(M^n \times (-\infty,0), o, g^j)$*, where* $g^j_{(x,t)} \doteqdot g(x, t+t_j)$*, converges locally uniformly in the smooth topology to a steady Ricci flow.*

12.4 Perelman's functional, noncollapsing, and the pointed Nash entropy

12.4.1 Perelman's functional

Given a compact Ricci flow $(M^n \times [0,T), g)$, define the functional

$$\mathscr{P}(f,g,\tau) \doteqdot \int_{M^n} \left[\tau\left(|\nabla f|^2 + \mathrm{R}\right) + f - n\right](4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-f}d\mu. \tag{12.14}$$

Observe that, when τ is identified with backwards time, PERELMAN'S FUNCTIONAL $\mathscr{P}$ is just a multiple of the functional F of (12.6) in the shrinking case, $\lambda > 0$ (with f replaced by $f - n$).

Just as in the two-dimensional setting, we find that $\mathscr{P}(f, g, \tau)$ is constant in time along a gradient self-similarly shrinking Ricci flow (M, g) with f taken to be the potential function and τ taken to be backwards time.[11]

[11] A Riemannian analogue of Perelman's functional was introduced by Tobias Holck Colding, "New monotonicity formulas for Ricci curvature and applications. I", and used to study Ricci flat manifolds. The connection between the two functionals was developed further by Bustamante and Martín, "Deriving Perelman's entropy from Colding's monotonic volume".

Theorem 12.10 (Perelman's monotonicity formula[12]). *Let $(M^n \times I, g)$ be a Ricci flow on a compact manifold M^n. If f and τ satisfy*

[12] Perelman, "The entropy formula for the Ricci flow and its geometric applications"

$$\begin{cases} (\partial_t + \Delta) f = |\nabla f|^2 - \mathrm{R} + \frac{n}{2\tau}, \\ \qquad \dfrac{d\tau}{dt} = -1, \end{cases}$$

then

$$\frac{d}{dt}\mathscr{P}(f, g, \tau) = 2\tau \int_{M^n} \left|\mathrm{Rc} + \nabla^2 f - \tfrac{1}{2\tau} g\right|^2 \mathrm{e}^{-f}\, d\mu \tag{12.15}$$

so long as $\tau > 0$. In particular, the PERELMAN ENTROPY

$$\mu(M^n, g_t, t_0 - t) \doteqdot \inf\left\{\mathscr{P}(g_t, f, t_0 - t) : \frac{1}{4\pi(t_0 - t)} \int_{M^n} \mathrm{e}^{-f}\, d\mu_t = 1\right\}$$

is nondecreasing for $t < t_0$ (strictly, unless (M^n, g_{t_0+t}) is a gradient shrinking soliton with potential $f(\cdot, t_0 + t)$).

Proof. Observe first that the function $\Phi \doteqdot (4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-f}$ satisfies the conjugate heat equation

$$0 = (\partial_t - \Delta)^*\Phi = -(\partial_t + \Delta - \mathrm{R})\Phi.$$

Note also that

$$\begin{aligned} \operatorname{div}(\Phi\nabla f) &= (\Delta f - |\nabla f|^2)\Phi \\ &= (\Delta f - \tfrac{1}{2}|\nabla f|^2 - \tfrac{1}{2}|\nabla f|^2)\Phi. \end{aligned}$$

Thus, after an integration by parts,

$$\mathscr{P}(f, g, \tau) = \int \left(2\tau\left[\Delta f - \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\mathrm{R}\right] + f - n\right)\Phi\, d\mu.$$

So consider the function

$$u \doteqdot 2\tau\left[\Delta f - \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\mathrm{R}\right] + f - n.$$

By the commutation formulae

$$\nabla_t \nabla v = \nabla\partial_t v + \mathrm{Rc}(\nabla v) \quad \text{and} \quad \partial_t \Delta v = \Delta\partial_t v + 2g(\mathrm{Rc}, \nabla^2 v),$$

we find that

$$\partial_t u = \partial_t f - 2\left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\mathrm{R}\right)$$
$$+2\tau\left(\Delta\partial_t f + 2g(\mathrm{Rc}, \nabla^2 f) - \mathrm{Rc}(\nabla f, \nabla f) - g(\nabla\partial_t f, \nabla f) + \tfrac{1}{2}\partial_t \mathrm{R}\right).$$

By the commutation formula[13]

[13] Recall Exercise 8.6.

$$\Delta\nabla v = \nabla\Delta v + \mathrm{Rc}(\nabla v),$$

we find that

$$\Delta u = 2\tau\left(\Delta\Delta f - |\nabla^2 f|^2 - g(\nabla\Delta f, \nabla f) - \mathrm{Rc}(\nabla f, \nabla f) + \tfrac{1}{2}\Delta\mathrm{R}\right) + \Delta f.$$

Recalling the evolution equation (9.10) for R, we thus obtain

$$\begin{aligned}(\partial_t - \Delta)u &= (\partial_t - \Delta)f - 2\left(\Delta f - \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\mathrm{R}\right)\\
&\quad + 2\tau\Big(\Delta(\partial_t - \Delta)f - g\big(\nabla(\partial_t - \Delta)f, \nabla f\big)\\
&\qquad\qquad + |\mathrm{Rc}|^2 + 2g(\mathrm{Rc}, \nabla^2 f) + |\nabla^2 f|^2\Big)\\
&= -2(\Delta f - |\nabla f|^2) + \tfrac{n}{2\tau} - 2(\Delta f + \mathrm{R})\\
&\quad + 2\tau\left(\frac{\operatorname{div}(\Phi\nabla(\partial_t - \Delta)f)}{\Phi} + |\mathrm{Rc} + \nabla^2 f|^2\right)\\
&= 2\tau\left(\frac{\operatorname{div}\left(\Phi\Big[\nabla\big(\partial_t - \Delta - \tfrac{1}{\tau}\big)f\Big]\right)}{\Phi} + \left|\mathrm{Rc} + \nabla^2 f - \tfrac{1}{2\tau}g\right|^2\right).\end{aligned}$$

We conclude that

$$\begin{aligned}\frac{d}{dt}\mathscr{P}(f, g, \tau) &= \frac{d}{dt}\int u\Phi\, d\mu\\
&= \int\left[\Phi(\partial_t - \Delta)u - u(\partial_t - \Delta)^*\Phi\right] d\mu\\
&= 2\tau\int\left|\mathrm{Rc} + \nabla^2 f - \tfrac{1}{2\tau}g\right|^2\Phi\, d\mu.\end{aligned}$$

□

12.4.2 *Noncollapsing of volume at the scale of the curvature*

Roughly speaking, a sequence of Riemannian manifolds (M_j^n, g_j) is said to COLLAPSE if some sequence of neighbourhoods $U_j \subset M_j^n$ and scales λ_j can be found such that $(U_j, \lambda_j g_j)$ resemble a lower dimensional manifold as $j \to \infty$. One precise way to quantify this is to ask for a sequence of points $p_j \in M_j^n$ such that

$$\mathrm{inj}_{g_j}(p_j) \sup_{B_{j\,\mathrm{inj}_{g_j}(p_j)}(p_j)} |\mathrm{Rm}|^{\frac{1}{2}} \le j^{-1}, \tag{12.16}$$

where $\mathrm{inj}_g(p)$ denotes the INJECTIVITY RADIUS of (M^n, g) at p—the radius of the largest ball in (T_pM^n, g_p) on which the exponential map is a diffeomorphism.

Note that $\mathrm{inj}_g |\mathrm{Rm}|^{\frac{1}{2}}$ is scale invariant. Thus, if (12.16) holds, then, at the scale of the *curvature*, the injectivity radius degenerates to zero. On the other hand, at the scale of the *injectivity radius*, the curvature is tending towards zero in arbitrarily large regions, and at this scale the regions converge to a flat space.

Perelman's monotonicity formula yields a lower bound for volumes at the scale of the curvature under Ricci flow.

Theorem 12.11. *Let $(M^n \times [0,T), g)$ be a Ricci flow on a compact manifold M^n. Given $(x,t) \in M^n \times [0,T)$ and $r \le 1$, if $|\mathrm{Rm}|^2 \le r^{-2}$ at time t on $B_r(x,t)$, then*

$$\mathrm{volume}(B_r(x,t),t) \ge \kappa r^n\,,$$

where $\kappa = \kappa(M^n, g_0, T)$.

Proof. Set $\tau \doteqdot t_0 + r^2 - t$ and let $\phi : [0,\infty) \to [0,1]$ be any fixed smooth function satisfying $\phi|_{[0,\frac{1}{2}]} = 1$, $\phi|_{[1,\infty)} = 0$, and $\frac{|\phi'(\xi)|}{\phi(\xi)} \le C$. Define

$$f(x,t_0) \doteqdot A - \log\left(\phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r}\right)\right)$$

and

$$\begin{aligned} u(x,t_0) &\doteqdot (4\pi r^2)^{-\frac{n}{2}} \mathrm{e}^{-f} \\ &= (4\pi r^2)^{-\frac{n}{2}} \phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r}\right) \mathrm{e}^{-A}\,, \end{aligned}$$

where A is chosen so that

$$\int_{M^n} u(\cdot,t_0)\, d\mu_{t_0} = 1\,.$$

Note that

$$\begin{aligned} A &= \log\left((4\pi r^2)^{-\frac{n}{2}} \int_{B_r(x_0,t_0)} \phi\left(\frac{d(x_0,\cdot,t_0)}{r}\right) d\mu_{t_0}\right) \\ &\le \log\left((4\pi)^{-\frac{n}{2}} \frac{\mathrm{volume}(B_r(x_0,t_0),t_0)}{r^n}\right). \end{aligned}$$

Thus, upper bounds for u will imply lower bounds for the volume ratio.

Observe that

$$
\begin{aligned}
&\mathscr{P}(g_{t_0}, f(\cdot,t_0), r^2) \\
&\quad = \int_{B_r(x_0,t_0)} \Big(r^2(\mathrm{R}^2 + |\nabla f|^2) + f - n \Big)\, u \, d\mu_{t_0} \\
&\quad \le \int_{B_r(x_0,t_0)} \left[r^2\left(\frac{n}{r^2} + \frac{C}{r^2} \right) - \log\left(\phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r} \right) \right) + A - n \right] u d\mu_{t_0} \\
&\quad = C + A - \int_{B_r(x_0,t_0)} \log\left(\phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r} \right) \right) u \, d\mu_{t_0} \\
&\quad = C + A - \frac{\int_{B_r(x_0,t_0)} \log\left(\phi\left(\frac{\mathrm{dist}(x_0,x,t_0)}{r} \right) \right) \frac{\phi(\mathrm{dist}(x_0,x,t_0))}{r} \, d\mu_{t_0}}{\int_{B_r(x_0,t_0)} \frac{\phi(\mathrm{dist}(x_0,\cdot,t_0))}{r} \, d\mu_{t_0}} \\
&\quad \le C + A + C' \frac{\mathrm{volume}(B_r(x_0,t_0),t_0)}{\mathrm{volume}(B_{\frac{r}{2}}(x_0,t_0),t_0)} \\
&\quad \le C'' + A
\end{aligned}
$$

due to the Bishop–Gromov inequality. The claim follows since, by the monotonicity of μ,

$$\mu(M^n, g_0, t_0 + r^2) \le \mu(M^n, g_{t_0}, r^2) \le \mathscr{P}(g_{t_0}, f(\cdot,t_0), r^2)$$

and $t_0 + r^2 \le T$. □

12.4.3 *The heat kernel*

Recall that the (classical) n-dimensional HEAT KERNEL K is given (for $x, y \in \mathbb{R}^n$ and $t > s$) by

$$K(x,t,y,s) \doteqdot (4\pi(t-s))^{-\frac{n}{2}} \mathrm{e}^{-\frac{|x-y|^2}{4(t-s)}} .$$

It satisfies the heat equation in the (x,t) variables for fixed (y,s) and the conjugate heat equation in the (y,s) variables for fixed (x,t); it converges in the distributional sense as $t \searrow s$ to the Dirac distribution centred at y, when y is fixed, and as $s \nearrow t$ to the Dirac distribution centred at x, when x is fixed. As such, it provides the REPRESENTATION FORMULAE

$$u(x,t) = \int_{\mathbb{R}^n} K(x,t,y,s) u(y,s) \, d\mathcal{L}(y) \quad \text{for} \quad t > s$$

for any distributional solution u to the heat equation, and

$$v(y,s) = \int_{\mathbb{R}^n} v(x,t) K(x,t,y,s) \, d\mathcal{L}(x) \quad \text{for} \quad s < t$$

for any distributional solution v to the conjugate heat equation. In particular, the heat kernel satisfies the REPRODUCTION FORMULA

$$K(x,t,y,s) = \int_{\mathbb{R}^n} K(x,t,\xi,\tau) K(\xi,\tau,y,s) \, d\mathcal{L}(\xi) \quad \text{for} \quad t > \tau > s.$$

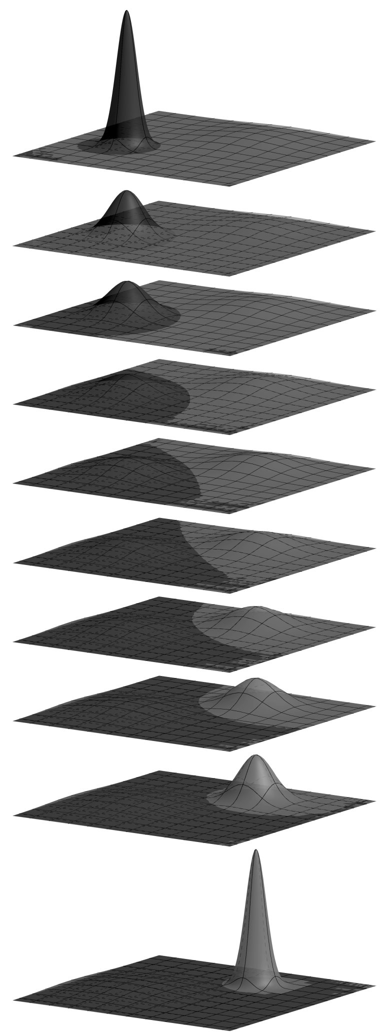


Figure 12.3: Euclidean heat kernel diffusing heat away from (x_0, t_0), via $(x,t) \mapsto K(x,t,x_0,t_0)$, and into (y_0, s_0), via $(y,s) \mapsto K(y_0,s_0,y,s)$.

It turns out that the heat equation also admits a kernel on any compact Riemannian manifold,[14] and, in fact, along any compact Ricci flow.

[14] Chavel, *Eigenvalues in Riemannian geometry*.

[15] Guenther, "The fundamental solution on manifolds with time-dependent metrics"

Theorem 12.12 (The heat kernel along a Ricci flow[15]). *Associated to any compact Ricci flow $(M^n \times I, g)$ is a unique function K (of pairs of spacetime points (x,t) and (y,s) subject to $t > s$) satisfying the following properties:*

1. *For any $(y,s) \in M^n \times I$,*
   (a) *the function $(x,t) \mapsto K(x,t,y,s)$ satisfies the heat equation,*
   (b) *$K(\cdot,t,y,s) \to \delta_y$ in the distributional sense as $t \searrow s$.*
2. *For any $x, y \in M^n$, $K(x,t,y,s) = \int_{M^n} K(x,t,\xi,\tau)K(\xi,\tau,y,s)\, d\mu_{g_\tau}(\xi)$ for $t > \tau > s$.*
3. *For any $(x,t) \in M^n \times I$,*
   (a) *the function $(y,s) \mapsto K(x,t,y,s)$ satisfies the conjugate heat equation,*
   (b) *$K(x,t,\cdot,s) \to \delta_x$ in the distributional sense as $s \nearrow t$.*
4. *$\int K(x,t,y,s)\, d\mu_{g_s}(y) = 1$ for all x, t and s.*

In particular, K provides the representation formulae

5. *$u(x,t) = \int_{M^n} K(x,t,y,s)u(y,s)\, d\mu_{g_s}(y)$ for all $t > s$ for any distributional solution u to the heat equation, and*
6. *$v(y,s) = \int_{M^n} v(x,t)K(x,t,y,s)\, d\mu_{g_t}(x)$ for all $s < t$ for any distributional solution v to the conjugate heat equation.*

Due to the fourth property, it is natural to introduce the probability measures

$$d\nu_{(x_0,t_0),t} \doteqdot K(x_0,t_0,\cdot,t)\, d\mu_{g_t}.$$

12.4.4 *The pointed Nash entropy of Hein and Naber*

Given any $(x_0,t_0) \in M^n \times \mathbb{R}$, we define the POINTED NASH ENTROPY[16] for $t < t_0$ by

[16] Hein and Naber, "New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow".

$$\mathscr{N}_{(x_0,t_0)}(t) \doteqdot (4\pi(t_0-t))^{-\frac{n}{2}} \int f_{(x_0,t_0)}(\cdot,t) e^{-f_{(x_0,t_0)}(\cdot,t)} d\mu_{g_t} - \frac{n}{2}, \quad (12.17)$$

where $f_{(x_0,t_0)}$ is the LOGARITHMIC FUNDAMENTAL SOLUTION TO THE CONJUGATE HEAT EQUATION based at (x_0,t_0); I.e.,

$$K(x_0,t_0,x,t) = (4\pi(t_0-t))^{-\frac{n}{2}} e^{-f_{(x_0,t_0)}(x,t)}.$$

This definition is natural in view of the Euclidean asymptotics[17]

[17] See, e.g. Chow, S.-C. Chu, et al., *The Ricci flow: techniques and applications. Part III. Geometric-analytic aspects*.

$$
\begin{aligned}
f_{(x_0,t_0)}(x,t) &\sim \frac{d^2(x_0,x,t_0)}{4(t_0-t)}, \\
\left|\nabla f_{(x_0,t_0)}(x,t)\right|^2 &\sim \frac{d^2(x_0,x,t_0)}{4(t_0-t)^2}, \\
\Delta f_{(x_0,t_0)}(x,t) &\sim \frac{n}{2(t_0-t)}
\end{aligned}
\tag{12.18}
$$

as $(x,t) \to (x_0,t_0)$.

Theorem 12.13. *Along any Ricci flow $(M^n \times I, g)$, for any $(x_0,t_0) \in M^n \times \mathbb{R}$,*

$$
-\frac{d}{dt}\Big((t_0-t)\,\mathscr{N}_{(x_0,t_0)}(t)\Big) = \mathscr{P}(f_{(x_0,t_0)}, g_t, t_0-t) \tag{12.19}
$$

for $t < t_0$. Thus,

$$
\frac{d^2}{dt^2}\Big((t_0-t)\,\mathscr{N}_{(x_0,t_0)}(t)\Big) \le 0 \tag{12.20}
$$

for $t < t_0$. Moreover,

$$
\mathscr{N}_{(x_0,t_0)}(t) \to 0 \ \text{ as } \ t \to t_0. \tag{12.21}
$$

It follows that

$$
\mathscr{N}_{(x_0,t_0)}(t) \ge \mu(g_t, t_0-t). \tag{12.22}
$$

Sketch of the proof. Since $(4\pi(t_0-t))^{-\frac{n}{2}}\mathrm{e}^{-f_{(x_0,t_0)}(x,t)}$ satisfies the conjugate heat equation,

$$
\begin{aligned}
\frac{d}{dt}\mathscr{N}_{(x_0,t_0)} &= \int (\partial_t - \Delta) f_{(x_0,t_0)}\, d\nu_{(x_0,t_0)} \\
&= \int \left(2\Delta f_{(x_0,t_0)} - |\nabla f_{(x_0,t_0)}|^2 + \mathrm{R} - \frac{n}{2(t_0-t)}\right) d\nu_{(x_0,t_0)} \\
&= -\int \Big(|\nabla f_{(x_0,t_0)}|^2 + \mathrm{R}\Big)\, d\nu_{(x_0,t_0)} - \frac{n}{2(t_0-t)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
-\frac{d}{dt}\Big((t_0-t)\,\mathscr{N}_{(x_0,t_0)}\Big) &= (t_0-t)\int \Big(|\nabla f_{(x_0,t_0)}|^2 + \mathrm{R}\Big)\, d\nu_{(x_0,t_0)} - \frac{n}{2} \\
&\quad + \int f_{(x_0,t_0)}\, d\nu_{(x_0,t_0)} - \frac{n}{2} \\
&= \mathscr{P}(f_{(x_0,t_0)}, g_t, t_0-t).
\end{aligned}
$$

This proves (12.19). The inequality (12.20) then follows from Perelman's monotonicity formula (12.15). The limit (12.21) follows from the aforementioned asymptotics for the logarithmic fundamental solutions to the conjugate heat equation. We may now conclude that

$$
\mathscr{N}_{(x_0,t_0)}(t) \ge \mathscr{P}(f_{(x_0,t_0)}, g_t, t_0-t).
$$

The inequality (12.22) follows. □

12.5 Perelman's $\mathcal{L}$-geometry

In this final (singularly voluminous) section of the chapter, we shall present a parabolic comparison-geometric perspective to Ricci flow,[18] which is due to Perelman.[19] It arises naturally out of the consideration of Perelman's functional, and replaces the Riemannian length functional/distance with a kind of spacetime length functional/distance.

[18] In fact, the framework can be applied, to some extent, to more general metric deformations, and one can sometimes think of the SUPER RICCI FLOW condition

$$\frac{dg_t}{dt} + 2\mathrm{Rc}_{g_t} \geq 0$$

introduced by McCann and P. M. Topping, "Ricci flow, entropy and optimal transportation" as playing the role of Ricci curvature lower bounds in Riemannian comparison geometry.

[19] Perelman, "The entropy formula for the Ricci flow and its geometric applications".

12.5.1 A Harnack inequality for the conjugate heat equation

The function

$$\begin{aligned} u &\doteqdot 2\tau \left[\Delta f - \tfrac{1}{2}|\nabla f|^2 + \tfrac{1}{2}\mathrm{R}\right] + f - n \\ &= 2\tau \left[-\partial_t f + \tfrac{1}{2}|\nabla f|^2 - \tfrac{1}{2}\mathrm{R}\right] + f \end{aligned}$$

which appears (after an integration by parts) in Perelman's functional is reminiscent of the terms which appear in the differential Harnack inequality for the heat equation.[20]

[20] See Exercise 12.4.

Theorem 12.14. *Along a Ricci flow $(M^n \times I, g)$, for any $(x_0, t_0) \in M^n \times \mathbb{R}$, the logarithmic fundamental solution to the conjugate heat equation based at (x_0, t_0) satisfies, for $\tau \doteqdot t_0 - t > 0$,*

$$2\tau \left(\partial_t f_{(x_0,t_0)} + \tfrac{1}{2}\left[\mathrm{R} - |\nabla f_{(x_0,t_0)}|^2\right]\right) - f \geq 0. \tag{12.23}$$

In particular,

$$(\partial_t - \Delta)\left(\tau\left[f_{(x_0,t_0)} - \tfrac{n}{2}\right]\right) \geq 0. \tag{12.24}$$

Sketch of the proof. Define

$$w \doteqdot 2\tau \left(\Delta f_{(x_0,t_0)} + \tfrac{1}{2}\left[\mathrm{R} - |\nabla f_{(x_0,t_0)}|^2\right]\right) + f - n.$$

Since $v \doteqdot (4\pi\tau)^{-\frac{n}{2}} e^{-f_{(x_0,t_0)}}$ satisfies the conjugate heat equation, the product vw can be shown to satisfy the inequality[21]

[21] Cf. the proof of Theorem 12.10.

$$(\partial_t - \Delta)^*(vw) = -2\tau \left|\mathrm{Rc} + \nabla^2 f_{(x_0,t_0)} - \tfrac{1}{2\tau} g\right|^2 v.$$

The claim then follows from the maximum principle (applied in backwards time) since

$$vw \to 0 \text{ as } t \to t_0,$$

which is a consequence of the heat kernel asymptotics (12.18). □

Integrating the differential Harnack inequality along spacetime curves yields a Harnack inequality.

Corollary 12.15. *Along a Ricci flow $(M^n \times I, g)$, for any $(x_0, t_0) \in M^n \times \mathbb{R}$, the logarithmic fundamental solution to the conjugate heat equation based at (x_0, t_0) satisfies, for any $t_2 < t_1 < t_0$*

$$2\sqrt{\tau(t_2)}\, f_{(x_0,t_0)}(x_2, t_2) - 2\sqrt{\tau(t_1)}\, f_{(x_0,t_0)}(x_1, t_1) \le \inf_\gamma \int_{\tau(t_1)}^{\tau(t_2)} \sqrt{\tau} \left(\mathrm{R}(\gamma(\tau), t_0 - \tau) + \left| \frac{d\gamma}{d\tau}(\tau) \right|^2_{g_{t_0-\tau}} \right) d\tau, \quad (12.25)$$

where $\tau(t) \doteqdot t_0 - t$ and the infimum is taken over regular curves $\gamma : [\tau(t_1), \tau(t_2)] \to M^n$ with $\gamma(\tau(t_1)) = x_1$ and $\gamma(\tau(t_2)) = x_2$.

Proof. Given a spacetime curve $\gamma : I \to M$ with $\gamma(\tau_1) = x_1$ and $\gamma(\tau_2) = x_2$, the Cauchy–Schwarz inequality and the differential Harnack inequality (12.23) yield

$$\begin{aligned} &\frac{d}{d\tau}\left(\sqrt{\tau} f_{(x_0,t_0)}(\gamma(\tau), t_0 - \tau)\right) \\ &\quad = \sqrt{\tau}\left(\frac{1}{\tau} f_{(x_0,t_0)} + 2g(\nabla f_{(x_0,t_0)}, \dot\gamma) - 2\partial_t f_{(x_0,t_0)} \right)\Big|_{(\gamma(\tau), t_0-\tau)} \\ &\quad \le \sqrt{\tau}\left(\frac{1}{\tau} f_{(x_0,t_0)} + |\nabla f_{(x_0,t_0)}|^2 + |\dot\gamma|^2 - 2\partial_t f_{(x_0,t_0)} \right)\Big|_{(\gamma(\tau), t_0-\tau)} \\ &\quad \le \sqrt{\tau}\left(|\dot\gamma(\tau)|^2_{g_{t_0-\tau}} + \mathrm{R}(\gamma(\tau), t_0 - \tau) \right), \end{aligned}$$

where $\dot\gamma \doteqdot \frac{d\gamma}{d\tau}$. Integrating yields

$$2\sqrt{\tau(t_2)}\, f_{(x_0,t_0)}(x_2, t_2) - 2\sqrt{\tau(t_1)}\, f_{(x_0,t_0)}(x_1, t_1) \le \int_{\tau(t_1)}^{\tau(t_2)} \sqrt{\tau} \left(\mathrm{R}(\gamma(\tau), t_0 - \tau) + |\dot\gamma(\tau)|^2_{g_{t_0-\tau}} \right) d\tau.$$

Optimizing with respect to γ yields the claim. □

12.5.2 The $\mathcal{L}$-functional

Given $t_0 \in \mathbb{R}$, we are led to consider the functional

$$\mathcal{L}_{t_0}(\gamma) \doteqdot \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(\left| \frac{d\gamma}{d\tau} \right|^2_{g_{t_0-\tau}} + \mathrm{R}(\gamma, t_0 - \tau) \right) d\tau, \quad (12.26)$$

which is reminiscent of the Dirichlet energy functional.[22] The resemblance is even more apparent if we parametrize by the distance-like parameter $r = \sqrt{\tau}$, for then

$$\mathcal{L}_{t_0}(\gamma) = \int_{r_1}^{r_2} \left(\frac{1}{2} \left| \frac{d\gamma}{dr} \right|^2_{g_{t_0-r^2}} + 2r^2\, \mathrm{R}(\gamma, t_0 - r^2) \right) dr, \quad (12.27)$$

where $r_j \doteqdot \sqrt{\tau_j}$.

[22] Recall that the Dirichlet energy of a curve $\gamma : I \to M^n$ in a Riemannian manifold is given by

$$E(\gamma) \doteqdot \frac{1}{2} \int_a^b \left| \frac{d\gamma}{dr} \right|^2 dr.$$

The critical points of E are arclength parametrized geodesics; indeed,

$$\frac{1}{2(b-a)} \operatorname{length}^2(\gamma) \le E(\gamma)$$

with equality precisely when the parametrization is proportional to arclength. See, for example, Klingenberg, *Lectures on closed geodesics*.

Continuing the analogy, the $\mathcal{L}$-GEODESICS—critical points of the functional $\mathcal{L}$ amongst variations with fixed endpoints—should play a role in a "Ricci flow spacetime" akin to that of geodesics in a Riemannian manifold. If $t < t_0$, then the infimum

$$L_{(x_0,t_0)}(x,t) \doteqdot \inf_{\gamma} \mathcal{L}_{t_0}(\gamma)$$

of $\mathcal{L}_{t_0}(\gamma)$ amongst regular curves $\gamma : [0, t_0 - t] \to M^n$ joining points $\gamma(0) = x_0$ and $\gamma(t_0 - t) = x$ should provide a kind of distance between the "events" (x_0, t_0) and (x, t) after division by twice the square root of backward time. This quantity,

$$\ell_{(x_0,t_0)} \doteqdot \frac{1}{2\sqrt{\tau}} L_{(x_0,t_0)},$$

is called the REDUCED DISTANCE.

Example 23. On the static Ricci flow $(\mathbb{R}^n \times (-\infty, \infty), g_{\mathbb{R}^n})$, for any $t_0 \in (-\infty, \infty)$ and $\gamma : [0, \overline{\tau}] \to \mathbb{R}^n$,

$$\mathcal{L}_{t_0}(\gamma) = \int_0^{\overline{\tau}} \sqrt{\tau} \left| \frac{d\gamma}{d\tau} \right|^2 d\tau = \frac{1}{2} \int_0^{\sqrt{\overline{\tau}}} \left| \frac{d\gamma}{dr} \right|^2 dr$$

is just the usual Dirichlet energy. So the $\mathcal{L}$-geodesics are the parametrized straight lines: $\gamma(\tau) = x_0 + \sqrt{\tau}\vec{v}$, $|\vec{v}| \neq 0$. Along such a curve, $\mathcal{L}_{t_0}(\gamma) = \frac{1}{2}\sqrt{\overline{\tau}}|\vec{v}|^2$ and $\vec{v} = \frac{\gamma(\tau)-x_0}{\sqrt{\tau}}$, so

$$L_{(x_0,t_0)}(x,t) = \frac{|x - x_0|^2}{2\sqrt{t_0 - t}} \quad \text{and} \quad \ell_{(x_0,t_0)}(x,t) = \frac{|x - x_0|^2}{4(t_0 - t)}. \qquad \blacksquare$$

Note that Corollary 12.15 implies that

$$f_{(x_0,t_0)} \leq \ell_{(x_0,t_0)},$$

where $f_{(x_0,t_0)}$ is the logarithmic fundamental solution to the conjugate heat equation based at (x_0, t_0). This yields the heat kernel estimate[23]

$$K(x_0, t_0, \cdot) \geq \left(4\pi(t_0 - t)\right)^{-\frac{n}{2}} \mathrm{e}^{-\ell_{(x_0,t_0)}}.$$

[23] Compare this with the heat kernel estimate on compact Riemannian manifolds of Li and Yau, "On the parabolic kernel of the Schrödinger operator".

Example 23 shows that equality holds on static Euclidean space.

12.5.3 *First variation of $\mathcal{L}$*

Proposition 12.16 (First variation of $\mathcal{L}$). *Given any $t_0 \in (\alpha, \omega)$, let $\gamma : [\tau_1, \tau_2] \to M^n$, $0 \leq \tau_1 < \tau_2 < t_0 - \alpha$, be a spacetime curve along a Ricci flow $(M^n \times (\alpha, \omega), g)$. For any variation $\{\gamma_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ of $\gamma_0 = \gamma$,*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}_{t_0}(\gamma_\varepsilon) = g_{(\gamma(\tau),t_0-\tau)} \left(\frac{1}{2\sqrt{\tau}} \frac{d\gamma}{d\tau}, V \right) \Bigg|_{\tau_1}^{\tau_2} \tag{12.28a}$$
$$- \int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\, g \left({}^{\gamma}\nabla_\tau \frac{d\gamma}{d\tau} + \frac{1}{2\tau} \frac{d\gamma}{d\tau} + 2\mathrm{Rc}\left(\frac{d\gamma}{d\tau}\right) - \frac{1}{2}\nabla \mathrm{R}, V \right) \Bigg|_{(\gamma(\tau),t_0-\tau)} d\tau.$$

where $V \doteqdot \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \gamma_\varepsilon$.

Equivalently (after reparametrizing by $r = \sqrt{\tau}$*),*

$$\begin{aligned}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) = &\ g_{(\gamma(r),t_0-r^2)}\left(\frac{d\gamma}{dr}, V\right)\Bigg|_{r_1}^{r_2} \\ &- \int_{r_1}^{r_2} g\left({}^{\gamma}\nabla_r \frac{d\gamma}{dr} + 4r\mathrm{Rc}\left(\frac{d\gamma}{dr}\right) - 2r^2\nabla \mathrm{R}, V\right)\Bigg|_{(\gamma(r),t_0-r^2)} dr.\end{aligned} \tag{12.28b}$$

Proof. If we define $\omega : (r_1, r_2) \times (-\varepsilon_0, \varepsilon_0) \to M^n$ by $\omega(r,\varepsilon) \doteqdot \gamma_\varepsilon(r)$, then

$$\nabla_{\frac{d\omega}{d\varepsilon}} \frac{d\omega}{dr} = \nabla_{\frac{d\omega}{dr}} \frac{d\omega}{d\varepsilon}$$

and

$$\begin{aligned}\frac{d}{dr}\left[g_{t_0-r^2}\left(V,\gamma'\right)\right] &= 4r\,\mathrm{Rc}_{t_0-r^2}(V,\gamma') + g_{t_0-r^2}\left({}^{\gamma}\nabla_r V, \gamma'\right) \\ &\quad + g_{t_0-r^2}\left(V, {}^{\gamma}\nabla_r \gamma'\right),\end{aligned}$$

where $\gamma' \doteqdot \frac{d\gamma}{dr}$ and ${}^{\gamma}\nabla$ denotes the pullback connection. Thus,

$$\begin{aligned}&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{r_1}^{r_2}\left(\frac{1}{2}\left|\frac{d\omega}{dr}\right|^2 + 2r^2\,\mathrm{R}\right) dr \\ &= \int_{r_1}^{r_2}\Big(g\left({}^{\gamma}\nabla_r V, \gamma'\right) + 2r^2\nabla_V \mathrm{R}\Big)\, dr \\ &= \int_{r_1}^{r_2}\Big(\partial_r\left[g\left(V,\gamma'\right)\right] - g\left(V, {}^{\gamma}\nabla_r\gamma'\right) - 4r\,\mathrm{Rc}(V,\gamma') + 2r^2\nabla_V \mathrm{R}\Big)\, dr \\ &= g(V,\gamma')\big|_{r_1}^{r_2} - \int_{r_1}^{r_2}\Big(g(V, {}^{\gamma}\nabla_r\gamma') + 4r\,\mathrm{Rc}(V,\gamma') - 2r^2\nabla_V \mathrm{R}\Big)\, dr,\end{aligned}$$

where we have suppressed the fact that the calculations are carried out along the spacetime curve $r \mapsto (\gamma(r), t_0 - r^2)$. The equation (12.28b) follows. A straightforward change of variables then yields (12.28a). □

So the $\mathcal{L}$-geodesics are characterized by the differential equation

$${}^{\gamma}\nabla_\tau \frac{d\gamma}{d\tau} + \frac{1}{2\tau}\frac{d\gamma}{d\tau} + 2\mathrm{Rc}_{t_0-\tau}\left(\frac{d\gamma}{d\tau}\right) - \frac{1}{2}\nabla \mathrm{R}_{t_0-\tau} = 0 \tag{12.29a}$$

or, equivalently, the equation

$${}^{\gamma}\nabla_r \frac{d\gamma}{dr} + 4r\mathrm{Rc}_{t_0-r^2}\left(\frac{d\gamma}{dr}\right) - 2r^2\nabla \mathrm{R}_{t_0-r^2} = 0. \tag{12.29b}$$

Note that, even though the equation (12.29a) is singular at $\tau = 0$, the equation (12.29b) is not. So we can always find a (unique) $\mathcal{L}_{t_0}$-geodesic γ emanating from (x_0, t_0) if we prescribe the initial velocity

$\frac{d\gamma}{dr}(0) = \lim_{\tau\to 0} 2\sqrt{\tau}\frac{d\gamma}{d\tau}$, and γ will depend continuously on the data (x_0, t_0, v).

Observe that

$$\frac{d}{dr}\frac{1}{2}g_{t_0-r^2}\left(\frac{d\gamma}{dr},\frac{d\gamma}{dr}\right) = g_{t_0-r^2}\left(\nabla_r\frac{d\gamma}{dr},\frac{d\gamma}{dr}\right) + 2r\,\mathrm{Rc}_{t_0-r^2}\left(\frac{d\gamma}{dr},\frac{d\gamma}{dr}\right)$$
$$= 2r^2\nabla_{\frac{d\gamma}{dr}}\mathrm{R}_{t_0-r^2} - 2r\,\mathrm{Rc}_{t_0-r^2}\left(\frac{d\gamma}{dr},\frac{d\gamma}{dr}\right).$$

Thus, if M^n is compact and the Ricci flow defined on $[\alpha, t_0]$, then the geodesic may be extended until $r^2 = \tau$ reaches $t_0 - \alpha$. Accordingly, we define the $\mathcal{L}$-EXPONENTIAL MAP AT (x, t) for $\tau < t_0 - \alpha$ by[24]

$$\mathcal{L}\exp^{\tau}_{(x,t)} v \doteqdot \gamma(\tau),$$

where γ is the unique solution to (12.29a) satisfying $\gamma(0) = x$ and $\lim_{\tau\to 0}\sqrt{\tau}\frac{d\gamma}{d\tau}(\tau) = v$.

Observe that, as $r \to 0$, the $\mathcal{L}_{t_0}$-geodesic equation (12.29b) tends to the geodesic equation for the metric g_{t_0}. Thus, by the continuous dependence of solutions to (12.29b) on the coefficients of the equation,

$$\mathcal{L}\exp^{\tau}_{(x_0,t_0)}\left(\frac{v}{2\sqrt{\tau}}\right) \to \exp_{(x_0,t_0)} v$$

and

$$2\sqrt{\tau}L_{(x_0,t_0)}(x, t_0 - \tau) \to \mathrm{dist}^2_{g_{t_0}}(x, x_0) \qquad (12.30)$$

as $\tau \to 0$.

By adapting the Riemannian theory, one may establish properties of $\mathcal{L}$-geodesics which are analogous to properties of their Riemannian counterparts.[25] In particular, given any $(x_0, t_0), (x_1, t_1) \in M^n \times I$ with $t_1 < t_0$, a minimizing $\mathcal{L}_{t_0}$-geodesic can be found joining $x_0 = \gamma(0)$ to $x_1 = \gamma(t_0 - t_1)$.[26]

We also have the following analogue of the Riemannian identity $|\nabla r| = 1$ for distance functions $r(x) \doteqdot \mathrm{dist}(x, x_0)$ (which should be compared with the differential Harnack inequality (12.23) for the conjugate heat kernel).

Proposition 12.17. *Along a Ricci flow $(M^n \times I, g)$ on a compact manifold M^n, for any $(x_0, t_0) \in M^n \times I$,*

$$\partial_\tau L_{(x_0,t_0)} + \frac{1}{4\sqrt{\tau}}|\nabla L_{(x_0,t_0)}|^2 - \sqrt{\tau}\,\mathrm{R} \geq 0, \qquad (12.31)$$

and hence

$$2\tau\left(\partial_\tau \ell_{(x_0,t_0)} - \tfrac{1}{2}\left[\mathrm{R} - |\nabla \ell_{(x_0,t_0)}|^2\right]\right) + \ell_{(x_0,t_0)} \geq 0, \qquad (12.32)$$

in the viscosity sense for $\tau > 0$.

[24] We may interpret $\mathcal{L}\exp^{\tau}$ as a map on the spatial tangent bundle via $(x, t, v) \mapsto \mathcal{L}\exp^{\tau}_{(x,t)} v$.

[25] The analogy is not perfect, however; as an example, observe that $\mathcal{L}\exp^{\tau}_{(x,t)}$ may not map the zero vector to x, due to the inhomogeneous term $\frac{1}{2}\nabla\mathrm{R}_{t-r^2}$ in (12.29a)—diffusion causes the base point to move, unless it is a critical point of $\mathrm{R}(\cdot, t)$.

[26] See, e.g. Kleiner and Lott, "Notes on Perelman's papers", §17; cf. e.g. Chavel, *Riemannian geometry*, §I.6-I.7.

Proof. Given any $(x,t) \in M^n \times I$ with $t < t_0$, we can find a minimizing $\mathcal{L}_{t_0}$-geodesic γ joining $x_0 = \gamma(0)$ to $x = \gamma(t_0 - t)$. Given any $v \in T_xM^n$, let $\{\gamma_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ be a variation of $\gamma_0 = \gamma$ with $\gamma_\varepsilon(0) = x_0$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon(\tau) = v$. If $\varphi(x,t)$ is a smooth function that satisfies $\varphi \leq L_{(x_0,t_0)}$ in a small *forwards*[27] neighbourhood of (x,t) with equality at (x,t), then

[27] I.e. backwards with respect to τ.

$$\varphi(\gamma_\varepsilon(t_0 - t), t) \leq L_{(x_0,t_0)}(\gamma_\varepsilon(t_0 - t), t) \leq \mathcal{L}_{t_0}(\gamma_\varepsilon),$$

with equality when $\varepsilon = 0$, and hence, by (12.28b),

$$\begin{aligned} 0 = {}& \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \left(\varphi(\gamma_\varepsilon(t_0 - t), t) - \mathcal{L}_{t_0}(\gamma_\varepsilon)\right) \\ ={}& \nabla_v\varphi(x,t) - g_t\left(\gamma', v\right) \end{aligned}$$

where $\gamma' \doteqdot \frac{d\gamma}{dr}$. Since v may be freely chosen, we conclude that

$$\nabla\varphi(x,t) = \gamma'\left(\sqrt{t_0 - t}\right) = 2\sqrt{t_0 - t}\,\dot{\gamma}(t_0 - t), \tag{12.33}$$

where $\dot{\gamma} \doteqdot \frac{d\gamma}{d\tau}$. On the other hand, if we set $\gamma_\varepsilon(\tau) \doteqdot \gamma(\frac{t_0 - t}{t_0 - t + \varepsilon}\tau)$, then $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon = -\dot{\gamma}$ and hence, for small $\varepsilon > 0$,

$$\begin{aligned} \varphi(x, t+\varepsilon) &\leq L_{(x_0,t_0)}(x, t+\varepsilon) \\ &\leq \int_0^{t_0 - t - \varepsilon} \sqrt{\tau}\left(\left|\frac{d\gamma_\varepsilon}{d\tau}\right|^2_{g_{t_0-\tau}} + \mathrm{R}(\gamma_\varepsilon, t_0 - \tau)\right) d\tau, \end{aligned}$$

and hence

$$\begin{aligned} 0 \geq{}& \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(\varphi(x, t+\varepsilon) - \int_0^{t_0 - t - \varepsilon}\sqrt{\tau}\left(\left|\frac{d\gamma_\varepsilon}{d\tau}\right|^2_{g_{t_0-\tau}} + \mathrm{R}(\gamma_\varepsilon, t_0 - \tau)\right) d\tau\right) \\ ={}& \partial_t\varphi(x,t) + \sqrt{t_0 - t}\left(|\dot{\gamma}|^2_{g_t} + \mathrm{R}(x,t)\right) \\ &- 2\int_0^{t_0 - t}\sqrt{\tau}\left(g_{t_0-\tau}\left(\nabla_\tau\dot{\gamma}, \dot{\gamma}\right) + \frac{1}{2}\nabla_{\dot{\gamma}}\mathrm{R}_{t_0-\tau}\right) d\tau. \end{aligned} \tag{12.34}$$

Since, by the $\mathcal{L}$-geodesic equation,

$$\begin{aligned} \partial_\tau\left[\sqrt{\tau}g_{t_0-\tau}\left(\dot{\gamma}, \dot{\gamma}\right)\right] ={}& \sqrt{\tau}\Big(\tfrac{1}{2\tau}g_{t_0-\tau}\left(\dot{\gamma}, \dot{\gamma}\right) \\ &+ 2\,\mathrm{Rc}_{t_0-\tau}\left(\dot{\gamma}, \dot{\gamma}\right) + 2g_{t_0-\tau}\left({}^{\gamma}\nabla_\tau\dot{\gamma}, \dot{\gamma}\right)\Big) \\ ={}& \sqrt{\tau}\left(g_{t_0-\tau}\left({}^{\gamma}\nabla_\tau\dot{\gamma}, \dot{\gamma}\right) + \tfrac{1}{2}\nabla_{\dot{\gamma}}\mathrm{R}_{t_0-\tau}\right), \end{aligned}$$

we conclude that

$$\partial_t\varphi(x,t) \leq \sqrt{t_0 - t}\left(\frac{1}{4(t_0 - t)}|\nabla\varphi(x,t)|^2 - \mathrm{R}(x,t)\right).$$

The inequality (12.31) follows. The inequality (12.43) then follows from the fact that any lower support ψ for $\ell_{(x_0,t_0)}$ induces a lower support $\varphi = 2\sqrt{\tau}\psi$ for $L_{(x_0,t_0)}$. □

12.5.4 *Second variation of $\mathcal{L}$*

We next consider the second variation of $\mathcal{L}$, and its consequences.

Proposition 12.18 (Second variation of $\mathcal{L}$)**.** *Given any $t_0 \in (\alpha, \omega)$, let $\gamma : [\tau_1, \tau_2] \to M^n$ be an $\mathcal{L}_{t_0}$-geodesic along a Ricci flow $(M^n \times (\alpha, \omega), g)$. For any variation $\{\gamma_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ of $\gamma_0 = \gamma$,*

$$\begin{aligned} &\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathcal{L}_{t_0}(\gamma_\varepsilon) \qquad (12.35a) \\ &\quad = g_{t_0-\tau}\left(2\sqrt{\tau}\dot{\gamma}, W\right)\Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\Big[|{}^{\gamma}\nabla_\tau V|^2 - \mathrm{Rm}(\dot{\gamma}, V, \dot{\gamma}, V) \\ &\qquad + \nabla_{\dot{\gamma}}\mathrm{Rc}(V,V) - 2\nabla_V \mathrm{Rc}(\dot{\gamma}, V) + \tfrac{1}{2}\nabla_V\nabla_V \mathrm{R}\Big]\Big|_{(\gamma(\tau), t_0-\tau)} d\tau. \end{aligned}$$

where $\dot{\gamma} \doteqdot \frac{d\gamma}{d\tau}$, $V \doteqdot \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon$ and $W \doteqdot \nabla_\varepsilon \frac{d\gamma_\varepsilon}{d\varepsilon}\big|_{\varepsilon=0}$.

Equivalently (after reparametrizing by $r = \sqrt{\tau}$)

$$\begin{aligned} &\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \mathcal{L}(\gamma_\varepsilon) \qquad (12.35b) \\ &\quad = g_{t_0-r^2}\left(\gamma', W\right)\Big|_{r_1}^{r_2} + \int_{r_1}^{r^2} \Big[|{}^{\gamma}\nabla_r V|^2 - \mathrm{Rm}(\gamma', V, \gamma', V) \\ &\qquad + 2r\nabla_{\gamma'}\mathrm{Rc}(V,V) - 4r\nabla_V \mathrm{Rc}(\gamma', V) + 2r^2\nabla_V\nabla_V \mathrm{R}\Big]\Big|_{(\gamma(r), t_0-r^2)} dr, \end{aligned}$$

where $\gamma' \doteqdot \frac{d\gamma}{dr}$.

Proof. As in the proof of Proposition 12.16, we define $\omega : (r_1, r_2) \times (-\varepsilon_0, \varepsilon_0) \to M^n$ by $\omega(r, \varepsilon) \doteqdot \gamma_\varepsilon(r)$ and work with the pullback connection ${}^{\omega}\nabla$. We begin with

$$\frac{d}{d\varepsilon}\mathcal{L}(\gamma_\varepsilon) = \int_{r_1}^{r_2} \left[g\left({}^{\omega}\nabla_r \frac{d\omega}{d\varepsilon}, \frac{d\omega}{dr}\right) + 2r^2\nabla_\varepsilon \mathrm{R}\right] dr.$$

Thus,

$$\begin{aligned} \frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0}\mathcal{L}(\gamma_\varepsilon) &= \int_{r_1}^{r_2}\Bigg[g\left({}^{\omega}\nabla_\varepsilon\left({}^{\omega}\nabla_r \frac{d\omega}{d\varepsilon}\right), \frac{d\omega}{dr}\right) + g\left({}^{\omega}\nabla_r \frac{d\omega}{d\varepsilon}, {}^{\omega}\nabla_\varepsilon \frac{d\omega}{dr}\right) \\ &\qquad + 2r^2\left(\nabla_{\frac{d\omega}{d\varepsilon}}\nabla_{\frac{d\omega}{d\varepsilon}}\mathrm{R} + \nabla_{{}^{\omega}\nabla_\varepsilon \frac{d\omega}{d\varepsilon}}\mathrm{R}\right)\Bigg]\Bigg|_{\varepsilon=0} dr \\ &= \int_{r_1}^{r_2}\Big[\mathrm{Rm}(\gamma', V, V, \gamma') + g\left({}^{\gamma}\nabla_r V, {}^{\gamma}\nabla_r V\right) \\ &\qquad + g\left({}^{\gamma}\nabla_r W, \gamma'\right) + 2r^2\left(\nabla_V\nabla_V \mathrm{R} + \nabla_W \mathrm{R}\right)\Big] dr. \end{aligned}$$

Recalling Exercise 8.3, we have

$$\begin{aligned} \frac{d}{dr}\Big[g\Big(W, \gamma'\Big)\Big] &= \frac{d}{dr}\left[g\left(\nabla_{\frac{d\omega}{d\varepsilon}}\frac{d\omega}{d\varepsilon}, \frac{d\omega}{dr}\right)\right]\bigg|_{\varepsilon=0} \\ &= 4r\,\mathrm{Rc}\left(W, \gamma'\right) + 4r\nabla_V\,\mathrm{Rc}\left(V, \gamma'\right) - 2r\nabla_{\gamma'}\,\mathrm{Rc}\left(V, V\right) \\ &\quad + g\left({}^{\gamma}\nabla_r W, \gamma'\right) + g\left(W, {}^{\gamma}\nabla_r \gamma'\right). \end{aligned}$$

Applying the $\mathcal{L}$-geodesic equation then yields

$$\frac{d}{dr}\Big[g\Big(W,\gamma'\Big)\Big] = 2r^2\nabla_W \mathrm{R} + 4r\nabla_V \operatorname{Rc}\left(V,\gamma'\right) - 2r\nabla_{\gamma'}\operatorname{Rc}\left(V,V\right) + g\left({}^{\gamma}\nabla_r W,\gamma'\right).$$

Putting this together yields (12.35b). The equation (12.35a) then follows by changing variables. □

Observe that some of the terms which appear in (12.35a) also appear in the matrix Harnack inequality (12.9). We should try to make the relationship more exact. To that end, we use the identity

$$\frac{d}{d\tau}\left[2\sqrt{\tau}\operatorname{Rc}(V,V)\right] = 2\sqrt{\tau}\Big(\tfrac{1}{2\tau}\operatorname{Rc}(V,V) + \nabla_{\dot\gamma}\operatorname{Rc}(V,V) + \nabla_\tau \operatorname{Rc}(V,V) + 2\operatorname{Rc}({}^{\gamma}\nabla_\tau V,V)\Big)$$

to rewrite (12.35a) as

$$\begin{aligned}
&\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0}\mathcal{L}_{t_0}(\gamma_\varepsilon)\\
&\quad= g_{t_0-\tau}\left(2\sqrt{\tau}\dot\gamma,W\right)\Big|_{\tau_1}^{\tau_2} - 2\sqrt{\tau}\operatorname{Rc}_{t_0-\tau}\left(V,V\right)\Big|_{\tau_1}^{\tau_2}\\
&\qquad+\int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\Big[\left|{}^{\gamma}\nabla_\tau V + \operatorname{Rc}(V)\right|^2 - \operatorname{Rm}(\dot\gamma,V,\dot\gamma,V)\\
&\qquad+2\nabla_{\dot\gamma}\operatorname{Rc}(V,V) - 2\nabla_V\operatorname{Rc}(\dot\gamma,V) + \tfrac{1}{2}\Big(\nabla_V\nabla_V \mathrm{R} + 2\operatorname{Rc}^2(V,V)\Big)\\
&\qquad\qquad+\tfrac{1}{2\tau}\operatorname{Rc}(V,V) + \nabla_\tau\operatorname{Rc}(V,V)\Big]\Big|_{(\gamma(\tau),t_0-\tau)}d\tau.\\
&\quad= g_{t_0-\tau}\left(2\sqrt{\tau}\dot\gamma,W\right)\Big|_{\tau_1}^{\tau_2} - 2\sqrt{\tau}\,g_{t_0-\tau}\left(V,V\right)\Big|_{\tau_1}^{\tau_2}\\
&\qquad+\int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\Big[\left|{}^{\gamma}\nabla_\tau V + \operatorname{Rc}(V)\right|^2 - H(\dot\gamma,V)\Big]\Big|_{(\gamma(\tau),t_0-\tau)}d\tau,
\end{aligned}\tag{12.36}$$

where

$$\begin{aligned}
H(U,V) \doteqdot{}& \nabla_t\operatorname{Rc}(V,V) - \tfrac{1}{2}\Big(\nabla_V\nabla_V \mathrm{R} + 2\operatorname{Rc}^2(V,V)\Big)\\
&+2\nabla_V\operatorname{Rc}(U,V) - 2\nabla_U\operatorname{Rc}(V,V) + \operatorname{Rm}(U,V,U,V)\\
&-\tfrac{1}{2(t_0-t)}\operatorname{Rc}(V,V).
\end{aligned}$$

These terms are *almost* identical to the (suitably contracted) left hand side of the matrix differential Harnack inequality (12.9), differing only in that the coefficient of the final term is centred at the *final* time, t_0, rather than the initial time.

The second variation of $\mathcal{L}_{t_0}$ leads to an inequality for the Hessian of $L_{(x_0,t_0)}$, which can be combined with the first variation identities to obtain the following analogue of the Laplacian comparison theorem from Riemannian geometry.

Proposition 12.19. *Along a Ricci flow $(M^n \times I, g)$ on a compact manifold M^n, for any $(x_0, t_0) \in M^n \times I$,*

$$(\partial_t - \Delta)^* \left[(4\pi(t_0 - t))^{-\frac{n}{2}} e^{-\ell_{(x_0,t_0)}} \right] \leq 0 \tag{12.37}$$

and

$$(\partial_t - \Delta) \left((t_0 - t) \left[\ell_{(x_0,t_0)} - \tfrac{n}{2} \right] \right) \geq 0 \tag{12.38}$$

in the viscosity sense[28] *for existence times $t < t_0$.*

Proof. Given any $(x, t) \in M^n \times I$ with $t < t_0$, we can find a minimizing $\mathcal{L}_{t_0}$-geodesic joining $x_0 = \gamma(0)$ to $x = \gamma(t_0 - t)$. Given any $v \in T_x M^n$, let $\{\gamma_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ be a variation of $\gamma_0 = \gamma$ with $\gamma_\varepsilon(0) = x_0$ and $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \gamma_\varepsilon(t_0 - t) = v$. If $\varphi(x, t)$ is a smooth function that satisfies $\varphi \leq L_{(x_0,t_0)}$ in a small backwards neighbourhood of (x, t) with equality at (x, t), then

$$\varphi(\gamma_\varepsilon(t_0 - t), t) \leq L_{(x_0,t_0)}(\gamma_\varepsilon(t_0 - t), t) \leq \mathcal{L}_{t_0}(\gamma_\varepsilon),$$

with equality when $\varepsilon = 0$, and hence, by (12.36),

$$\begin{aligned} 0 \geq \; & \frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \left(\varphi(\gamma_\varepsilon(t_0 - t), t) - \mathcal{L}_{t_0}(\gamma_\varepsilon) \right) \\ = \; & \left(\nabla_V \nabla_V \varphi + \nabla_W \varphi \right)\big|_{(x,t)} - g_{t_0 - \tau}\left(2\sqrt{\tau}\dot{\gamma}, W\right)\big|_0^{\overline{\tau}} + 2\sqrt{\tau}\, \mathrm{Rc}_{t_0-\tau}(V, V)\big|_0^{\overline{\tau}} \\ & - \int_0^{\overline{\tau}} 2\sqrt{\tau} \left[|{}^\gamma\nabla_\tau V + \mathrm{Rc}(V)|^2 - H(\dot{\gamma}, V) \right] \bigg|_{(\gamma(\tau), t_0 - \tau)} d\tau, \end{aligned} \tag{12.39}$$

where $\dot{\gamma} \doteqdot \frac{d\gamma}{d\tau}$. Recalling (12.33), we may equate

$$\nabla_W \varphi(x, t) = g_{t_0 - \tau}\left(2\sqrt{\tau}\dot{\gamma}, W\right)\big|_0^{\overline{\tau}}.$$

Now, given any g_t-orthonormal basis $\{v_i\}_{i=1}^n$ for $T_x M^n$, we can find variation fields $\{V_i(\tau)\}_{i=1}^n$ by solving

$$\begin{cases} {}^\gamma\nabla_\tau V_i = \frac{1}{2\tau} V_i - \mathrm{Rc}(V_i) & \text{for } \tau \in [0, \overline{\tau}] \\ V_i(\overline{\tau}) = v_i . \end{cases} \tag{12.40}$$

This yields the orthonormal frame $\left\{ \sqrt{\frac{\overline{\tau}}{\tau}} V_i(\tau) \right\}_{i=1}^n$ along $(\gamma(\tau), t_0 - \tau)$. (In particular, $V_i(0) = 0$). Applying (12.39) to variations generated by these fields and summing the result then yields

$$\begin{aligned} \Delta\varphi(x, t) \leq \; & -2\sqrt{\overline{\tau}} \mathrm{R}(x, t) + \tfrac{n}{\sqrt{\overline{\tau}}} \\ & + \frac{1}{\overline{\tau}} \int_0^{\overline{\tau}} \tau^{\frac{3}{2}} \left(\partial_\tau \mathrm{R} + \tfrac{1}{\tau}\mathrm{R} + 2\nabla_{\dot{\gamma}}\mathrm{R} - 2\,\mathrm{Rc}(\dot{\gamma}, \dot{\gamma}) \right) \Big|_{(\gamma(\tau), t_0 - \tau)} d\tau. \end{aligned} \tag{12.41}$$

On the other hand, recalling (12.33) and (12.34), we have (see Exercise 12.9)

$$\begin{aligned} |\nabla\varphi(x, t)|^2 = \; & -4\overline{\tau}\mathrm{R}(x, t) + \tfrac{2}{\sqrt{\overline{\tau}}}\varphi(x, t) \\ & + \frac{4}{\sqrt{\overline{\tau}}} \int_0^{\overline{\tau}} \tau^{\frac{3}{2}} \left(\partial_\tau \mathrm{R} + \tfrac{1}{\tau}\mathrm{R} + 2\nabla_{\dot{\gamma}}\mathrm{R} - 2\,\mathrm{Rc}(\dot{\gamma}, \dot{\gamma}) \right) \Big|_{(\gamma(\tau), t_0 - \tau)} d\tau \end{aligned} \tag{12.42}$$

[28] We would be remiss not to be a bit more precise about this. To wit: every smooth φ which supports $\ell_{(x_0,t_0)}$ from below at (x, t) in a *forwards* parabolic neighbourhood of $(x, t) \in M^n \times I \cap (-\infty, t_0)$ (backwards with respect to τ) satisfies

$$(\partial_t - \Delta)^* \left[(4\pi(t_0 - t))^{-\frac{n}{2}} e^{-\varphi} \right] \leq 0$$

at (x, t), and every smooth φ which supports $\ell_{(x_0,t_0)}$ from below at (x, t) in a *backwards* parabolic neighbourhood of $(x, t) \in M^n \times I \cap (-\infty, t_0]$ satisfies

$$(\partial_t - \Delta) \left((t_0 - t) \left[\ell_{(x_0,t_0)} - \tfrac{n}{2} \right] \right) \geq 0$$

at (x, t).

and

$$\partial_t\varphi(x,t) \leq -2\sqrt{\overline{\tau}}\mathrm{R}(x,t) + \tfrac{1}{2\overline{\tau}}\varphi(x,t) \qquad (12.43)$$
$$+\frac{1}{\overline{\tau}}\int_0^{\overline{\tau}} \tau^{\frac{3}{2}}\left(\partial_\tau\mathrm{R} + \tfrac{1}{\tau}\mathrm{R} + 2\nabla_{\dot\gamma}\mathrm{R} - 2\,\mathrm{Rc}(\dot\gamma,\dot\gamma)\right)\Big|_{(\gamma(\tau),t_0-\tau)} d\tau.$$

Putting these together yields the inequality

$$(\partial_t + \Delta)\left(\tfrac{1}{2\sqrt{\tau}}\varphi\right) \leq \left|\nabla\left(\tfrac{1}{2\sqrt{\tau}}\varphi\right)\right|^2 - \mathrm{R} + \tfrac{n}{2\tau}$$

at (x,t). The first claim follows, since any upper support, ψ, for $(4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-\ell}$ induces a lower support, φ, for L via

$$\psi \doteqdot (4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-\frac{1}{2\sqrt{\tau}}\varphi}.$$

The second claim may be established similarly, since (12.43) holds with the opposite inequality for backwards lower supporting functions. □

Applying the trace differential Harnack inequality to (12.42) yields the following useful estimate for Ricci flows with positive curvature.

Proposition 12.20. *Let $(M^n \times I, g)$ be a Ricci flow with positive curvature operator on a compact manifold M^n. Given any $(x_0,t_0) \in M^n \times I$ and any τ_0 such that $t_0 - \tau_0 \in I$, the inequality*

$$\left|\nabla\ell_{(x_0,t_0)}\right|^2 + \mathrm{R} \leq \frac{10\ell_{(x_0,t_0)}}{t_0 - t} \qquad (12.44)$$

holds in the viscosity sense for $t \in [t_0 - \frac{\tau_0}{2}, t_0]$.

Proof. Let φ be a smooth function which supports $\ell_{(x_0,t_0)}$ from below at (x,t). Applying the trace differential Harnack inequality (12.11) to (12.42) yields, at (x,t),

$$\begin{aligned}
|\nabla\varphi|^2 + \mathrm{R} - \frac{1}{2\tau}\varphi &\leq \frac{1}{\tau^{\frac{3}{2}}}\int_0^\tau \sigma^{\frac{3}{2}}\left(\frac{1}{\tau_0-\sigma} + \frac{1}{\sigma}\right)\mathrm{R}\,d\sigma\\
&\leq \frac{2}{\tau^{\frac{3}{2}}}\int_0^\tau \sigma^{\frac{1}{2}}\,\mathrm{R}\,d\sigma\\
&\leq \frac{2}{\tau^{\frac{3}{2}}}\int_0^\tau \sigma^{\frac{1}{2}}\left(|\dot\gamma|^2 + \mathrm{R}\right)d\sigma\\
&= \frac{4}{\tau}\ell\\
&= \frac{4}{\tau}\varphi.
\end{aligned}$$

The claim follows. □

Dropping the gradient term in (12.44), we see that $(t_0 - t)\,\mathrm{R}$ is bounded by[29] $\ell_{(x_0,t_0)}$. On the other hand, dropping the scalar cur-

[29] This is not immediately obvious, since (12.44) only holds (globally) in a weak sense. It can nonetheless be deduced from the viscosity sense. In any case, we will restrict to regions where $\ell_{(x_0,t_0)}$ is smooth here.

vature term, integrating (12.44) yields a bound for $\ell_{(x_0,t_0)}$ in terms of distance to x_0 and time until t_0, at least in regions where $\ell_{(x_0,t_0)}$ is smooth. Indeed, if $\ell_{(x_0,t_0)}(\cdot,t)$ is smooth along a minimizing g_t-geodesic[30] $\gamma:[0,\tau]\to M^n$, $\overline{\tau}=t_0-t$ joining $\gamma(0)=y$ and $\gamma(\tau)=x$, then, under the assumption $\mathrm{R}>0$ (so that $\ell_{(x_0,t_0)}>0$), the function $f(\sigma)\doteqdot\ell^{\frac{1}{2}}_{(x_0,t_0)}(\gamma(\sigma),t)$ is smooth for $t<t_0$, and we may estimate

[30] It will be sufficient that $\sigma\mapsto\ell_{(x_0,t_0)}(\gamma(\sigma),t)$ be piecewise C^1.

$$\begin{aligned} f' &= \frac{1}{2}\frac{\nabla_{\gamma'}\ell_{(x_0,t_0)}|_{(\gamma,t)}}{\ell^{\frac{1}{2}}_{(x_0,t_0)}(\gamma,t)}\\ &\le \frac{1}{2}|\gamma'|_{g_t}\frac{\left|\nabla\ell_{(x_0,t_0)}|_{(\gamma,t)}\right|_{g_t}}{\ell^{\frac{1}{2}}_{(x_0,t_0)}(\gamma,t)}\\ &\le \frac{2|\gamma'|_{g_t}}{\sqrt{\tau}}. \end{aligned}$$

Since $|\gamma'|_{g_t}\equiv\frac{d(x,y,t)}{t_0-t}$, we find that

$$\begin{aligned} \ell_{(x_0,t_0)}(x,t) &\le \left(\ell^{\frac{1}{2}}_{(x_0,t_0)}(y,t)+\frac{2d(x,y,t)}{\sqrt{t_0-t}}\right)^2\\ &\le 5\left(\ell_{(x_0,t_0)}(y,t)+\frac{d^2(x,y,t)}{t_0-t}\right). \end{aligned}$$

We conclude that

$$\sup_{\frac{d^2(x,y,t)}{t_0-t}\le\rho^2}\left[\ell_{(x_0,t_0)}(\cdot,t)+(t_0-t)\,\mathrm{R}(\cdot,t)\right]\le C(\rho,\lambda) \tag{12.45}$$

for all $t\in[t_0-\frac{\tau_0}{2},t_0]$ and all $y\in M^n$ such that $\ell_{(x_0,t_0)}(y,t)\le\lambda^2$ and $\ell_{(x_0,t_0)}(\cdot,t)$ is smooth in $B_{\rho\sqrt{t_0-t}}(y,t)$.

Note that the requirement $\ell_{(x_0,t_0)}(y,t)\le\lambda^2$ in the estimate (12.45) is not vacuous.

Lemma 12.21. *Let $(M^n\times I,g)$ be a Ricci flow with on a compact manifold M^n. Given $t_0\in I$, there exists, for any $t\in I\cap(-\infty,t_0)$, some $x\in M^n$ such that $\ell_{(x_0,t_0)}(x,t)\le\frac{n}{2}$.*

Proof. If $\ell_{(x_0,t_0)}(x,t)>\frac{n}{2}$ for all $x\in M^n$ for some $t_*\in I\cap(-\infty,t_0)$, then we can find $\varepsilon>0$ such that $(t_0-t_*)\big(\ell_{(x_0,t_0)}(x,t_*)-\frac{n}{2}\big)\ge\varepsilon$ for all $x\in M^n$. But then (12.38) ensures that $(t_0-t)\big(\ell_{(x_0,t_0)}(x,t)-\frac{n}{2}\big)\ge\varepsilon$ (and in particular $(t_0-t)\ell_{(x_0,t_0)}(x,t)\ge\varepsilon$) for all $x\in M^n$ and $t\in[t_*,t_0)$. But this violates (12.30). □

12.5.5 $\mathcal{L}$-Jacobi fields and the $\mathcal{L}$-cut locus

Next, we wish to relate the bilinear form

$$Q(V,V) \doteqdot \int_{\tau_1}^{\tau_2} 2\sqrt{\tau}\Big[|{}^\gamma\nabla_\tau V|^2 - \mathrm{Rm}(\dot\gamma, V, \dot\gamma, V) \\ + \nabla_{\dot\gamma}\mathrm{Rc}(V,V) - 2\nabla_V \mathrm{Rc}(\dot\gamma, V) + \tfrac{1}{2}\nabla_V\nabla_V \mathrm{R}\Big]\Big|_{(\gamma(\tau), t_0-\tau)} d\tau.$$

acting on vector fields $V \in \Gamma(\gamma^* TM)$ along γ to an L^2-self-adjoint $\mathbb{R}$-linear map $T : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$. To that end (recalling Exercise 8.3) we apply the identity

$$\frac{d}{d\tau}\big[2\sqrt{\tau} g({}^\gamma\nabla_\tau V, V)\big] = 2\sqrt{\tau}\Big[\tfrac{1}{2\tau} g({}^\gamma\nabla_\tau V, V) + 2\,\mathrm{Rc}({}^\gamma\nabla_\tau V, V) \\ + \nabla_{\dot\gamma}\,\mathrm{Rc}(V,V) + g({}^\gamma\nabla_\tau({}^\gamma\nabla_\tau V), V) + |{}^\gamma\nabla_\tau V|^2\Big],$$

to write

$$Q(V,V) = 2\sqrt{\tau}\, g_{t_0-\tau}\,({}^\gamma\nabla_\tau V, V)\big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} 2\sqrt{\tau} g(TV, V)\, d\tau,$$

where

$$TV \doteqdot {}^\gamma\nabla_\tau({}^\gamma\nabla_\tau V) + \tfrac{1}{2\tau}{}^\gamma\nabla_\tau V + 2\nabla_V \mathrm{Rc}(\dot\gamma) + 2\mathrm{Rc}({}^\gamma\nabla_\tau V) \\ + \mathrm{Rm}(\dot\gamma, V)\dot\gamma - \tfrac{1}{2}\nabla_V(\nabla \mathrm{R}).$$

We conclude that, for any vector field V along γ satisfying symmetric boundary conditions[31],

$$Q(V,V) = -(TV, V)_{L^2(2\sqrt{\tau} d\tau)}.$$

[31] Meaning that

$$2\sqrt{\tau}\, g_{t_0-\tau}\,({}^\gamma\nabla_\tau V, V)\big|_{\tau_1}^{\tau_2} = 0.$$

The operator $T : \Gamma(\gamma^*TM) \to \Gamma(\gamma^*TM)$ is called the $\mathcal{L}$-Jacobi operator. A solution $V \in \Gamma(\gamma^*TM)$ to the $\mathcal{L}$-Jacobi equation

$$-TV = 0$$

along an $\mathcal{L}$-geodesic γ is called an $\mathcal{L}$-Jacobi field (along γ).

Observe that $\mathcal{L}$-Jacobi fields correspond to variation fields of $\mathcal{L}$-geodesic variations. Indeed, if $\{\gamma_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ is a variation of the $\mathcal{L}$-geodesic $\gamma = \gamma_0$ through $\mathcal{L}$-geodesics γ_ε, then, by the $\mathcal{L}$-geodesic equation (12.29a), the variation field $V \doteqdot \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon$ satisfies

$$\begin{aligned} {}^\gamma\nabla_\tau\big({}^\gamma\nabla_\tau V\big) &= {}^\gamma\nabla_\tau\,({}^\gamma\nabla_V\dot\gamma) \\ &= \nabla_V\,({}^\gamma\nabla_\tau\dot\gamma) - \mathrm{Rm}(\dot\gamma, V)\dot\gamma \\ &= \nabla_V\left(\tfrac{1}{2}\nabla \mathrm{R} - \tfrac{1}{2\tau}\dot\gamma - 2\mathrm{Rc}(\dot\gamma)\right) - \mathrm{Rm}(\dot\gamma, V)\dot\gamma \\ &= \tfrac{1}{2}\nabla_V(\nabla \mathrm{R}) - \tfrac{1}{2\tau}\nabla_V\dot\gamma - 2\nabla_V\mathrm{Rc}(\dot\gamma) - 2\,\mathrm{Rc}(\nabla_V\dot\gamma) \\ &\quad - \mathrm{Rm}(\dot\gamma, V)\dot\gamma \\ &= \tfrac{1}{2}\nabla_V(\nabla \mathrm{R}) - \tfrac{1}{2\tau}{}^\gamma\nabla_\tau V - 2\nabla_V\mathrm{Rc}(\dot\gamma) - 2\,\mathrm{Rc}({}^\gamma\nabla_\tau V) \\ &\quad - \mathrm{Rm}(\dot\gamma, V)\dot\gamma. \end{aligned}$$

We may therefore (again mirroring the Riemannian setting) characterize the derivative of the $\mathcal{L}$-exponential map in terms of the $\mathcal{L}$-Jacobi fields. To achieve this, we set up a geodesic variation $\{\gamma_\varepsilon\}_{\varepsilon\in(\varepsilon_0,\varepsilon_0)}$ as follows: start with a minimizing $\mathcal{L}_{t_0}$-geodesic $\gamma(\tau) = \mathcal{L}\exp^{\tau}_{(x_0,t_0)} u$ joining $\gamma(0) = x_0$ to $\gamma(\overline{\tau}) = x$. Given any $v \in T_{x_0}M^n$, we can find a (short) g_t-geodesic $\{\varepsilon \to x(\varepsilon)\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ by solving the g_t-geodesic equation with initial data (x, v); i.e. $x(\varepsilon) \doteqdot \exp_{(x,t)} \varepsilon v$. If γ is minimizing on $[0, \overline{\tau} + \delta)$ for some $\delta > 0$, then ε_0 may be chosen small enough that a *unique* minimizing $\mathcal{L}_{t_0}$-geodesic $\gamma_\varepsilon : [0, \overline{\tau}_\varepsilon] \to M^n$ joining $\gamma_\varepsilon(0) = x_0$ to $\gamma_\varepsilon(\overline{\tau}_\varepsilon) = x(\varepsilon)$ may be found for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. But then $\gamma_\varepsilon(\tau) = \mathcal{L}\exp^{\tau}_{(x_0,t_0)} u_\varepsilon$ for some variation $\{u_\varepsilon\}_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ of $u_0 = u$. Since the kernel of the linear map $T : \Gamma(\gamma^*TM^n) \to \Gamma(\gamma^*TM^n)$ is $2n$-dimensional, we conclude that

$$V(\tau) \doteqdot \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \gamma_\varepsilon(\tau) = d\mathcal{L}\exp^{\tau}_{(x_0,t_0)}\Big|_u \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} u_\varepsilon \qquad (12.46)$$

is the unique $\mathcal{L}$-Jacobi field along γ satisfying $V(0) = 0$ and $V(\overline{\tau}) = v$. Since $T_u(T_{x_0}M^n)$ and $T_{\gamma(\overline{\tau})}M^n$ have the same dimension, we also find that $d\mathcal{L}\exp^{\overline{\tau}}_{(x_0,t_0)}\big|_u$ is an isomorphism.

Denote by $\mathsf{D}^{\tau}_{(x_0,t_0)}$ the set of tangent vectors $v \in T_{x_0}M^n$ which define $\mathcal{L}_{t_0}$-geodesics $\sigma \mapsto \mathcal{L}\exp^{\sigma}_{(x_0,t_0)} v$ that are minimizing up to some value of σ which *exceeds* τ. By smoothness of minimizers of $\mathcal{L}_{t_0}$, such geodesics *uniquely* minimize $\mathcal{L}_{t_0}$ up to τ, and we may thus conclude that the restriction of $\mathcal{L}\exp^{\tau}_{(x_0,t_0)}$ to $\mathsf{D}^{\tau}_{(x_0,t_0)}$ is a diffeomorphism onto its image under $\mathcal{L}\exp^{\tau}_{(x_0,t_0)}$ (which we denote by $D^{\tau}_{(x_0,t_0)} \subset M^n$).

As in the Riemannian setting, it can be shown[32] that the $\mathcal{L}$-CUT LOCUS, $M^n \setminus D^{\tau}_{(x_0,t_0)}$, has measure zero in M^n.

[32] See, e.g. Kleiner and Lott, "Notes on Perelman's papers", §17; cf. e.g. Chavel, *Riemannian geometry*, §III.3.

12.5.6 *The reduced volume and noncollapsing*

Define the REDUCED VOLUME $\tilde{V}$ along a Ricci flow $(M^n \times I, g)$ by

$$\tilde{V}_{(x_0,t_0)}(\tau) \doteqdot \int_{M^n} (4\pi\tau)^{-\frac{n}{2}} \mathrm{e}^{-\ell_{(x_0,t_0)}(\cdot,t_0-\tau)}\, d\mu_{t_0-\tau}$$

for any basepoint $(x_0, t_0) \in M^n \times I$ and $t_0 - \tau \in I$. Inspired by the Bishop–Gromov volume comparison theorem[33], we find that the reduced volume is monotone in τ.

[33] See, e.g., ibid., Proposition III.4.1.

Proposition 12.22. *Along any Ricci flow $(M^n \times I, g)$ on a compact manifold M^n,*

$$\frac{d}{d\tau}\tilde{V}_{(x_0,t_0)}(\tau) \le 0$$

for any basepoint $(x_0, t_0) \in M^n \times I$, with strict inequality unless

$$\mathrm{Rc} + \nabla^2 \ell_{(x_0,t_0)} = \tfrac{1}{2\tau} g \;\; in \;\; D^{\tau}_{(x_0,t_0)}.$$

Proof. Since $\mathcal{L}\exp^{\tau}_{(x_0,t_0)}$ is a diffeomorphism from $\mathsf{D}^{\tau}_{(x_0,t_0)}$ to $D^{\tau}_{(x_0,t_0)}$ and $M^n \setminus D^{\tau}_{(x_0,t_0)}$ has measure zero in M^n, we may compute the reduced volume by pulling back to $T_{x_0}M^n$ using $\mathcal{L}\exp^{\tau}_{(x_0,t_0)}$. Indeed,

$$\tilde{V}_{(x_0,t_0)}(\tau) = \int_{\mathsf{D}^{\tau}_{(x_0,t_0)}} (4\pi\tau)^{-\frac{n}{2}} \mathrm{e}^{-\ell_{(x_0,t_0)}(\gamma_u(\tau),t_0-\tau)} \mathcal{J}(u,\tau)\, d\mathrm{m}(u),$$

where $\gamma_u(\tau) = \mathcal{L}\exp^{\tau}_{(x_0,t_0)} u$, $\mathcal{J}(u,\tau)$ is the Jacobian determinant of the coordinate change $\mathcal{L}\exp^{\tau}_{(x_0,t_0)}$ at u, and $d\mathrm{m}$ denotes the Lebesgue measure on $T_{x_0}M^n$. By (12.46), $\mathcal{J}^2(u,\tau)$ is equal to the determinant of the matrix whose entries are $g_{t_0-\tau}(V_i(\tau), V_j(\tau))$, where $\{V_j\}_{j=1}^n$ are a basis for the $\mathcal{L}$-Jacobi fields along γ_u which vanish at 0. Thus,

$$\begin{aligned}
\frac{d}{d\tau} \log \mathcal{J}(u,\tau) &= \frac{1}{2}\frac{d}{d\tau}\sum_{j=1}^{n} |V_j|^2 \\
&= \sum_{j=1}^{n} \left(\mathrm{Rc}(V_j, V_j) + g({}^{\gamma_u}\nabla_{\tau} V_j, V_j) \right) \\
&= \sum_{j=1}^{n} \left(\mathrm{Rc}(V_j, V_j) + \tfrac{1}{2\sqrt{\tau}} \nabla^2 L_{(x_0,t_0)}(V_j, V_j) \right).
\end{aligned}$$

We may arrange that the basis $\{V_j\}_{j=1}^n$ is orthonormal at a given choice of τ, yielding

$$\frac{d}{d\tau} \log \mathcal{J} = \mathrm{R} + \Delta\ell$$

at that point. If we write

$$d\tilde{\mathrm{m}}(u) \doteqdot (4\pi\tau)^{-\frac{n}{2}} \mathrm{e}^{-\ell_{(x_0,t_0)}(\gamma_u(\tau),t_0-\tau)} \mathcal{J}(u,\tau) \chi_{\mathsf{D}^{\tau}_{(x_0,t_0)}}(u)\, d\mathrm{m}(u),$$

where $\chi_{\mathsf{D}^{\tau}_{(x_0,t_0)}}$ is the characteristic function of $\mathsf{D}^{\tau}_{(x_0,t_0)}$, then, recalling (12.34) and observing that $\tau \mapsto \chi_{\mathsf{D}^{\tau}_{(x_0,t_0)}}(u)$ is nonincreasing, we conclude that

$$\begin{aligned}
\frac{d}{dt}\tilde{V}_{(x_0,t_0)}(\tau) &= \frac{d}{dt}\int_{T_{x_0}M^n} d\tilde{\mathrm{m}} \\
&\geq -\int_{T_{x_0}M^n} \left((\partial_t + \Delta)\ell_{(x_0,t_0)} - |\nabla \ell_{(x_0,t_0)}|^2 + \mathrm{R} - \tfrac{n}{2\tau} \right) d\tilde{m} \\
&= -\int_{T_{x_0}M^n} (\partial_t - \Delta)^* \left((4\pi(t_0 - t))^{-\frac{n}{2}} \mathrm{e}^{-\ell_{(x_0,t_0)}} \right) d\tilde{m} \\
&\geq 0
\end{aligned}$$

due to Proposition 12.19 (where it is understood that the integrands are pulled back to $T_{x_0}M^n$ via $\mathcal{L}\exp$). In fact, the inequality is strict unless the test vector fields (solutions to (12.40) along γ_u with $v_i = V_i(\overline{\tau})$) coincide with the $\mathcal{L}$-Jacobi fields V_i for each u. But then $|V_i|^2 \equiv \frac{\tau}{\overline{\tau}}$, and hence

$$\frac{1}{2\overline{\tau}} = \frac{d}{d\tau}\bigg|_{\tau=\overline{\tau}} \frac{1}{2}|V_i|^2 = \left(\mathrm{Rc}(V_i, V_i) + \nabla^2 \ell_{(x_0,t_0)}(V_i, V_i) \right)\Big|_{\tau=\overline{\tau}}.$$

The rigidity claim follows since u and $\overline{\tau}$ are arbitrary and $\{V_i(\overline{\tau})\}_{i=1}^n$ is a basis for $T_{\gamma_u(\overline{\tau})}M^n$. $\square$

The monotonicity of reduced volume yields, for admissible $\tau \leq \tau_0$,

$$\begin{aligned}\tilde{V}_{(x_0,t_0)}(\tau) &\geq V_{(x_0,t_0)}(\tau_0)\\ &= (4\pi\tau_0)^{-\frac{n}{2}}\int_{M^n} \mathrm{e}^{-\ell_{(x_0,t_0)}(\cdot,t_0-\tau_0)}\,d\mu_{t_0-\tau_0}\\ &\geq \kappa,\end{aligned}$$

so long as $\ell_{(x_0,t_0)}(\cdot,t_0-\tau_0)$ can be suitably bounded from above (at least on a set of fixed $g_{t_0-\tau_0}$-size).

On the other hand, setting $\tau = r^2$, we can write

$$\tilde{V}_{(x_0,t_0)}(\tau) = (4\pi)^{-\frac{n}{2}} r^{-n}\int_{B_r(x_0,t_0)} \mathrm{e}^{-\ell_{(x_0,t_0)}(\cdot,t_0-r^2)}\,d\mu_{t_0-r^2} + \mathcal{E}(\tau),$$

where the "error term" is given by

$$\mathcal{E}(\tau) \doteqdot \int_{M^n\setminus B_r(x_0,t_0)} (4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-\ell_{(x_0,t_0)}(\cdot,t_0-\tau)}\,d\mu_{t_0-\tau}.$$

Now, if $\mathrm{R} \gtrsim -r^{-2}$ in $B_r(x_0,t_0)\times[t_0-r^2,t_0]$, then $\ell_{(x_0,t_0)} \gtrsim -1$ in $B_r(x_0,t_0)\times[t_0-r^2,t_0]$, and hence

$$\tilde{V}_{(x_0,t_0)}(\tau) \lesssim \frac{\mathrm{volume}\left(B_r(x_0,t_0),g_{t_0-r^2}\right)}{r^n} + \mathcal{E}(\tau).$$

Moreover, if $\mathrm{R} \lesssim r^{-2}$ in $B_r(x_0,t_0)\times[t_0-r^2,t_0]$, then

$$-\frac{d}{dt}\log\mathrm{volume}(B_r(x_0,t_0),t) \lesssim r^{-2}$$

and we may therefore relate

$$\mathrm{volume}\left(B_r(x_0,t_0),g_{t_0-r^2}\right) \lesssim \mathrm{volume}\left(B_r(x_0,t_0),g_{t_0}\right).$$

If the error term can be absorbed, then we will obtain a volume non-collapsing estimate.

Theorem 12.23. *Let $(M^n\times[0,T),g)$ be a Ricci flow on a compact manifold M^n. Given $(x,t)\in M^n\times[0,T)$ and $r^2\leq t$, if $|\mathrm{Rm}|^2 \leq r^{-2}$ on $B_r(x,t)\times[t-r^2,t]$, then*

$$\mathrm{volume}(B_r(x,t),t) \geq \kappa r^n,$$

where $\kappa = \kappa(M^n,g_0,T)$.

Proof. Proceeding as above, we may estimate

$$\tilde{V}_{(x,t)}(\tau) \geq (4\pi T)^{-\frac{n}{2}}\int_{M^n}\mathrm{e}^{-\ell_{(x,t)}(\cdot,0)}\,d\mu_0.$$

To estimate the integral from below, it suffices to estimate $\ell_{(x,t)}(\cdot, 0)$ from above on some set of nontrivial g_0-measure. To that end, choose (in accordance with (9.13)) a time $t_* = t_*\big(n, \max_{M^n\times\{0\}}|\mathrm{Rm}|\big)$ such that $\max_{M^n\times[0,t_*]}|\mathrm{Rm}| \le \frac{1}{t_*}$ and (in accordance with Lemma 12.21) a point $x_* \in M^n$ such that $\ell_{(x,t)}(x_*, t_*) \le \frac{n}{2}$. Let $\alpha : [0, t-t_*] \to M^n$ be a minimizing $\mathcal{L}_t$-geodesic joining $\alpha(0) = x$ to $\alpha(t-t_*) = x_*$ and, given any $y \in B_{\sqrt{t_*}}(x_*, t_*)$, let $\beta_y : [t-t_*, t] \to M^n$ be a minimizing g_{t_*}-geodesic joining $\beta_y(t-t_*) = x_*$ to $\beta_y(t) = y$. Taking $\gamma : [0,t] \to M^n$ to be the curve joining $\gamma(0) = x$ to $\gamma(t) = y$ via the concatenation of α and β_y, we find that

$$\begin{aligned}
L_{(x,t)}(y,0) &\le \mathcal{L}_t(\gamma)\\
&= \mathcal{L}_t(\alpha) + \int_{t-t_*}^{t}\sqrt{\tau}\left(|\beta_y'|^2 + \mathrm{R}(\beta_y(\tau), t-\tau)\right)d\tau\\
&\le n\sqrt{t-t_*} + n^2\sqrt{t}\int_{t-t_*}^{t}\frac{d\tau}{t_*}\\
&\le C(n)\sqrt{t},
\end{aligned}$$

from which we deduce that

$$\ell_{(x,t)}(\cdot, 0) \le C(n) \;\text{ in }\; B_{r_*}(x_*, t_*),$$

where $r_* = \sqrt{t_*}$. Since $|\mathrm{Rm}| \le \frac{1}{t_*}$ for $t < t_*$, Proposition 9.6 ensures that

$$B_{r_*/C}(x_*, t) \subset B_{r_*}(x_*, t_*),$$

where $C = C(n, M^n, g_0)$, at which point may conclude that

$$\tilde{V}(\tau) \ge \kappa(n, M^n, g_0, T).$$

On the other hand, proceeding as above, we may estimate

$$\tilde{V}_{(x,t)}(\tau) \le C\frac{\mathrm{volume}\left(B_\rho(x,t), g_t\right)}{r^n} + \mathcal{E}(\tau)$$

for any $\tau = \rho^2 \le r^2$, where the constant $C = C(n)$ depends only on the dimension n. To estimate the error term, note first that, for any given $D > 0$, the image under $\mathcal{L}\exp^{\tau}_{(x,t)}$ of $\mathsf{B}_D = \mathsf{B}_D(0,t)$ (the origin centred $g_{(x,t)}$-ball in T_xM^n of radius D) will be contained in $B_r(x,t)$ when τ is sufficiently small[34] (less than $\tau_D = \delta r^2$ for some $\delta = \delta(n, D) > 0$). Monotonicity of the integrand in the $\mathcal{L}$-exponential coordinates then yields

$$\begin{aligned}
\mathcal{E}(\tau) &\le \int_{T_xM^n\setminus\mathsf{B}_D}(4\pi\tau)^{-\frac{n}{2}}\mathrm{e}^{-\ell_{(x,t)}(\gamma_u(\tau), t-\tau)}\mathcal{J}(u,\tau)\,d\mathrm{m}(u)\\
&\le \lim_{\sigma\to 0}\int_{T_xM^n\setminus\mathsf{B}_D}(4\pi\sigma)^{-\frac{n}{2}}\mathrm{e}^{-\ell_{(x,t)}(\gamma_u(\sigma), t-\sigma)}\mathcal{J}(u,\sigma)\,d\mathrm{m}(u)\\
&= \int_{T_xM^n\setminus\mathsf{B}_D}(4\pi)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|u|^2}{4}}\,d\mathrm{m}(u).
\end{aligned}$$

[34] This is a consequence of continuous dependence of solutions to the $\mathcal{L}$-geodesic equation (12.29b) on the equation data. (Note that $|\nabla \mathrm{R}_{t-\tau}|$ is bounded by $C(n)r^{-3}$ in $B_r(x,t)$ for $\tau \le r^2$ due to the assumed curvature bound and the Bernstein estimates.

Taking D suitably large, we may then conclude that

$$\frac{\text{volume}\left(B_\rho(x,t),g_t\right)}{\rho^n} \geq \kappa,$$

where $\kappa = \kappa(n, M^n, g_0, T)$, so long as $\rho^2 \doteqdot \tau \leq \tau_D = \delta r^2$. Taking $\tau = \tau_D$, we now find that

$$\begin{aligned}\text{volume}\left(B_r(x,t),g_t\right) &\geq \text{volume}\left(B_\rho(x,t),g_t\right)\\ &\geq \kappa\rho^n\\ &= \kappa\delta^{\frac{n}{2}} r^n,\end{aligned}$$

which completes the proof. □

Observe that the hypothesis of Theorem 12.23 is stronger than that of Theorem 12.11 in that the curvature bound is assumed on the whole parabolic cylinder $B_r(x,t) \times [t - r^2, t]$, rather than only at the final time. This is no detriment in practice, as such curvature bounds are already needed in order to obtain convergence of rescaled flows. On the other hand, it turns out that the argument of Theorem 12.23 can be *localized*. This is crucial for controlling Perelman's RICCI FLOW WITH SURGERY[35].

[35] See Perelman, "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds.", "Ricci flow with surgery on three-manifolds." or, e.g., Kleiner and Lott, "Notes on Perelman's papers"; Morgan and Tian, *Ricci flow and the Poincaré conjecture*.

12.6 Exercises

Exercise 12.1. Verify the soliton identities (12.4a), (12.4b) and (12.5a).

Exercise 12.2. Show that the triple $(\mathbb{R}^n, g_{\mathbb{R}^n}, \frac{1}{2}\lambda|x|^2)$ defines a shrinking/steady/expanding soliton according to the sign of λ.

Exercise 12.3. Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$. Suppose that

$$\int_{M^n} h^2\, d\mu = 1.$$

1. Using Jensen's inequality, show for any $\gamma > 0$ that

$$\int_{M^n} h^2 \log h\, d\mu \leq \frac{1}{\gamma} \log\left(\int_{M^n} h^{2+\gamma}\, d\mu\right).$$

2. Using the interpolation inequality for Lebesgue spaces, Young's inequality, and Hölder's inequality show, for a suitable choice of $\gamma = \gamma(n) > 0$, that

$$\left(\int_{M^n} h^{2+\gamma}\, d\mu\right)^{\frac{1}{2+\gamma}} \leq \varepsilon \left(\int_{M^n} h^{2^*}\, d\mu\right)^{\frac{1}{2^*}} + C_\varepsilon$$

for any any $\varepsilon > 0$, where $2^* = \frac{2n}{n-2}$ is the Sobolev conjugate of 2 and C_ε depends on n, volume(M^n) and ε.

3. Deduce from the Sobolev inequality that

$$\int_{M^n} h^2 \log h \, d\mu \le \varepsilon \left(\int_{M^n} |\nabla h|^2 \, d\mu \right)^{\frac{1}{2}} + C_\varepsilon$$

for any $\varepsilon > 0$, where C_ε depends on n, volume(M^n) and ε.

Exercise 12.4. Let $v = \big(-4\pi(t_0 - t)\big)^{-\frac{n}{2}} \mathrm{e}^{-f} : \mathbb{R}^n \times (-\infty, t_0] \to \mathbb{R}$ be a solution to the conjugate heat equation. Show that

$$2\tau \left(\Delta f - \tfrac{1}{2}|\nabla f|^2 \right) + f - n \le 0,$$

or, equivalently,

$$2\tau \left(\partial_t f - \tfrac{1}{2}|\nabla f|^2 \right) - f \ge 0,$$

where $\tau \doteqdot t_0 - t$.

Exercise 12.5. It is well known that, in dimensions $n \ge 3$, the critical points of the Einstein–Hilbert functional

$$\mathcal{H}(g) \doteqdot \int_{M^n} \mathrm{R} \, d\mu$$

for Riemannian metrics g on a manifold M^n are the Einstein metrics; i.e., those metrics satisfying

$$\mathrm{Rc} = 0.$$

Indeed, if $\{g_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0}$ is a one-parameter family of metrics on M^n with $g_0 = g$ and $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_\varepsilon = h$, then (by (9.6) and (9.12))

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{H}(g_\varepsilon) = -\int_{M^n} g\left(\mathrm{Rc} - \tfrac{1}{2}\mathrm{R}\, g, h \right) d\mu.$$

But, in dimensions $n \ge 3$, the Einstein tensor $\mathrm{Rc} - \frac{1}{2}\mathrm{R}\, g$ can only vanish if Rc vanishes.

(a) Prove that, on any differentiable manifold of dimension at least three, any metric whose Einstein tensor vanishes is Ricci flat.

All of this is to say that, while the Ricci flat metrics are the critical points of $\mathcal{H}$, the gradient flow of $\mathcal{H}$ is *not* the Ricci flow.[36]

The problem term, $\frac{1}{2}\mathrm{R}g$, arises from the variation of the measure. So consider, instead of the Einstein–Hilbert functional, the functional

$$\mathcal{F}(g) \doteqdot \int_{M^n} \mathrm{R} \, d\nu,$$

where ν is now some *fixed* measure. We may write $d\nu = \mathrm{e}^{-f_\varepsilon} d\mu_{g_\varepsilon}$ for some family of functions f_ε. Set $f = f_0$.

[36] So, instead of Ricci flow, we should consider the Einstein flow,

$$\frac{d}{dt} g_t = \left(\mathrm{Rc}_{g_t} - \tfrac{1}{2}\mathrm{R}_{g_t} g_t \right),$$

right? Wrong: this flow is not parabolic (not even weakly), so it's not clear where to begin! (Changing the sign does not help either, so don't even bother!)

(b) Show that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{F}(g_\varepsilon) = -\int_{M^n} \Big[g\left(\mathrm{Rc} + \nabla^2 f, h\right) - h(\nabla f, \nabla f) + \left(|\nabla f|^2 - \Delta f\right) \mathrm{tr}_g(h) \Big] \, \mathrm{e}^{-f} \, d\mu.$$

The cost of eliminating the scalar curvature term is four new terms involving derivatives of f! This does not seem like much of an improvement but, on the upside, we do know the variation of f:

(c) Show that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f_\varepsilon = \tfrac{1}{2} \, \mathrm{tr}_g(h).$$

The term

$$\int_{M^n} \Delta f \, \mathrm{tr}_g(h) \, \mathrm{e}^{-f} d\mu = 2 \int_{M^n} \Delta f \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f_\varepsilon \, dv$$

is reminiscent of the first variation of the Dirichlet energy. So consider

$$\mathcal{E}(u) \doteqdot \frac{1}{2} \int_{M^n} |\nabla u|^2 \, dv.$$

(d) Show that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{E}(f_\varepsilon) = \frac{1}{2} \int_{M^n} \Big[\left(|\nabla f|^2 - \Delta f\right) \mathrm{tr}_g(h) - h(\nabla f, \nabla f) \Big] \, \mathrm{e}^{-f} \, d\mu.$$

Set

$$\mathcal{F}(f, g) \doteqdot \int_{M^n} \left(|\nabla f|^2 + \mathrm{R}\right) \mathrm{e}^{-f} \, d\mu.$$

(e) Deduce that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{F}(f_\varepsilon, g_\varepsilon) = \int_{M^n} g(\mathrm{Rc} + \nabla^2 f, h) \mathrm{e}^{-f} \, d\mu.$$

if $(f_0, g_0) = (f, g)$ and $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (f_\varepsilon, g_\varepsilon) = (\frac{1}{2} \, \mathrm{tr}_g(h), h)$.

(f) Conclude that a solution (f, g) to the system

$$\begin{cases} \mathcal{L}_{\partial_t} g = -2(\mathrm{Rc} + \nabla^2 f) \\ \partial_t f = -\left(\mathrm{R} + \Delta f\right) \end{cases} \tag{12.47}$$

will satisfy

$$\frac{d}{dt} \mathcal{F}(f, g) = -\int_{M^n} \left| \mathrm{Rc} + \nabla^2 f \right| \mathrm{e}^{-f} \, d\mu.$$

So the system (12.47) is the (formal) gradient flow of the functional $\mathcal{F}$ (subject to the constant mass constraint).

Note that the gradient flow system (12.47) is geometrically equivalent to Ricci flow coupled with a (logarithmic) conjugate heat flow.

(g) Show that, after pulling back by the flow of ∇f, the gradient flow system (12.47) becomes

$$\begin{cases} \mathcal{L}_{\partial_t} g = -2\mathrm{Rc} \\ (\partial_t - \Delta)^* \mathrm{e}^{-f} = 0. \end{cases}$$

Define the Nash entropy of a pair (f, g) by

$$\mathcal{N}(f,g) \doteqdot -\int_{M^n} f \, d\nu .$$

(h) Show that, along a solution (f, g) to the gradient flow system (12.47),

$$\frac{d}{dt}\mathcal{N}(f,g) = \mathcal{F}(f,g).$$

Exercise 12.6. Consider the gradient self-similarly shrinking Ricci flow $(\mathbb{R}^n \times (-\infty, 0), g, f)$, where $g_t = g_{\mathbb{R}^n}$ is the static Euclidean metric and $f(x,t) = \frac{|x|^2}{-4t}$ is the Gaussian potential. Set $\tau(t) = -t$.

(a) Show that $2\tau(\Delta f - \frac{1}{2}|\nabla f|^2) + f - n = 0$.

(b) Deduce that $\mathscr{P}(f, g, \tau) \equiv 0$.

(c) Deduce that $\mathscr{N}_{(x_0,0)}(t) = 0$ for all $(x_0, t_0) \in \mathbb{R}^n \times (-\infty, \infty)$ and $t < t_0$.

Exercise 12.7 (Gaussian L^2-Poincaré inequality[37]). Consider the gradient self-similarly shrinking Ricci flow $(\mathbb{R}^n \times (-\infty, 0), g, f)$, where $g_t = g_{\mathbb{R}^n}$ is the static Euclidean metric and $f(x,t) = \frac{|x|^2}{-4t}$ is the Gaussian potential, and let u be a solution to the heat equation along the flow satisfying

$$\int_{\mathbb{R}^n} u(\cdot, -1) \, d\nu_{-1} = 0,$$

where $d\nu_t = K(0, 0, \cdot, t) d\mu_{g_t}$ is the heat kernel measure based at $(0, 0)$.

(a) Show that

$$u(0,0) = 0.$$

(b) Deduce that

$$\int_{\mathbb{R}^n} u^2(\cdot, -1) \, d\nu_{-1} - 2\int_{-1}^{0}\int_{\mathbb{R}^n} |\nabla u|^2(\cdot, -1) \, d\nu_t = 0.$$

(c) Deduce that

$$\int_{\mathbb{R}^n} u^2(\cdot, -1) \, d\nu_{-1} \leq 2\int_{\mathbb{R}^n} |\nabla u|^2(\cdot, -1) \, d\nu_{-1}.$$

[37] Hein and Naber, "New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow"

(d) Conclude that any (sufficiently smooth) function u on $\mathbb{R}^n$ which satisfies
$$\int_{\mathbb{R}^n} u\,d\nu = 0,$$
satisfies
$$\int_{\mathbb{R}^n} u^2\,d\nu \le 2\int_{\mathbb{R}^n} |\nabla u|^2\,d\nu,$$
where $d\nu = (4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4}}$ is the GAUSSIAN MEASURE.

Exercise 12.8 (Gaussian L^2-log-Sobolev inequality[38]). Consider the gradient self-similarly shrinking Ricci flow $(\mathbb{R}^n \times (-\infty, 0), g, f)$, where $g_t = g_{\mathbb{R}^n}$ is the static Euclidean metric and $f(x,t) = \frac{|x|^2}{-4t}$ is the Gaussian potential, and let u be a positive solution to the heat equation along the flow satisfying
$$\int_{\mathbb{R}^n} u(\cdot,-1)\,d\nu_{-1} = 1,$$
where $d\nu_t = K(0,0,\cdot,t)d\mu_{g_t}$ is the heat kernel measure based at $(0,0)$.

[38] ibid.

(a) Show that
$$\log u(0,0) = 0.$$

(b) Deduce that
$$\int_{\mathbb{R}^n} u\log u(\cdot,-1)\,d\nu_{-1} - \int_{-1}^{0}\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{u}(\cdot,-1)\,d\nu_t = 0.$$

(c) Deduce that
$$\int_{\mathbb{R}^n} u\log u(\cdot,-1)\,d\nu_{-1} \le \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{u}(\cdot,-1)\,d\nu_{-1}.$$

(d) Conclude that any (sufficiently smooth) function u on $\mathbb{R}^n$ which satisfies
$$\int_{\mathbb{R}^n} u\,d\nu = 1,$$
satisfies
$$\int_{\mathbb{R}^n} u\log u\,d\nu \le 2\int_{\mathbb{R}^n} \frac{|\nabla u|^2}{u}\,d\nu,$$
where $d\nu = (4\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4}}$ is the Gaussian measure.

Exercise 12.9. Given $t_0 \in I$, let $\gamma : [0,\overline{\tau}] \to M^n$ be a minimizing $\mathcal{L}_{t_0}$-geodesic along a Ricci flow $(M^n \times I, g)$ joining $\gamma(0) = x_0$ to some arbitrary point $\gamma(t_0 - t) = x$.

(a) Show that
$$\begin{aligned}\frac{d}{d\tau}\left[\tau^{\frac{3}{2}}\left(\mathrm{R} + |\dot\gamma|^2\right)\right] &= \tau^{\frac{3}{2}}\left(\partial_\tau \mathrm{R} + \tfrac{1}{\tau}\mathrm{R} + 2\nabla_{\dot\gamma}\mathrm{R} - 2\,\mathrm{Rc}(\dot\gamma,\dot\gamma)\right)\\ &\quad - \tfrac{\sqrt{\tau}}{2}(\mathrm{R} + |\dot\gamma|^2),\end{aligned}$$
where all quantities are evaluated along $(\gamma(\tau), t_0 - \tau)$.

(b) Deduce that

$$\overline{\tau}^{\frac{3}{2}}\left(\mathrm{R}+|\dot{\gamma}(\overline{\tau})|^2\right)+\tfrac{1}{2}L=\int_0^{\overline{\tau}}\tau^{\frac{3}{2}}\left(\partial_\tau\mathrm{R}+\tfrac{1}{\tau}\mathrm{R}+2\nabla_{\dot{\gamma}}\mathrm{R}-2\,\mathrm{Rc}(\dot{\gamma},\dot{\gamma})\right)d\tau,$$

where L denotes the L-distance from (x_0,t_0) and the left hand side is evaluated at (x,t).

(c) Conclude that every smooth lower support φ for L at (x,t) satisfies

$$|\nabla\varphi|^2+4\overline{\tau}\mathrm{R}-\tfrac{2}{\sqrt{\overline{\tau}}}\varphi=\frac{4}{\sqrt{\overline{\tau}}}\int_0^{\overline{\tau}}\tau^{\frac{3}{2}}\left(\partial_\tau\mathrm{R}+\tfrac{1}{\tau}\mathrm{R}+2\nabla_{\dot{\gamma}}\mathrm{R}-2\,\mathrm{Rc}(\dot{\gamma},\dot{\gamma})\right)d\tau$$

and

$$\partial_t\varphi+2\overline{\tau}\mathrm{R}-\frac{1}{2\sqrt{\overline{\tau}}}\varphi\leq\frac{1}{\sqrt{\overline{\tau}}}\int_0^{\overline{\tau}}\tau^{\frac{3}{2}}\left(\partial_\tau\mathrm{R}+\tfrac{1}{\tau}\mathrm{R}+2\nabla_{\dot{\gamma}}\mathrm{R}-2\,\mathrm{Rc}(\dot{\gamma},\dot{\gamma})\right)d\tau,$$

where, again, the left hand sides are evaluated at (x,t).

13
Towards a classification of ancient solutions

Let $(M^n \times [0,T), g)$ be a maximal Ricci flow on a compact manifold M^n. Suppose that $T < \infty$, so that $\limsup_{t \nearrow T} \max_{M^n \times \{t\}} |\mathrm{Rm}| \to \infty$. If we choose (x_j, t_j) so that $\lambda_j^2 \doteqdot |\mathrm{Rm}_{(x_j,t_j)}| = \max_{M^n \times [0, T-j^{-1}]} |\mathrm{Rm}|$, then the pointed Ricci flows $(M^n \times I_j, g_j, x_j)$ defined by

$$(g_j)_{(x,t)} \doteqdot \lambda_j^2 g_{(x, \lambda_j^{-2} t + t_j)}, \quad I_j \doteqdot [-\lambda_j^2 t_j, 0]$$

will satisfy $|\mathrm{Rm}| \leq 1$ and $|\mathrm{Rm}_{(x_j,t_j)}| = 1$. Moreover, by Theorem 12.11, $\mathrm{volume}(B_r(x,t), t) \geq \kappa r^n$ whenever $|\mathrm{Rm}| \leq r^{-1}$ in $B_r(x,t)$ and $r \leq \lambda_j$. By the compactness theorem (Theorem 9.19), we can then find a complete *ancient* (subsequential) limit flow $(M^n \times (-\infty, 0], g, o)$, on which

1. $|\mathrm{Rm}| \leq K < \infty$;
2. If $|\mathrm{Rm}| \leq r^{-1}$ in $B_r(x,t)$, then $\mathrm{volume}(B_r(x,t),t) \geq \kappa r^n$.

In dimensions two and three, we will also have

3. $\mathrm{Rm} \geq 0$ and $\mathrm{R} > 0$

due to Proposition 9.11 and Theorem 12.2 (and the fact that $|\mathrm{Rm}| = 1$ at $(o, 0)$). But then the differential Harnack inequality will also hold[1]:

4. (a) $M_{ij} w_i w_j + 2P_{ijk} u_{ij} w_k + \mathrm{Rm}_{ikjl} u_{ik} u_{jl} \geq 0$ for all $w \in TM$ and $u \in \Lambda^2(TM)$, where $M_{ij} \doteqdot \Delta \mathrm{Rc}_{ij} + 2\mathrm{Rm}_{ikjl}\mathrm{Rc}_{kl} - \frac{1}{2}\left(\nabla_i \nabla_j \mathrm{R} + 2\mathrm{Rc}_{ij}^2\right)$ and $P_{ijk} \doteqdot \nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik}$;

   in particular,

   (b) $\partial_t \mathrm{R} + 2\nabla_v \mathrm{R} + 2\mathrm{Rc}(v,v) \geq 0$ for all $v \in TM$.

A good understanding of such solutions will thus provide a good understanding of singularity formation in three-dimensional Ricci flow on compact manifolds. Confidence that this is genuine progress towards an understanding of singularity formation can be taken from the following classical theorem of Hirschman.[1]

Theorem 13.1 (Appell's theorem[2]). *Any positive ancient solution u to the heat equation on $\mathbb{R}^n$ satisfying $u(x,0) = \mathrm{e}^{o(|x|)}$ must be constant.*

[1] Hamilton showed that the argument sketched in Theorem 12.6 may still be applied when M^n is noncompact, so long as the flow has bounded curvature on compact time intervals. See Richard S. Hamilton, "The Harnack estimate for the Ricci flow".

[1] This is the caloric counterpart of Liouville's theorem for harmonic functions. As for Liouville's theorem, the hypotheses are necessary—consider the solutions $\mathrm{e}^{x_1 + t}$ and $|x|^2 + 2nt$, for example. Note that Widder's theorem guarantees that a positive solution to the heat equation on $\mathbb{R}^n \times [\alpha, \omega)$ can be extended uniquely (amongst positive solutions) to $\mathbb{R}^n \times [\alpha, \infty)$. See Widder, "Positive solutions of the heat equation".

[2] Appell, "Sur l'équation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$ et la Théorie de la chaleur"; Hirschman, "A note on the heat equation"

We will present an overview of the landscape and structure of positively curved ancient solutions to Ricci flow. The proofs of many of these results are highly technical. In such cases, we either only sketch the arguments, or omit the proof entirely.

13.1 *Ancient solutions in two space dimensions*

So far, the only ancient Ricci flows we have seen in two dimensions are (highly symmetric) solitons. Namely, the static/shrinking plane, the shrinking sphere, and the cigar soliton (modulo quotients). There is a further (non-soliton) example, which was discovered independently by Fateev–Onofri–Zamolodchikov,[3] King[4] and Rosenau.[5]

[3] Fateev, Onofri, and Al. B. Zamolodchikov, "Integrable deformations of the O(3) sigma model. The sausage model".

[4] King, "Exact polynomial solutions to some nonlinear diffusion equations".

[5] Rosenau, "On fast and super-fast diffusion".

Example 24 (The ancient sausage solution)**.** The time-dependent metric

$$g = \chi^2\, dr^2 + \psi^2\, d\theta^2\,, \tag{13.1a}$$

where

$$\chi^2(r,t) \doteqdot \frac{\tanh(-2t)}{1-\sin^2 r\tanh^2(-2t)} \quad\text{and}\quad \psi^2(r,t) \doteqdot \cos^2 r\chi^2(r,t)\,, \tag{13.1b}$$

extends to a (time-dependent) metric on S^2 and evolves by Ricci flow. Indeed, ψ is smoothly odd at $r = \pm\frac{\pi}{2}$ and, introducing the arclength coordinate

$$s(r,t) \doteqdot \int_0^r \chi(\rho,t)\, d\rho\,,$$

we find that

$$-\mathrm{K} = \frac{\psi_{ss}}{\psi} = \frac{-1}{\sinh(-2t)\cosh(-2t)}\frac{1+\sin^2 r\tanh^2(-2t)}{1-\sin^2 r\tanh^2(-2t)} = \frac{\chi_t}{\chi} = \frac{\psi_t}{\psi}\,.$$

So $(S^2\times(-\infty,0), g)$ indeed satisfies Ricci flow. We also see that its curvature is positive everywhere at all times.

Observe that, for any fixed $r \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$s(r,t) \to 2\,\mathrm{arctanh}(\tan\tfrac{r}{2}) \quad\text{and}\quad \psi(r,t)\to 1$$

as $t \to -\infty$. So, away from the poles, the solution looks like a flat cylinder of radius one when $t \sim -\infty$. (In fact, since the curvature converges to zero away from the poles, the Bernstein estimates and interpolation can be exploited to obtain local uniform convergence in the smooth topology.)

On the other hand, near each pole, the sausage resembles a cigar soliton of the same scale as the asymptotic cylinder. There are various ways to see this; for instance, one may apply the rigidity case of the differential Harnack inequality to obtain an asymptotic steady soliton as explained in §12.3; this soliton will be rotationally symmetric with curvature $\lim_{t\to-\infty} K(\pm\frac{\pi}{2},t) = 2$ at the centre of symmetry, and must therefore be the (unit scale) cigar (see Example 17).

One way to "derive" the ancient sausage is as follows:[6] recall that solutions $u : S^2 \times I \to \mathbb{R}$ to the logarithmic fast diffusion equation

$$u_t = \Delta_{S^2} \log u - 2 \tag{13.2}$$

give rise to Ricci flows via $g = ug_{S^2}$. We seek a solution to (13.2) which is rotationally symmetric. So suppose that $u(r,\theta,t) = u(r,t)$, where (r,θ) are standard polar coordinates on S^2. In that case, (13.2) becomes

$$\begin{aligned} u_t &= \frac{1}{\cos r}\big(\cos r(\log u)_r\big)_r - 2 \\ &= \frac{1}{\cos^2 r}(\log u)_{\xi\xi} - 2, \end{aligned}$$

where $\partial_\xi = \cos r\partial_r$. Setting[7] $v = \cos^2 ru$ (and $\xi = 2\,\mathrm{arctanh}\tan\frac{r}{2}$), we find that v must satisfy

$$v_t = (\log v)_{\xi\xi}, \tag{13.3}$$

the one-dimensional logarithmic fast diffusion equation! Observe now that any antiderivative, say $V(\xi,t) \doteqdot \int_0^\xi v(x,t)\,dx$, of a solution v to (13.3) satisfies the equation

$$V_t = (\log V_\xi)_\xi - C \tag{13.4}$$

for some function C of t only. This does not seem much of an improvement, but consider the following remarkable fact (which is readily verified): if $(\xi,t) \mapsto X(\xi+\lambda t)$ and $(\xi,t) \mapsto Y(\xi-\lambda t)$ are similarity solutions to (13.4), then their sum is also a solution (even though (13.4) is nonlinear!) Consider, then, the ansatz

$$V(\xi,t) = F(\xi+\lambda t) - F(\xi-\lambda t)$$

for some univariate function F. This will solve (13.4) if and only if $f = F'$ satisfies

$$f' = \lambda f^2 + Cf + D$$

for some constants C, $D \in \mathbb{R}$. The solutions are given by

$$f(z) = \alpha - \beta\tanh(\lambda\beta(z-z_0))$$

for $\alpha,\beta,z_0 \in \mathbb{R}$, which yields the solution

$$v(\xi,t) = \beta\tanh\big(\beta\lambda(\xi-\lambda(t-t_0))\big) - \beta\tanh\big(\beta\lambda(\xi+\lambda(t-t_0))\big)$$

to (13.3). Smooth extensibility to the sphere demands that $\beta = \lambda^{-1}$. The constants λ and t_0 then correspond to parabolic dilations and

[6] Ibid.

[7] The map $(r,\theta) \mapsto (\xi,\theta)$ is an isometry from $((-\frac{\pi}{2},\frac{\pi}{2}) \times S^1, u(dr^2+\cos^2 r d\theta^2))$ to $(\mathbb{R}\times S^1, v(d\xi^2+d\theta^2))$.

time-translations, respectively. Taking $t_0 = 0$ and $\lambda = 2$, we arrive at

$$\begin{aligned} g &= \frac{\tanh\left(\xi - 2t)\right) - \tanh\left(\xi + 2t)\right)}{2}(d\xi^2 + d\theta^2) \\ &= \frac{\tanh\left(2(\operatorname{arctanh}\tan\frac{r}{2} - t)\right) - \tanh\left(2(\operatorname{arctanh}\tan\frac{r}{2} + t)\right)}{2\cos^2 r} \\ &\qquad\qquad \cdot (dr^2 + \cos^2 r d\theta^2)\,. \end{aligned}$$

Applying the addition law

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

recovers (13.1).

Consider now the time-dependent diffeomorphisms $\phi_\pm(\cdot, t) : \mathbb{R} \times S^1 \to \mathbb{R} \times S^2$ defined by $\phi_\pm(\xi, \theta, t) \doteqdot (\xi \pm 2t, \theta)$. Observe that

$$\phi_\pm(\cdot, \tau)^* g_{(\xi,\theta,t+\tau)} \to u_\pm(\xi, t)(d\xi^2 + d\theta^2)$$

pointwise as $\tau \to -\infty$, where

$$u_\pm(\xi, t) \doteqdot \frac{1}{2}(\tanh(\pm\xi - 2t) + 1)\,.$$

For each $t \in \mathbb{R}$, this extends to a metric on the plane ($\xi = \mp\infty$ corresponding to the origin) and the convergence can be bootstrapped to smooth convergence on compact subsets of $\mathbb{R}^2 \times (-\infty, \infty)$. We leave it to the reader to verify that the metrics $g_\pm \doteqdot u_\pm(d\xi^2 + d\theta^2)$ are both isometric to the unit scale cigar solution. (Indeed they *must* be, since they are rotationally symmetric steady Ricci flows on the plane with "asymptotic radius" $\lim_{\xi\to\pm\infty} u = 1$.) ■

The ancient sausage example completes the list of two-dimensional ancient Ricci flows!

Theorem 13.2 (Classification of ancient Ricci flows in two-dimensions[8]). *Every maximal, complete ancient Ricci flow $(M^2 \times (-\infty, \omega), g)$ on a connected surface M^2 is either*

- *a shrinking round sphere,*
- *a static flat plane,*
- *a cigar solution,*
- *an ancient sausage, or*
- *an isometric quotient of one of the above examples.*

[8] S. Chu, "Type II ancient solutions to the Ricci flow on surfaces"; Daskalopoulos and Richard S. Hamilton, "Geometric estimates for the logarithmic fast diffusion equation"; Daskalopoulos, Richard S. Hamilton, and Sesum, "Classification of ancient compact solutions to the Ricci flow on surfaces"; Daskalopoulos and Sesum, "Eternal solutions to the Ricci flow on $\mathbb{R}^2$"

Sketch of the proof. Even though we consider potentially noncompact surfaces, it can be shown that our ancient Ricci flow $(M^2 \times (-\infty, \omega), g)$ has nonnegative curvature, and thus positive curvature everywhere unless it is flat.[9] Moreover, if M^2 is not compact, then the timeslices of $(M^2 \times (-\infty, \omega), g)$ must have curvature tending to zero at infinity. Indeed, for any t_0 and any sequence of points x_j such that $d(x_j, o, t_0) \to \infty$, the sequence (M^n, x_j, g_{t_0}) subconverges in the pointed Gromov–Hausdorff sense to a limit space which contains a line, and hence splits off a line. But in two-dimensions, this limit must be locally isometric to $\mathbb{R}^2$. Thus, for j sufficiently large, $B_r(x_j, t_0)$ is close to a Euclidean ball in the Gromov–Hausdorff sense after passing to the universal cover. In particular, its volume (in the universal cover) is close to πr^2. So Perelman's curvature estimate implies that $\mathrm{K}(x_j, t_0) \leq Cr^{-2}$, and we conclude that $\mathrm{K}(x_j, t_0) \to 0$.

[9] B.-L. Chen, Xu, and Zhang, "Local pinching estimates in 3-dim Ricci flow".

Bounded curvature at infinity is sufficient to establish the differential Harnack inequality. By exploiting the differential Harnack inequality and a type-I vs type-II analysis, Chu and Daskalopoulos–Šešum were able to show that the cigar is the only possibility in the noncompact case.

The compact examples were classified by Daskalopoulos–Hamilton–Šešum. The key ideas are a monotonicity formula,

$$\frac{d}{dt}\int_{S^2}\left(\frac{|\nabla^{S^2}v|^2}{v} - 4v\right) d\mu_{S^2} \leq 0,$$

for the PRESSURE FUNCTION $v \doteqdot u^{-1}$ of $g = ug_{S^2}$, and an analysis of the backwards limits of solutions to the equation

$$v_t = v^2(\Delta_{S^2}\log v + 2). \qquad \square$$

13.2 Noncollapsing ancient solutions with positive curvature operator

Let us refer to an ancient solution to Ricci flow satisfying properties 1.–4. at the beginning of this chapter as a κ-SOLUTION.

13.2.1 A nontrivial example

So far, our only examples of ancient solutions are either solitons with a high degree of symmetry (obtained by reduction to an ODE) or the ancient sausage solution (an explicit non-soliton solution obtained by imposing an *ad hoc* ansatz on the logarithmic fast diffusion equation). Perelman provided the first truly "parabolic" (in the sense of PDE methods) construction of an ancient Ricci flow.[10]

[10] Perelman, "Ricci flow with surgery on three-manifolds."

Theorem 13.3 (The ancient Steeden[11]). *There exists a non-round ancient Ricci flow* $(S^3 \times (-\infty, 0), g)$ *which has positive curvature and on which* $O(3) \times O(1)$ *acts by isometries.*

Sketch of the proof. The idea is to take a limit of "very old" solutions constructed by evolving suitable initial data. We begin by evolving a sequence of ($O(3) \times O(1)$-invariant) smoothly capped cylinders $C_k = S^2 \times [-k, k]$ of radius one and length $2k$. When $k = 0$, the solution is the round sphere of radius one, which shrinks to a point after time ~ 1. For other values of k, C_k still shrinks to a point in time ~ 1 (since $\mathrm{R} \sim 1$ at the initial time), becoming round in the process (in accordance with Hamilton's theorem). After translating time, we can arrange that the final time is $t = 0$. By the trace Harnack inequality (Theorem 12.7) and the linear distance distortion estimate (Proposition 9.7), it can be shown that the "perigee" and "apogee" take a fixed time to decrease by $1/2$. So we can parabolically rescale so that, for $k \geq 1$, the "eccentricity" is ~ 2 and the diameter is $\sim 1/2$ at time $t = -1$, and that the initial time α_k goes to $-\infty$ as $k \to \infty$. Since the volumes are uniformly controlled from below, Perelman's curvature estimate (Theorem 9.21) and the Bernstein estimates ensure that the curvature and its derivatives are uniformly bounded along the sequence. We can now take a limit using the compactness theorem. Since we ensured that the eccentricity is ~ 2 at time -1, the limit cannot be the shrinking sphere. □

[11] Steeden are the producers of the iconic Australian Rugby League football (which is more oval than a European football and less pointy than a North American football). Evidently, I am a Rugby League fan; followers of the Rugby Union may prefer the "ancient Gilbert"; followers of Australian Rules Football may prefer the "ancient Sherrin". Followers of American or Canadian football should consider orbifolds.

13.2.2 *Structure of noncollapsing ancient solutions with positive curvature*

The following two theorems, established by Perelman,[12] are key tools in the analysis of κ-solutions.

[12] Perelman, "The entropy formula for the Ricci flow and its geometric applications".

Theorem 13.4. *Let* $(M^n \times (-\infty, 0], g)$ *be a* κ*-solution. If* M^n *is noncompact, then the* ASYMPTOTIC CURVATURE RATIO[13]

[13] This number is independent of the choice of point x_0.

$$\mathcal{R}(M^n, g_0) \doteqdot \limsup_{\mathrm{dist}(x, x_0, 0) \to \infty} \mathrm{R}(x, 0)\, \mathrm{dist}^2(x, x_0, 0)$$

is infinite.

Sketch of the proof. Suppose, contrary to the claim, that $\mathcal{R}(M^n, g_0) < \infty$. Consider the rescaled flow $(M^n \times (-\infty, 0], \lambda^2 g_{\lambda^{-2}t})$. Note that at time zero, the rescaled metrics $(M^n, \lambda^2 g_0)$ always limit to some metric cone (C, d, o) as $\lambda \searrow 0$ in the Gromov–Hausdorff sense. Due to the curvature bound (and noncollapsing) the limit and the convergence will be smooth away from the tip, o. But since the radial direction must be a null eigenvalue of Rc, we deduce (as before) that the limit splits off a line. But this is only possible if the limit cone is flat, and this violates positive curvature on the original flow (by Toponogov's theorem). □

Corollary 13.5. *Let $(M^n \times (-\infty, 0], g)$ be a κ-solution. If M^n is noncompact, then there are points $x_j \in M^n$ and scales λ_j such that $(M^n \times (-\infty, 0], x_j, g_j)$, $(g_j)_{(x,t)} \doteqdot \lambda_j^2 g_{(x, \lambda_j^{-2} t)}$ converges to a κ-solution which splits off a line.*

Sketch of the proof. Since the asymptotic curvature ratio is infinite, we can find points $x_j \in M^n$ such that

$$d_j^2 \doteqdot 10\mathrm{R}(x_j, 0)\,\mathrm{dist}^2(x_j, x_0, 0) \to \infty.$$

In particular, $\mathrm{dist}^2(x_j, x_0, 0) \to \infty$. By point-picking, we can find $y_j \in B_{2d_j/\sqrt{\mathrm{R}(x_j,0)}}(x_j, 0)$ that $\mathrm{R}(y_j, 0) \geq \mathrm{R}(x_j, 0)$ and $\mathrm{R} \leq 2\mathrm{R}(x_j, 0)$ in $B_{d_j/\sqrt{\mathrm{R}(y_j,0)}}(y_j, 0)$. Since $d_j \to \infty$, the pointed rescaled flows $(M^n \times (-\infty, 0], y_j, Q_j g_{(\cdot, Q_j^{-1} t)})$ converge locally smoothly to a limit κ-solution. But (since $y_j \to \infty$) this solution must contain a line, and hence split off a line. □

In particular,

Corollary 13.6. *all two-dimensional κ-solutions are compact.*

This fact of course agrees with the classification of two-dimensional ancient solutions described above.

Perelman's second key observation is the vanishing of the *asymptotic volume ratio*.

Theorem 13.7. *Let $(M^n \times (-\infty, 0], g)$ be a κ-solution. The* ASYMPTOTIC VOLUME RATIO[14]

$$\mathcal{V}(M^n, g_0) \doteqdot \limsup_{r \to \infty} \frac{\mathrm{volume}(B_r(x_0, 0))}{r^n}$$

is zero.

[14] This number is independent of the choice of point x_0.

Sketch of the proof. If $n = 2$, then M^n is compact, and the claim is true. So suppose that the claim is true for some dimension $n \geq 2$ and let $(M^{n+1} \times (-\infty, 0], g)$ be a noncompact κ-solution. By Corollary (13.5), $(M^{n+1} \times (-\infty, 0], g)$ splits off a line at infinity after rescaling. The claim then follows from the inductive hypothesis, since, by the Bishop–Gromov volume comparison theorem, $\mathrm{volume}(B_r(x)) \geq \mathcal{V} r^{n+1}$, which is invariant under rescaling, and hence passes to the limit. □

One consequence of Theorem 13.7 is that, in a κ-solution, the curvature and the normalized volume control each other. Using this fact, in conjunction with Theorem 13.7, the following precompactness property is established by a contradiction argument.

Theorem 13.8 (Precompactness of the space of three-dimensional noncompact κ-solutions[15]). *Given any $\kappa > 0$, the space of three-dimensional noncompact κ-solutions is compact modulo scaling: if $(M_k \times (-\infty, 0], g_k, p_k)$ is a sequence of pointed three-dimensional noncompact κ-solutions and $\lambda_k \doteqdot \sqrt{\mathrm{R}_k(p_k, 0)}$, then a subsequence of the sequence of pointed, rescaled κ-solutions $(M_k \times (-\infty, 0], \tilde{g}_k, p_k)$, where $(\tilde{g}_k)_t \doteqdot \lambda_k^2 (g_k)_{\lambda_k^{-2} t}$, converges locally uniformly in the smooth topology to a κ-solution.*

[15] Perelman, "The entropy formula for the Ricci flow and its geometric applications". See also Kleiner and Lott, "Notes on Perelman's papers", §46.

13.2.3 *Noncollapsing ancient solutions in three space dimensions*

Perelman established the following characterization of κ-solutions in three dimensions[16] by an intricate contradiction argument.

[16] Perelman, "The entropy formula for the Ricci flow and its geometric applications".

Theorem 13.9. *Every connected oriented three-dimensional κ-solution is one of the following.*

1. *A shrinking round spherical space form;*
2. *A shrinking round cylinder or finite quotient;*
3. *A C-component: an S^3 or $\mathbb{R}P^3$ whose diameter, curvature and volume are all bounded uniformly (between C^{-1} and C) after rescaling to normalize any one of them;*
4. *A C-capped ε-tube (after removing one C-cap and rescaling, it is ε close to a unit round cylinder of length ε^{-1}); or*
5. *A doubly C-capped ε-tube.*

Sketch of the proof. After BLOWING DOWN (taking the limit of $\lambda^2 g_{(\cdot, \lambda^{-2} t)}$ as $\lambda \searrow 0$ about points $x_*(t)$ of bounded $\ell_{(x_0,0)}(\cdot, t)$-distance to a fixed x_0 using (12.45)) we see an "asymptotic shrinker" (since the reduced volume will be constant on the limit).[17]

[17] See Perelman, "The entropy formula for the Ricci flow and its geometric applications", Proposition 11.2 or Kleiner and Lott, "Notes on Perelman's papers", Proposition 39.1.

The only asymptotic shrinking solitons are finite quotients of shrinking round spheres or cylinders, so every solution of sufficiently large normalized diameter is made up of ε-tubes and regions of uniformly bounded diameter.

Any example which is not a shrinking cylinder or quotient must satisfy $\mathrm{Rm} > 0$. By the soul theorem, such examples must be either compact or diffeomorphic to $\mathbb{R}^3$; using the compactness of the space of κ-solutions, it can be shown that a noncompact example with $\mathrm{Rm} > 0$ must be C-capped.

A similar argument shows that a compact example either has uniformly bounded diameter, or is a doubly-capped ε-tube. In every case $\mathrm{Rm} > 0$, so Hamilton's theorem implies that the manifold is diffeomorphic to a spherical space form. The uniformly bounded diameter components are either round or C-components. □

In fact, there is now a complete list of such solutions.

Theorem 13.10 (Angenent–Brendle–Daskalopoulos–Šešum[18]). *Every κ-solution in three dimensions is one of the following:*

1. *a static/shrinking* $\mathbb{R}^3$.
2. *a shrinking sphere.*
3. *a shrinking cylinder.*
4. *a radio-dish soliton.*
5. *an ancient Steeden.*
6. *an isometric quotient of one of the above.*

[18] S. Angenent, Brendle, et al., "Unique asymptotics of compact ancient solutions to three-dimensional Ricci flow"; Brendle, "Ancient solutions to the Ricci flow in dimension 3"; Brendle, Daskalopoulos, and Sesum, "Uniqueness of compact ancient solutions to three-dimensional Ricci flow"

13.3 *Further examples of ancient solutions with positive curvature operator*

There are a great many further examples[19] of ancient Ricci flows, even under the assumption of positive curvature.

[19] The below list is *not* exhaustive.

Example 25 (The ancient hypersausage[20]). The time-dependent metric

$$g = \chi^2(r,t)\,dr^2 + \psi^2(r,t)\,d\theta^2 + \varphi^2(r,t)\,d\omega^2$$

[20] Fateev, "The duality between two-dimensional integrable field theories and sigma models"

defined on $(r,\theta,\omega) \in (0,\frac{\pi}{2}) \times S^1 \times S^1$ for $t \in (-\infty, 0)$ by

$$\chi^2(r,t) \doteqdot \frac{\cosh(-4t)\sinh(-4t)}{[\cos^2 r + \sin^2 r\cosh(-4t)][\sin^2 r + \cos^2 r\cosh(-4t)]} \tag{13.5a}$$

$$\psi^2(r,t) \doteqdot \frac{\cos^2 r\sinh(-4t)}{\sin^2 r + \cos^2 r\cosh(-4t)} \tag{13.5b}$$

$$\varphi^2(r,t) \doteqdot \frac{\sin^2 r\sinh(-4t)}{\cos^2 r + \sin^2 r\cosh(-4t)} \tag{13.5c}$$

extends to S^3 and satisfies Ricci flow. Mapping $(0,\frac{\pi}{2}) \times S^1 \times S^1$ into $S^3 \subset \mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$ via the Hopf map $(r,\theta,\omega) \mapsto (\cos r\,\mathrm{e}^{i\theta}, \sin r\,\mathrm{e}^{i\omega})$, we see that g is invariant under the induced action of $U(1) \times U(1)$. It admits a further nontrivial isometric $\mathbb{Z}_2$-action induced by

$$(r,\theta,\omega) \to (\tfrac{\pi}{2} - r, \omega, \theta).$$

Introducing the orthonormal basis $e_1 = \chi^{-1}\partial_r$, $e_2 = \psi^{-1}\partial_\theta$, $e_3 = \varphi^{-1}\partial_\omega$, the curvature operator is diagonalized, with diagonal components

$$\begin{aligned}\sec(e_1\wedge e_2) &= -\frac{\psi_{ss}}{\psi}\\ &= \frac{1}{\sinh(-4t)}\left(2\frac{\cos^2 r+\sin^2 r\cosh(-4t)}{\sin^2 r+\cos^2 r\cosh(-4t)}-\frac{1}{\cosh(-4t)}\right) \quad (13.6a)\\ \sec(e_1\wedge e_3) &= -\frac{\varphi_{ss}}{\varphi}\\ &= \frac{1}{\sinh(-4t)}\left(2\frac{\sin^2 r+\cos^2 r\cosh(-4t)}{\cos^2 r+\sin^2 r\cosh(-4t)}-\frac{1}{\cosh(-4t)}\right) \quad (13.6b)\\ \sec(e_2\wedge e_3) &= -\frac{\psi_s\varphi_s}{\psi\varphi} = \frac{1}{\cosh(-4t)\sinh(-4t)}, \quad (13.6c)\end{aligned}$$

where

$$s(r,t) \doteqdot \int_0^r \chi(\rho,t)\,d\rho.$$

Since the function

$$r\mapsto \frac{\cos^2 r+\sin^2 r\cosh(-4t)}{\sin^2 r+\cos^2 r\cosh(-4t)}$$

is nondecreasing for $r\in[0,\frac{\pi}{2}]$, we find that

$$\sec(e_i\wedge e_j)\geq \frac{1}{\cosh(-4t)\sinh(-4t)}$$

for each $i\neq j$. In particular, g has positive curvature. Since it is not on the list from Theorem 13.10, its volume must collapse (relative to the scale of the curvature) as $t\to-\infty$. Indeed, as $t\to-\infty$,

$$\chi(r,t)\to\frac{1}{\sin r\cos r},\quad \psi(r,t)\to 1 \text{ and } \varphi(r,t)\to 1$$

for any $r\in(0,\frac{\pi}{2})$ and hence, for any point o on, say, the Clifford torus

$$\{p\in S^3 : r(p)=\tfrac{\pi}{4}\}=\{(\tfrac{1}{\sqrt{2}}e^{i\theta},\tfrac{1}{\sqrt{2}}e^{i\omega}) : (\theta,\omega)\in S^1\times S^1\},$$

we have

$$\max_{B_r(o,t)} \mathrm{R}(\cdot,t)\sim 0 \text{ but } \operatorname{volume}(B_r(o,t),t)\sim 8\pi^2 r$$

as $t\to-\infty$ for any large r.

Finally, let us rewrite the hypersausage as

$$g=\sinh(-4t)\left(\frac{dr^2+\cos^2 r\,d\theta^2}{\sin^2 r+\cos^2 r\cosh(-4t)}+\frac{dr^2+\sin^2 r\,d\omega^2}{\cos^2 r+\sin^2 r\cosh(-4t)}\right).$$

Consider the new coordinate ξ defined in $(2\tau,\infty)$ for a given $\tau<0$ by

$$\tanh\tfrac{\xi-2\tau}{2}=\tan\tfrac{r}{2}.$$

Under this transformation, the hypersausage metric at time $t+\tau$ is given by

$$g_{t+\tau} = \sinh(-4(t+\tau))\Bigg(\frac{d\xi^2 + d\theta^2}{\sinh^2(\xi - 2\tau) + \cosh(-4(t+\tau))} + \frac{d\xi^2 + \sinh^2(\xi-2\tau)\,d\omega^2}{1+\sinh^2(\xi-2\tau)\cosh(-4(t+\tau))}\Bigg).$$

Observe that, as $\tau \to -\infty$,

$$\frac{\sinh(-4(t+\tau))}{1+\sinh^2(\xi-2\tau)\cosh(-4(t+\tau))} = \frac{\tanh(-4(t+\tau))}{\frac{1}{\cosh(-4(t+\tau))} + \sinh^2(\xi-2\tau)} \to 0,$$

$$\frac{1}{1+\sinh^2(\xi-2\tau)\cosh(-4(t+\tau))} \to 1$$

and[21]

$$\begin{aligned}
&\frac{\sinh(-4(t+\tau))}{\sinh^2(\xi-2\tau)+\cosh(-4(t+\tau))}\\
&= \frac{2\tanh(-2(t+\tau))}{[\sinh(\xi+2t)+\tanh(-2(t+\tau))\cosh(\xi+2t)]^2 + 1 + \tanh^2(-2(t+\tau))}\\
&\to \frac{2}{[\sinh(\xi+2t)+\cosh(\xi+2t)]^2+2}\\
&= \frac{1}{1+\frac{1}{2}e^{2(\xi+2t)}}.
\end{aligned}$$

[21] Recall the hyperbolic "angle sum" formulae

$$\sinh(2T) = 2\sinh(T)\cosh(T),$$

$$\cosh(2T) = \cosh^2(T) + \sinh^2(T),$$

and

$$\sinh(X+T) = \sinh(X)\cosh(T) + \cosh(X)\sinh(T).$$

We conclude that

$$g_{t+\tau} \to \frac{d\xi^2 + d\theta^2}{1+\frac{1}{2}e^{2(\xi+2t)}} + d\omega^2$$

as $\tau \to -\infty$ (locally uniformly in the smooth topology since the curvature is bounded on compact time intervals), which we recognize as a fixed time-translation of the standard cigar metric.

Similarly (or by the isometric $\mathbb{Z}_2$ action),

$$g_{t+\tau} \to \frac{d\eta^2 + d\omega^2}{1+\frac{1}{2}e^{2(\eta+2t)}} + d\theta^2$$

locally uniformly in the smooth topology, where for a given $\tau < 0$ $\eta \in (-2\tau, \infty)$ is defined by

$$e^{\eta - 2\tau} = \tan\frac{r}{2}. \qquad \blacksquare$$

In fact, the hypersausage is part of a one-parameter family of (geometrically distinct) ancient Ricci flows on S^3.

Example 26 (Twisted ancient hypersausages[22]). For each $k \in (-1,1)$ and $\lambda > 0$, define a function $\xi : (-\infty, 0) \to \mathbb{R}$ by

$$-4\lambda^{-2}(1-k)(1+k)t = \xi - \frac{k}{2}\log\frac{\cosh\xi + k\sinh\xi}{\cosh\xi - k\sinh\xi}.$$

The time-dependent metric[23]

$$g = \chi^2(r,t)\,dr^2 + \psi^2(r,t)\,d\theta^2 + \varphi^2(r,t)\,d\omega^2 + 2v(r,t)d\theta\,d\omega$$

defined in Hopf coordinates $(r,\theta,\omega) \in (0,\frac{\pi}{2}) \times S^1 \times S^1$ for $t \in (-\infty, 0)$ by

$$\begin{aligned}
\chi^2(r,t) &\doteqdot \frac{\lambda^2 \sinh\xi\cosh\xi}{A(r,\xi)B(r,\xi)}\\
\psi^2(r,t) &\doteqdot \frac{\lambda^2\cos^2 r\tanh\xi\left(\sin^2 r\cosh^2\xi + \cos^2 r\sqrt{\cosh^2\xi - k^2\sinh^2\xi}\right)}{A(r,\xi)B(r,\xi)}\\
\varphi^2(r,t) &\doteqdot \frac{\lambda^2\sin^2 r\tanh\xi\left(\cos^2 r\cosh^2\xi + \sin^2 r\sqrt{\cosh^2\xi - k^2\sinh^2\xi}\right)}{A(r,\xi)B(r,\xi)}\\
v(r,t) &\doteqdot \frac{-k\lambda^2\cos^2 r\sin^2 r\tanh\xi\sinh^2\xi}{A(r,\xi)B(r,\xi)},
\end{aligned}$$

where

$$\begin{aligned}
A(r,\xi) &\doteqdot \cos^2 r + \sin^2 r\sqrt{\cosh^2\xi - k^2\sinh^2\xi}\\
B(r,\xi) &\doteqdot \sin^2 r + \cos^2 r\sqrt{\cosh^2\xi - k^2\sinh^2\xi},
\end{aligned}$$

extends to S^3 and satisfies Ricci flow.

The parameter λ corresponds to parabolic rescaling. Taking $\lambda = 1$, the solution corresponding to $k = 0$ is the ancient hypersausage described in the previous example. For each $k \neq 0$, the examples corresponding to $\pm k$ are isometric, but otherwise the members of the family are all geometrically distinct. Indeed, as $t \to -\infty$, the restriction of g to the principal domain converges to

$$g_{-\infty} = \frac{1}{1-k^2}\left(\frac{dr^2}{\sin^2 r\cos^2 r} + d\theta^2 + d\omega^2 - 2kd\theta\,d\omega\right),$$

which is isometric to the standard metric on $\mathbb{R} \times S^1_1 \times S^1_{\frac{1}{\sqrt{1-k^2}}}$. ■

[22] Bakas, Kong, and Ni, "Ancient solutions of Ricci flow on spheres and generalized Hopf fibrations"

[23] Juxtaposition of forms denotes the symmetric tensor product.

Maximally twisting the ancient hypersausage yields a further example.

Example 27 (The ancient Hopf fibration[24]). Setting $\lambda = 1$ and taking $k \to 1$ in the hypersausage family yields another ancient solution on

[24] ibid.

S^3. Noting that $\xi \mapsto -8t$ as $k \to 1$, we see that the limit metric takes the form

$$g = \chi^2(r,t)\,dr^2 + \psi^2(r,t)\,d\theta^2 + \varphi^2(r,t)\,d\omega^2 + 2v(r,t)d\theta\,d\omega$$

with

$$\chi^2(r,t) \doteqdot \sinh(-8t)\cosh(-8t) \tag{13.7a}$$

$$\psi^2(r,t) \doteqdot \cos^2 r\sinh(-8t)\cosh(-8t)\left(1-\cos^2 r\tanh^2(-8t)\right) \tag{13.7b}$$

$$\varphi^2(r,t) \doteqdot \sin^2 r\sinh(-8t)\cosh(-8t)\left(1-\sin^2 r\tanh^2(-8t)\right) \tag{13.7c}$$

$$v(r,t) \doteqdot -\cos^2 r\sin^2 r\sinh(-8t)\cosh(-8t)\tanh^2(-8t)\,. \tag{13.7d}$$

This example is related to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. To see this, we introduce the coordinates

$$R \doteqdot 2r\,,\;\; \Theta \doteqdot \theta+\omega\,,\;\; \Omega \doteqdot \theta-\omega\,,$$

with respect to which the Hopf map is given by

$$\left(\cos\tfrac{R}{2}\,\mathrm{e}^{i\frac{\Theta+\Omega}{2}},\sin\tfrac{R}{2}\,\mathrm{e}^{i\frac{\Theta-\Omega}{2}}\right)\mapsto\left(\sin R\,\mathrm{e}^{2i\Omega},\cos R\right)\;;$$

so the fibres are parametrized by Θ. Defining the one-forms

$$\begin{aligned} X &\doteqdot \sin\Theta\,dR - \sin R\cos\Theta\,d\Omega\,,\\ Y &\doteqdot \cos\Theta\,dR + \sin R\sin\Theta\,d\Omega\,,\\ Z &\doteqdot d\Theta + \cos R\,d\Omega\,, \end{aligned}$$

the ancient Hopf fibration may be expressed as

$$g = \frac{1}{4}\left(\sinh(-8t)\cosh(-8t)\left(X^2+Y^2\right)+\tanh(-8t)Z^2\right).$$

The first term arises from the standard metric on the base space, S^2:

$$X^2+Y^2 = dR^2 + \sin^2 R\,d\Omega^2\,,$$

while the second term may be viewed as a connection one-form on the total space, since

$$dZ = -X\wedge Y\,.$$

Note that the fibres all have the same length, $2\pi\tanh(-8t)$, and collapse at the scale of the curvature as $t\to-\infty$. ∎

The ancient Hopf fibration generalizes[25] to a family of "explicit" ancient Ricci flows on the total spaces S^{2m+1} of the higher dimensional Hopf fibrations $S^1 \hookrightarrow S^{2m+1} \to \mathbb{C}P^m$. Further analogous "fibration-compatible" examples exist on the total spaces of the quaternionic and octonionic Hopf fibrations $S^3 \hookrightarrow S^{4n+1} \to \mathbb{H}P^n$ and $S^7 \hookrightarrow S^{15} \to S^8$,

[25] Bakas, Kong, and Ni, "Ancient solutions of Ricci flow on spheres and generalized Hopf fibrations".

respectively. More generally, it is known that a compact homogeneous space admits a collapsing ancient homogeneous Ricci flow if and only if it is the total space of a homogeneous torus bundle. All known homogeneous examples are invariant under a corresponding torus action (this is known to be necessary under certain assumptions) and, after appropriately rescaling, collapse the torus fibres as time tends to minus infinity and Gromov–Hausdorff converge to an Einstein metric on the base.[26]

The ancient hypersausage may be viewed as a three-dimensional analogue of the ancient sausage solution on S^2; it is not known at present whether or not there exist ancient hypersausages on higher dimensional spheres (i.e. positively curved ancient solutions on S^n which collapse a T^{n-1}).

The following pair of examples do not arise from "cohomogeneity one" structures and provide a different generalization of the ancient sausage.

[26] Buzano, "Ricci flow on homogeneous spaces with two isotropy summands"; Cao and Saloff-Coste, "Backward Ricci flow on locally homogeneous 3-manifolds"; Krishnan, Pediconi, and Sbiti, "Toral symmetries of collapsed ancient solutions to the homogeneous Ricci flow"; Pediconi and Sbiti, "Collapsed ancient solutions of the Ricci flow on compact homogeneous spaces"; Sbiti, "On the Ricci flow of homogeneous metrics on spheres".

Example 28 (Generalized Steedens[27])**.** Perelman's construction generalizes to spheres S^n of any dimension $n \geq 3$ and any bisymmetry class $O(k) \times O(n+1-k)$, $k = 3, \ldots, n$. These examples have positive curvature and their volume does not collapse at any scale as $t \to -\infty$. ∎

[27] The four dimensional case is treated explicitly by Buttsworth, "$SO(2) \times SO(3)$-invariant Ricci solitons and ancient flows on S^4".

Note that, while the symmetry groups $O(k) \times O(n+1-k)$ and $O(\ell) \times O(n+1-\ell)$ agree (up to a congruence of $S^n \subset \mathbb{R}^{n+1}$) when $\ell = n+1-k$, the two corresponding examples in the above construction are *not* congruent (since, for instance, the blow-down of the example with symmetry group $O(k) \times O(n+1-k)$ is the shrinking cylinder $(S^{k-1} \times \mathbb{R}^{n-k} \times (-\infty, 0), -2(k-2)tg_{S^{k-1}} \oplus g_{\mathbb{R}^{n-k}}))$. This begs the question of the whereabouts of the "missing" example: the one corresponding to the symmetry group $O(2) \times O(n-1)$ (whose blow-down should be $\{\mathrm{pt}\} \times \mathbb{R}^{n-1}$).

Example 29 (The ancient pancake[28])**.** For each $n \geq 3$, there is an $O(2) \times O(n-1)$ invariant ancient Ricci flow

$$g = \chi^2(r,t)dr^2 + \psi^2(r,t)d\theta^2 + \varphi^2(r,t)g_{S^{n-2}}$$

on $(-\frac{\pi}{2}, \frac{\pi}{2}) \times S^1 \times S^{n-2}$ which extends to a Ricci flow on S^n with positive curvature (where $O(2) \times O(n-1)$ acts in the standard way). This example is not the shrinking sphere. Indeed, its "girth" $h(t) \doteqdot 2\pi \max \psi(\cdot, t)$ satisfies $h(t) = 1 - o(1)$ as $t \to -\infty$. Nor is g congruent to the ancient hypersausage in case $n = 3$, since its "radius" $r(t) \doteqdot \max \varphi(\cdot, t)$ satisfies $r(t) = -2t + o(-t)$ as $t \to -\infty$.

[28] Bourni, Buttsworth, et al., "Ancient Ricci flows of bounded girth."

This example is constructed by extending the time $t = -R$ slice of the ($O(2) \times O(1)$-invariant) ancient sausage solution on S^2 to an $O(2) \times O(n-1)$-invariant metric on S^n, evolving this metric by Ricci flow to obtain, after time-translation, an "old-but-not-ancient" Ricci flow $(S^n \times [-\alpha_r, 0), g)$ which shrinks to a round point at time zero in accordance with Hamilton's theorem, and (after establishing a number of uniform-in-R estimates) taking a limit as $R \to \infty$. ■

Shrinking and steady solitons are "trivial" examples of ancient Ricci flows. The gradient shrinking solitons with nonnegative curvature operator are relatively easily classified.

Theorem 13.11 (Munteanu–Wang[29])**.** *Every n-dimensional gradient shrinking soliton with nonnegative curvature operator is either:*

1. *a flat* $\mathbb{R}^n$*,*
2. *a compact symmetric space,*
3. *an orthogonal product of a k-dimensional compact symmetric space with* $\mathbb{R}^{n-k}$ *for some* $k = 2, \ldots, n-1$*, or*
4. *an admissible*[30] *isometric quotient of one of these.*

[29] Munteanu and J. Wang, "Positively curved shrinking Ricci solitons are compact"

[30] I.e. compatible with the shrinker potential.

In dimension three, there are only the "obvious" ones: the shrinking sphere, the shrinking cylinder, shrinking Euclidean space, and admissible quotients.

On the other hand, there are a great many steady soliton examples. An important family of examples in the positive curvature setting are the FLYING WINGS.[31]

[31] So named, by Richard Hamilton, for their resemblance to the Northrop and Grumman "Flying Wing" aircraft.

Example 30 (Flying wings[32])**.** For every $n \geq 3$ and each $\theta \in (0, \frac{\pi}{2})$, there is an $O(2) \times O(n-2)$ invariant steady soliton on $\mathbb{R}^n$ which has positive curvature and girth $\sim 2\pi$. It is asymptotic to an S^{n-3} family of cigar hyperplanes "tilted through angle θ" (in the sense that the metric cone at infinity is the round cone of dimension $n-1$ with exterior angle 2θ). ■

[32] Lai, "A family of 3D steady gradient solitons that are flying wings"

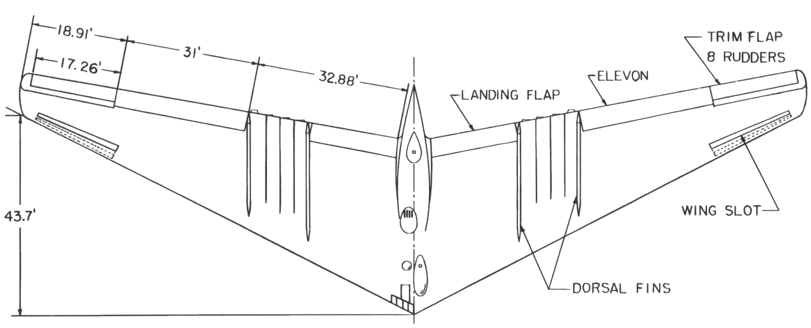


Figure 13.1: Northrop–Grumman YB-49 "flying wing". United States Air Force, *USAF Series YB-35, YB-35A, and YB-49 Aircraft, AN 01-15EAA-4*.

The flying wing family interpolates between a hyperplane of cigars ($\theta = 0$) and the radio-dish ($\theta = \frac{\pi}{2}$). There is also a family of entire analogues of the flying wings, for which the radio dish plays the role of the cigar.

Example 31 (Noncollapsing wings[33]). For every $n \geq 4$ and each pair of numbers $0 < \lambda < \mu$ satisfying $\lambda + (n-1)\mu = 1$, there is an $O(n-1) \times O(1)$ invariant gradient steady soliton on $\mathbb{R}^n$ which is noncollapsed with positive curvature operator, and has Ricci curvature equal to $\operatorname{diag}(\lambda, \mu, \ldots, \mu)$ at its "tip" (critical point of its potential function). ■

[33] Lai, "A family of 3D steady gradient solitons that are flying wings"

A more thorough investigation of Ricci solitons in low dimensions may be found in Chow, *Ricci solitons in low dimensions*.

There is also a family of examples which interpolate between the generalized ancient Steedens, constructed by Haslhofer.

Example 32 (Deformed Steedens[34]). There exists a one-parameter family of noncollapsing ancient Ricci flows on S^4 with positive curvature operator that are "only" $O(3) \times O(1) \times O(1)$-invariant. The blowdown of each member of the family is the shrinking cylinder $(S^2 \times \mathbb{R}^2 \times (-\infty, 0), -2tg_{S^2} \oplus g_{\mathbb{R}^2})$. These examples are produced by a careful modification of Perelman's construction, and conjecturally complete the list of κ-solutions in dimension four.[35] ■

[34] Haslhofer, "On κ-solutions and canonical neighborhoods in 4d Ricci flow"

[35] See ibid., Conjecture 1.3.

A good classification of ancient solutions is thus a very difficult problem in general, even under the assumption of positive curvature. The three dimensional case may be within reach, however.

13.4 *Exercises*

Exercise 13.1. Suppose that $(\xi, t) \mapsto X(\xi + \lambda t)$ and $(\xi, t) \mapsto Y(\xi + \lambda t)$ are solutions to (13.4). Show that $(\xi, t) \mapsto X(\xi + \lambda t) + Y(\xi + \lambda t)$ is also a solution to (13.4).

Exercise 13.2. Given $\lambda, C, D \in \mathbb{R}$, find all solutions to the equation

$$f' = \lambda f + Cf + D$$

by separating variables.

Exercise 13.3. Show that the eternal time-dependent metric on $\mathbb{R} \times S^1$ defined by

$$g_{(\xi,\theta,t)} = \tfrac{1}{2}(\tanh(\xi - 2t) + 1)(d\xi^2 + d\theta^2)$$

extends to $\mathbb{R}^2$ (upon identifying $\{\xi = -\infty\}$ with a point) and is isometric to the standard cigar. *Hint: consider the variable* $r = \operatorname{arcsinh} e^{\xi}$.

Exercise 13.4. Verify equations (13.6). Deduce that the metric defined by (13.5) does indeed satisfy Ricci flow and has positive curvature operator.

Exercise 13.5. Prove that the only shrinking soliton metrics on S^n, $n \geq 2$, with nonnegative curvature operator are the shrinking round metrics, $t \mapsto \frac{1}{2(\omega - t)} g_{S^n}$.

Epilogue

And how did I get onto the Ricci flow? Well, that has to do with Jimmy Carter and the oil crisis. I had bought a nice water ski boat on Lake Cayuga, but it was taking all this money to fill the gas tank. So I had to get something going for my NSF [research proposal] and started thinking about the Ricci flow; I thought about it for a while and it sounded like it might go somewhere. So, I sent that in for my NSF proposal and this was the first time I got rejected. And I think I know why—because after Yau talked to me; he said: "Oh, when I first heard your idea about the Ricci flow, I thought you were a madman." I thought that was the nicest compliment I ever had!

I didn't know much geometry at the time, and I was trying to imitate Eells–Sampson where they have a Dirichlet energy. So, I wanted to take an integral of the first derivative of the metric squared and minimize that. And that wasn't working because I finally found out that in any sort of invariant sense, the first derivative of the metric was zero (the covariant derivative). But then one day I had this bright idea—what if there were such a thing? What would I do next? And I figured I'd integrate by parts and get the d/dt of the metric is something that would be two derivatives of the metric. And I say, "Oh, the only thing that's intrinsic about two derivatives in metric is the curvature." And then I thought, "Well, which curvature?" You got scalar curvature, Ricci curvature, Riemannian curvature... the scalar curvature didn't have enough indices, the Riemannian curvature had too many, and the Ricci curvature was just right. It looked like the metric. So, I wrote down d/dt of the metric is the Ricci curvature. I computed out the evolution of the Riemannian curvature, and I realized it was a *backwards* parabolic equation. So, I thought, "Okay, I'll just put in d/dt of the metric is *minus* the Ricci curvature." And then I put a two in to get rid of the unpleasant one half and started working on that.

And then I knew that the Riemannian curvature was evolving via a nice parabolic type equation, and the Ricci curvature [too], and I made a curious decision to start working on three dimensions instead of two. One person was to say, "Well, you should start on two and if you can't do that, you should give up." But see, I was always quite vain, and I thought, well, I should do something better than just reprove the 100-year-old [uniformization] theorem. So, I had read this thing in Eisenhart that said that in three dimensions you can capture all the curvature from the Ricci curvature, and I thought, "Oh, well, that sounds like a good place to start." Which turned out to be a lucky guess, because positive

Ricci curvature in three dimensions is in some ways stronger than positive scalar in two dimensions. It kind of has more constraints to it. So, then I started working on it and the real breakthrough came one day when I had a girlfriend who was teaching at Gettysburg College; the only thing to do in Gettysburg was to walk around the cemeteries—and I think it was raining that day and you couldn't even do that! So, I just cranked on and I got two good estimates and came up with that good theorem and I kind of got it started. I mean, back then it seemed nearly impossible that you could actually do Poincaré with it. But, you know, a lot of success in math is being lucky, being in the right place at the right time, and trying the right thing.

– R. Hamilton, *And Quiet Goes the Ricci Flow: A Conversation with Richard Hamilton.*

Bibliography

A Perfect Circle. "Vanishing". In: *Thirteenth Step*. Virgin Records, 2003.

Abresch, U. and J. Langer. "The normalized curve shortening flow and homothetic solutions". In: *J. Differential Geom.* 23.2 (1986), pp. 175–196.

Altschuler, Steven, Sigurd B. Angenent, and Yoshikazu Giga. "Mean curvature flow through singularities for surfaces of rotation". In: *J. Geom. Anal.* 5.3 (1995), pp. 293–358.

Altschuler, Steven J. "Singularities of the curve shrinking flow for space curves". In: *J. Differential Geom.* 34.2 (1991), pp. 491–514.

Andrews, Ben. "Classification of limiting shapes for isotropic curve flows". In: *J. Amer. Math. Soc.* 16.2 (2003), pp. 443–459.

– "Contraction of convex hypersurfaces in Euclidean space". In: *Calc. Var. Partial Differential Equations* 2.2 (1994), pp. 151–171.

– "Contraction of convex hypersurfaces in Riemannian spaces". In: *J. Differential Geom.* 39.2 (1994), pp. 407–431.

– "Harnack inequalities for evolving hypersurfaces". In: *Math. Z.* 217.2 (1994), pp. 179–197.

– "Noncollapsing in mean-convex mean curvature flow". In: *Geom. Topol.* 16.3 (2012), pp. 1413–1418.

Andrews, Ben and Charles Baker. "Mean curvature flow of pinched submanifolds to spheres". In: *J. Differential Geom.* 85.3 (2010), pp. 357–395.

Andrews, Ben and Paul Bryan. "A comparison theorem for the isoperimetric profile under curve-shortening flow". In: *Comm. Anal. Geom.* 19.3 (2011), pp. 503–539.

– "Curvature bound for curve shortening flow via distance comparison and a direct proof of Grayson's theorem". In: *J. Reine Angew. Math.* 653 (2011), pp. 179–187.

– "Curvature bounds by isoperimetric comparison for normalized Ricci flow on the two-sphere". In: *Calc. Var. Partial Differential Equations* 39.3-4 (2010), pp. 419–428.

Andrews, Ben, Bennett Chow, et al. *Extrinsic geometric flows*. Vol. 206. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, [2020] ©2020, pp. xxviii+759.

Andrews, Ben and Christopher Hopper. *The Ricci flow in Riemannian geometry*. Vol. 2011. Lecture Notes in Mathematics. A complete proof of the differentiable 1/4-pinching sphere theorem. Springer, Heidelberg, 2011, pp. xviii+296.

Angenent, S. B. and J. J. L. Velázquez. "Degenerate neckpinches in mean curvature flow". In: *J. Reine Angew. Math.* 482 (1997), pp. 15–66.

Angenent, Sigurd, Simon Brendle, et al. "Unique asymptotics of compact ancient solutions to three-dimensional Ricci flow". In: *Comm. Pure Appl. Math.* 75.5 (2022), pp. 1032–1073.

Angenent, Sigurd, Panagiota Daskalopoulos, and Natasa Sesum. "Uniqueness of two-convex closed ancient solutions to the mean curvature flow". In: *Ann. of Math. (2)* 192.2 (2020), pp. 353–436.

Angenent, Sigurd and Dan Knopf. "An example of neckpinching for Ricci flow on S^{n+1}". In: *Math. Res. Lett.* 11.4 (2004), pp. 493–518.

Angenent, Sigurd B. "Nodal properties of solutions of parabolic equations". In: *Rocky Mountain J. Math.* 21.2 (1991). Current directions in nonlinear partial differential equations (Provo, UT, 1987), pp. 585–592.

– "On the formation of singularities in the curve shortening flow". In: *J. Differential Geom.* 33.3 (1991), pp. 601–633.

– "Shrinking doughnuts". In: *Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989)*. Vol. 7. Progr. Nonlinear Differential Equations Appl. Birkhäuser Boston, Boston, MA, 1992, pp. 21–38.

– "The zero set of a solution of a parabolic equation". In: *J. Reine Angew. Math.* 390 (1988), pp. 79–96.

Angenent, Sigurd B., James Isenberg, and Dan Knopf. "Degenerate neckpinches in Ricci flow". In: *J. Reine Angew. Math.* 709 (2015), pp. 81–117.

Appell, Paul. "Sur l'équation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$ et la Théorie de la chaleur". In: *J. Math. Pures Appl.* 4.8 (1892), pp. 187–216.

Bakas, Ioannis, Shengli Kong, and Lei Ni. "Ancient solutions of Ricci flow on spheres and generalized Hopf fibrations". In: *J. Reine Angew. Math.* 663 (2012), pp. 209–248.

Baker, Charles and Huy The Nguyen. "Codimension two surfaces pinched by normal curvature evolving by mean curvature flow". In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 34.6 (2017), pp. 1599–1610.

Ball, J. M. "Differentiability properties of symmetric and isotropic functions". In: *Duke Math. J.* 51.3 (1984), pp. 699–728.

Bando, Shigetoshi. "Real analyticity of solutions of Hamilton's equation". In: *Math. Z.* 195.1 (1987), pp. 93–97.

Berger, Marcel. "Les variétés riemanniennes $(1/4)$-pincées". In: *C. R. Acad. Sci. Paris* 250 (1960), pp. 442–444.

Böhm, Christoph and Burkhard Wilking. "Manifolds with positive curvature operators are space forms". In: *Ann. of Math. (2)* 167.3 (2008), pp. 1079–1097.

Bourni, Theodora, Timothy Buttsworth, et al. "Ancient Ricci flows of bounded girth." Preprint, arXiv:2302.04964.

Bourni, Theodora, Mat Langford, and Stephen Lynch. "Collapsing and noncollapsing in convex ancient mean curvature flow". In: *J. Reine Angew. Math.* 801 (2023), pp. 273–305.

Bourni, Theodora, Mat Langford, and Giuseppe Tinaglia. "Ancient mean curvature flows out of polytopes". In: *Geom. Topol.* 26.4 (2022), pp. 1849–1905.

– "Collapsing ancient solutions of mean curvature flow". In: *J. Differential Geom.* 119.2 (2021), pp. 187–219.

– "Convex ancient solutions to curve shortening flow". In: *Calc. Var. Partial Differential Equations* 59.4 (2020), Paper No. 133, 15.

– "Convex ancient solutions to mean curvature flow". In: *Differential geometry in the large*. Vol. 463. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2021, pp. 50–74.

– "On the existence of translating solutions of mean curvature flow in slab regions". In: *Anal. PDE* 13.4 (2020), pp. 1051–1072.

Brakke, Kenneth A. *The motion of a surface by its mean curvature*. Vol. 20. Mathematical Notes. Princeton University Press, Princeton, N.J., 1978, pp. i+252.

Brendle, Simon. "Ancient solutions to the Ricci flow in dimension 3". In: *Acta Math.* 225.1 (2020), pp. 1–102.

Brendle, Simon and Kyeongsu Choi. "Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$". In: *Invent. Math.* 217.1 (2019), pp. 35–76.

Brendle, Simon, Panagiota Daskalopoulos, and Natasa Sesum. "Uniqueness of compact ancient solutions to three-dimensional Ricci flow". In: *Invent. Math.* 226.2 (2021), pp. 579–651.

Brendle, Simon and Gerhard Huisken. "A fully nonlinear flow for two-convex hypersurfaces in Riemannian manifolds". In: *Invent. Math.* 210.2 (2017), pp. 559–613.

Brendle, Simon and Richard Schoen. "Manifolds with $1/4$-pinched curvature are space forms". In: *J. Amer. Math. Soc.* 22.1 (2009), pp. 287–307.

Bryan, Paul. "Curvature bounds via an isoperimetric comparison for Ricci flow on surfaces". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 16.3 (2016), pp. 729–757.

Bustamante, Ignacio and Reiris Martín. "Deriving Perelman's entropy from Colding's monotonic volume". Preprint, arXiv:2501.12949.

Buttsworth, Timothy. "$SO(2) \times SO(3)$-invariant Ricci solitons and ancient flows on $\mathbb{S}^4$". In: *J. Lond. Math. Soc. (2)* 106.2 (2022), pp. 1098–1130.

Buzano, Maria. "Ricci flow on homogeneous spaces with two isotropy summands". In: *Ann. Glob. Anal. Geom.* 45.1 (2014), pp. 25–45.

Cao, Xiaodong and Laurent Saloff-Coste. "Backward Ricci flow on locally homogeneous 3-manifolds". In: *Commun. Anal. Geom.* 2 (2009), pp. 305–325.

Carapetis, Anthony. *Curve shortening demo*. GitHub repository, https://github.com/acarapetis/curve-shortening-demo. 2019.

Chang, Kenneth. *In Bubbles and Metal, the Art of Shape-Shifting*. New York Times. May 22, 2007.

Chavel, Isaac. *Eigenvalues in Riemannian geometry*. Vol. 115. Pure and Applied Mathematics. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. Academic Press, Inc., Orlando, FL, 1984, pp. xiv+362.

– *Riemannian geometry*. Second. Vol. 98. Cambridge Studies in Advanced Mathematics. A modern introduction. Cambridge University Press, Cambridge, 2006, pp. xvi+471.

Chen, Bing-Long, Guoyi Xu, and Zhuhong Zhang. "Local pinching estimates in 3-dim Ricci flow". In: *Math. Res. Lett.* 20.5 (2013), pp. 845–855.

Chen, Bing-Long and Xi-Ping Zhu. "Complete Riemannian manifolds with pointwise pinched curvature". In: *Invent. Math.* 140.2 (2000), pp. 423–452.

Chen, Xiuxiong, Peng Lu, and Gang Tian. "A note on uniformization of Riemann surfaces by Ricci flow". In: *Proc. Amer. Math. Soc.* 134.11 (2006), pp. 3391–3393.

Choi, Beomjun et al. "Classification of bubble-sheet ovals in $\mathbb{R}^4$". Preprint, arXiv:2209.04931.

Choi, Kyeongsu and Robert Haslhofer. "Classification of ancient noncollapsed flows in $\mathbb{R}^4$". Preprint, arXiv:2412.10581.

Choi, Kyeongsu, Robert Haslhofer, and Or Hershkovits. "Classification of noncollapsed translators in $\mathbb{R}^4$". In: *Camb. J. Math.* 11.3 (2023), pp. 563–698.

Chow, Bennett. *Ricci solitons in low dimensions*. Vol. 235. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2023, pp. xvi+339.

– "The Ricci flow on the 2-sphere". In: *J. Differential Geom.* 33.2 (1991), pp. 325–334.

Chow, Bennett, Sun-Chin Chu, et al. *The Ricci flow: techniques and applications. Part III. Geometric-analytic aspects*. Vol. 163. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2010, pp. xx+517.

Chow, Bennett and Dan Knopf. *The Ricci flow: an introduction*. Vol. 110. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004, pp. xii+325.

Chow, Bennett, Peng Lu, and Lei Ni. *Hamilton's Ricci flow*. Vol. 77. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006, pp. xxxvi+608.

Chu, Sun-Chin. "Type II ancient solutions to the Ricci flow on surfaces". In: *Comm. Anal. Geom.* 15.1 (2007), pp. 195–215.

Clutterbuck, Julie, Oliver C. Schnürer, and Felix Schulze. "Stability of translating solutions to mean curvature flow". In: *Calc. Var. Partial Differential Equations* 29.3 (2007), pp. 281–293.

Colding, Tobias H. and William P. Minicozzi II. "Generic mean curvature flow I: generic singularities". In: *Ann. of Math. (2)* 175.2 (2012), pp. 755–833.

Colding, Tobias Holck. "New monotonicity formulas for Ricci curvature and applications. I". In: *Acta Math.* 209.2 (2012), pp. 229–263.

Daskalopoulos, Panagiota, Richard Hamilton, and Natasa Sesum. "Classification of compact ancient solutions to the curve shortening flow". In: *J. Differential Geom.* 84.3 (2010), pp. 455–464.

Daskalopoulos, Panagiota and Richard S. Hamilton. "Geometric estimates for the logarithmic fast diffusion equation". In: *Comm. Anal. Geom.* 12.1-2 (2004), pp. 143–164.

Daskalopoulos, Panagiota, Richard S. Hamilton, and Natasa Sesum. "Classification of ancient compact solutions to the Ricci flow on surfaces". In: *J. Differential Geom.* 91.2 (2012), pp. 171–214.

Daskalopoulos, Panagiota and Mariel Saez. "Uniqueness of entire graphs evolving by mean curvature flow". In: *J. Reine Angew. Math.* 796 (2023), pp. 201–227.

Daskalopoulos, Panagiota and Natasa Sesum. "Eternal solutions to the Ricci flow on $\mathbb{R}^2$". In: *Int. Math. Res. Not.* (2006), Art. ID 83610, 20.

Deruelle, Alix, Felix Schulze, and Miles Simon. "Initial stability estimates for Ricci flow and three dimensional Ricci-pinched manifolds." Preprint, arXiv:2203.15313.

Du, Wenkui and Robert Haslhofer. "On uniqueness and nonuniqueness of ancient ovals". Preprint, arXiv:2105.13830.

Ecker, Klaus. "A local monotonicity formula for mean curvature flow". In: *Ann. of Math. (2)* 154.2 (2001), pp. 503–525.

– *Regularity theory for mean curvature flow*. Vol. 57. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, 2004, pp. xiv+165.

Ecker, Klaus and Gerhard Huisken. "Interior estimates for hypersurfaces moving by mean curvature". In: *Invent. Math.* 105.3 (1991), pp. 547–569.

Ecker, Klaus and Gerhard Huisken. "Mean curvature evolution of entire graphs". In: *Ann. of Math. (2)* 130.3 (1989), pp. 453–471.

Epstein, C. L. and M. I. Weinstein. "A stable manifold theorem for the curve shortening equation". In: *Comm. Pure Appl. Math.* 40.1 (1987), pp. 119–139.

Eschenburg, J.-H. "Local convexity and nonnegative curvature—Gromov's proof of the sphere theorem". In: *Invent. Math.* 84.3 (1986), pp. 507–522.

Fateev, V. A. "The duality between two-dimensional integrable field theories and sigma models". In: *Phys. Lett. B* 357.3 (1995), pp. 397–403.

Fateev, V. A., E. Onofri, and Al. B. Zamolodchikov. "Integrable deformations of the O(3) sigma model. The sausage model". In: *Nuclear Phys. B* 406.3 (1993), pp. 521–565.

Freedman, Michael Hartley. "Afterword: PC4 at age 40". In: *The disc embedding theorem*. Oxford Univ. Press, Oxford, 2021, pp. 447–455.

– "The topology of four-dimensional manifolds". In: *J. Differential Geometry* 17.3 (1982), pp. 357–453.

Friedan, D. "Nonlinear models in $2+\varepsilon$ dimensions". In: *Phys. Rev. Lett.* 45.13 (1980), pp. 1057–1060.

Gage, M. and R. S. Hamilton. "The heat equation shrinking convex plane curves". In: *J. Differential Geom.* 23.1 (1986), pp. 69–96.

Gossip. "Heavy Cross". In: *Music for Men*. Columbia, 2009.

Grayson, Matthew. "The heat equation shrinks embedded plane curves to round points." In: *J. Differential Geom.* 26.2 (1987), pp. 285–314.

Guenther, Christine M. "The fundamental solution on manifolds with time-dependent metrics". In: *J. Geom. Anal.* 12.3 (2002), pp. 425–436.

Halldorsson, Hoeskuldur P. "Self-similar solutions to the curve shortening flow". In: *Trans. Amer. Math. Soc.* 364.10 (2012), pp. 5285–5309.

Hamilton, Richard. *And Quiet Goes the Ricci Flow: A Conversation with Richard Hamilton.* `https://scgp.stonybrook.edu/archives/38013`. Simons Center for Geometry and Physics. Interview by John Morgan. Edited by Maria Shtilmark. 2022.

Hamilton, Richard S. "An isoperimetric estimate for the Ricci flow on the two-sphere". In: *Modern methods in complex analysis (Princeton, NJ, 1992)*. Vol. 137. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1995, pp. 191–200.

– "Convex hypersurfaces with pinched second fundamental form". In: *Comm. Anal. Geom.* 2.1 (1994), pp. 167–172.

– "Harnack estimate for the mean curvature flow". In: *J. Differential Geom.* 41.1 (1995), pp. 215–226.

– "Isoperimetric estimates for the curve shrinking flow in the plane". In: *Modern methods in complex analysis (Princeton, NJ, 1992)*. Vol. 137.

Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1995, pp. 201–222.
– "Non-singular solutions of the Ricci flow on three-manifolds". In: *Comm. Anal. Geom.* 7.4 (1999), pp. 695–729.
– "The Harnack estimate for the Ricci flow". In: *J. Differential Geom.* 37.1 (1993), pp. 225–243.
– "The Ricci flow on surfaces". In: *Mathematics and general relativity (Santa Cruz, CA, 1986)*. Vol. 71. Contemp. Math. Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.
– "Three-manifolds with positive Ricci curvature". In: *J. Differential Geometry* 17.2 (1982), pp. 255–306.
Haslhofer, Robert. "On κ-solutions and canonical neighborhoods in 4d Ricci flow". In: *J. Reine Angew. Math.* 811 (2024), pp. 257–265.
Haslhofer, Robert and Or Hershkovits. "Ancient solutions of the mean curvature flow". In: *Comm. Anal. Geom.* 24.3 (2016), pp. 593–604.
Haslhofer, Robert and Bruce Kleiner. "Mean curvature flow of mean convex hypersurfaces". In: *Comm. Pure Appl. Math.* 70.3 (2017), pp. 511–546.
Hein, Hans-Joachim and Aaron Naber. "New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow". In: *Comm. Pure Appl. Math.* 67.9 (2014), pp. 1543–1561.
Hirschman Jr., I. I. "A note on the heat equation". In: *Duke Math. J.* 19 (1952), pp. 487–492.
Hoffman, D. et al. "Graphical translators for mean curvature flow". In: *Calc. Var. Partial Differential Equations* 58.4 (2019), Paper No. 117, 29.
Huisken, Gerhard. "A distance comparison principle for evolving curves". In: *Asian J. Math.* 2.1 (1998), pp. 127–133.
– "Asymptotic behavior for singularities of the mean curvature flow". In: *J. Differential Geom.* 31.1 (1990), pp. 285–299.
– "Flow by mean curvature of convex surfaces into spheres". In: *J. Differential Geom.* 20.1 (1984), pp. 237–266.
– *Mathmedia Interview: Prof. Gerhard Huisken*. `https://www.math.sinica.edu.tw/interviewindexe/journals/4820`. Mathmedia 41.3. Interview by Tai-Ping Liu and Jih-Hsin Cheng. 2017.
Huisken, Gerhard and Alexander Polden. "Geometric evolution equations for hypersurfaces". In: *Calculus of variations and geometric evolution problems (Cetraro, 1996)*. Vol. 1713. Lecture Notes in Math. Springer, Berlin, 1999, pp. 45–84.
Huisken, Gerhard and Carlo Sinestrari. "Mean curvature flow singularities for mean convex surfaces". In: *Calc. Var. Partial Differential Equations* 8.1 (1999), pp. 1–14.
– "Mean curvature flow with surgeries of two-convex hypersurfaces". In: *Invent. Math.* 175.1 (2009), pp. 137–221.

Ivey, Thomas. "Ricci solitons on compact three-manifolds". In: *Differential Geom. Appl.* 3.4 (1993), pp. 301–307.

Kinderlehrer, David and Guido Stampacchia. *An introduction to variational inequalities and their applications*. Vol. 31. Classics in Applied Mathematics. Reprint of the 1980 original. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000, pp. xx+313.

King, J. R. "Exact polynomial solutions to some nonlinear diffusion equations". In: *Phys. D* 64.1-3 (1993), pp. 35–65.

King Gizzard & The Lizard Wizard. "Murder of the Universe". In: *Murder of the Universe*. Flightless, 2017.

Kleiner, Bruce and John Lott. "Notes on Perelman's papers". In: *Geom. Topol.* 12.5 (2008), pp. 2587–2855.

Klingenberg, Wilhelm. *Lectures on closed geodesics*. Grundlehren der Mathematischen Wissenschaften, Vol. 230. Springer-Verlag, Berlin-New York, 1978, pp. x+227.

– "Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung". In: *Comment. Math. Helv.* 35 (1961), pp. 47–54.

Krishnan, Anusha, Francesco Pediconi, and Sammy Sbiti. "Toral symmetries of collapsed ancient solutions to the homogeneous Ricci flow". Preprint, arXiv:2312.01469.

Kühnel, Wolfgang. *Differential geometry*. German. Vol. 77. Student Mathematical Library. Curves—surfaces—manifolds. American Mathematical Society, Providence, RI, 2015, pp. xii+402.

Lai, Yi. "A family of 3D steady gradient solitons that are flying wings". In: *J. Differential Geom.* 126.1 (2024), pp. 297–328.

Lambert, Ben and Elena Mäder-Baumdicker. "A note on Alexandrov immersed mean curvature flow". In: *J. Geom. Anal.* 34.9 (2024), Paper No. 268, 25.

Lee, Man-Chun and Peter Topping. "Manifolds with PIC1 pinched curvature". Preprint, arXiv:2211.07623.

– "Three-manifolds with non-negatively pinched Ricci curvature". Preprint, arXiv:2204.00504.

Li, Peter and Shing-Tung Yau. "On the parabolic kernel of the Schrödinger operator". In: *Acta Math.* 156.3-4 (1986), pp. 153–201.

Lott, John. "On 3-manifolds with pointwise pinched nonnegative Ricci curvature". In: *Math. Ann.* 388.3 (2024), pp. 2787–2806.

Lukyanov, S. L., E. S. Vitchev, and A. B. Zamolodchikov. "Integrable model of boundary interaction: the paperclip". In: *Nuclear Phys. B* 683.3 (2004), pp. 423–454.

Lynch, Stephen. "Convexity and gradient estimates for fully nonlinear curvature flows". Available at `https://publikationen.uni-tuebingen.de/xmlui/handle/10900/109737`. PhD thesis. Mathematisch-Naturwissenschaftliche Fakultät: Eberhard Karls Universität Tübingen, Nov. 2020.

Mantegazza, Carlo. *Lecture notes on mean curvature flow*. Vol. 290. Progress in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2011, pp. xii+166.

McCann, Robert J. and Peter M. Topping. "Ricci flow, entropy and optimal transportation". In: *Amer. J. Math.* 132.3 (2010), pp. 711–730.

Micallef, Mario J. and John Douglas Moore. "Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes". In: *Ann. of Math. (2)* 127.1 (1988), pp. 199–227.

Michael, J. H. and L. M. Simon. "Sobolev and mean-value inequalities on generalized submanifolds of R^n". In: *Comm. Pure Appl. Math.* 26 (1973), pp. 361–379.

Morgan, John and Gang Tian. *Ricci flow and the Poincaré conjecture*. Vol. 3. Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007, pp. xlii+521.

Mullins, W. W. "Two-dimensional motion of idealized grain boundaries". In: *J. Appl. Phys.* 27 (1956), pp. 900–904.

Munteanu, Ovidiu and Jiaping Wang. "Positively curved shrinking Ricci solitons are compact". In: *J. Differential Geom.* 106.3 (2017), pp. 499–505.

Nakayama, Kazuaki, Takeshi Iizuka, and Miki Wadati. "Curve lengthening equation and its solutions". In: *J. Phys. Soc. Japan* 63.4 (1994), pp. 1311–1321.

Nash, J. "Continuity of solutions of parabolic and elliptic equations". In: *Amer. J. Math.* 80 (1958), pp. 931–954.

Nguyen, Huy T. "Isotropic curvature and the Ricci flow". In: *Int. Math. Res. Not. IMRN* 3 (2010), pp. 536–558.

Ni, Lei and Baoqiang Wu. "Complete manifolds with nonnegative curvature operator". In: *Proc. Amer. Math. Soc.* 135.9 (2007), pp. 3021–3028.

Olver, Peter J. *Introduction to partial differential equations*. Undergraduate Texts in Mathematics. Springer, Cham, 2014, pp. xxvi+635.

Pediconi, Francesco and Sammy Sbiti. "Collapsed ancient solutions of the Ricci flow on compact homogeneous spaces". In: *Proceedings of the London Mathematical Society* 125.5 (2022), pp. 1130–1151.

Perelman, Grisha. "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds." Preprint, arXiv:math/0307245.

– "Ricci flow with surgery on three-manifolds." Preprint, arXiv:0303109.

– "The entropy formula for the Ricci flow and its geometric applications". Preprint, arXiv:0211159.

Puscifer. "A Singularity". In: *Existential Reckoning*. Alchemy Recordings, 2020.

Rauch, H. E. "A contribution to differential geometry in the large". In: *Ann. of Math. (2)* 54 (1951), pp. 38–55.

Rengaswami, Sathyanarayanan. "Classification of bowl-type translators to fully nonlinear curvature flows". In: *J. Geom. Anal.* 34.6 (2024), Paper No. 186, 23.

Rosenau, P. "On fast and super-fast diffusion". In: *Phys Rev Lett.* 74.7 (1995), pp. 1056–1059.

Sacksteder, Richard. "On hypersurfaces with no negative sectional curvatures". In: *Amer. J. Math.* 82 (1960), pp. 609–630.

Sbiti, Sammy. "On the Ricci flow of homogeneous metrics on spheres". In: *Annals of Global Analysis and Geometry* 61.3 (2022), pp. 499–517.

Shi, Wan-Xiong. "Deforming the metric on complete Riemannian manifolds". In: *J. Differential Geom.* 30.1 (1989), pp. 223–301.

Smale, Stephen. "Generalized Poincaré's conjecture in dimensions greater than four". In: *Ann. of Math. (2)* 74 (1961), pp. 391–406.

Smith, Cyril Stanley. "Shape of metal grains". In: *Metal Interfaces*. American Society for Testing Materials, Cleveland, 1952, p. 108.

Spruck, Joel and Ling Xiao. "Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature". In: *Amer. J. Math.* 142.3 (2020), pp. 993–1015.

Tonegawa, Yoshihiro. *Brakke's mean curvature flow*. Springer Briefs in Mathematics. An introduction. Springer, Singapore, 2019, pp. xii+100.

Tool. "Parabola". In: *Lateralus*. Volcano Entertainment, 2001.

Tso (Chou), Kaising. "Deforming a hypersurface by its Gauss–Kronecker curvature". In: *Comm. Pure Appl. Math.* 38.6 (1985), pp. 867–882.

United States Air Force. *USAF Series YB-35, YB-35A, and YB-49 Aircraft, AN 01-15EAA-4*. https://commons.wikimedia.org/w/index.php?curid=151723696. Provided by the National Museum of the United States Air Force, Dayton, Ohio. Public Domain. 1948.

Unknown Illustrator. *Manuscript of Saint-Esprit*. http://initiale.irht.cnrs.fr/en/codex/1719. Foundation of the hospital of the Holy Spirit of Dijon in pictures, IRHT 062062 2. In the collection of the Carmelite Convent, Dijon, France. Public Domain. Ca. 1450.

Urbas, John. "Complete noncompact self-similar solutions of Gauss curvature flows. II. Negative powers". In: *Adv. Differential Equations* 4.3 (1999), pp. 323–346.

Wang, Xu-Jia. "Convex solutions to the mean curvature flow". In: *Ann. of Math. (2)* 173.3 (2011), pp. 1185–1239.

White, Brian. "A local regularity theorem for mean curvature flow". In: *Ann. of Math. (2)* 161.3 (2005), pp. 1487–1519.

– "The nature of singularities in mean curvature flow of mean-convex sets". In: *J. Amer. Math. Soc.* 16.1 (2003), pp. 123–138.

Widder, D. V. "Positive solutions of the heat equation". In: *Bull. Amer. Math. Soc.* 69 (1963), pp. 111–112.

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