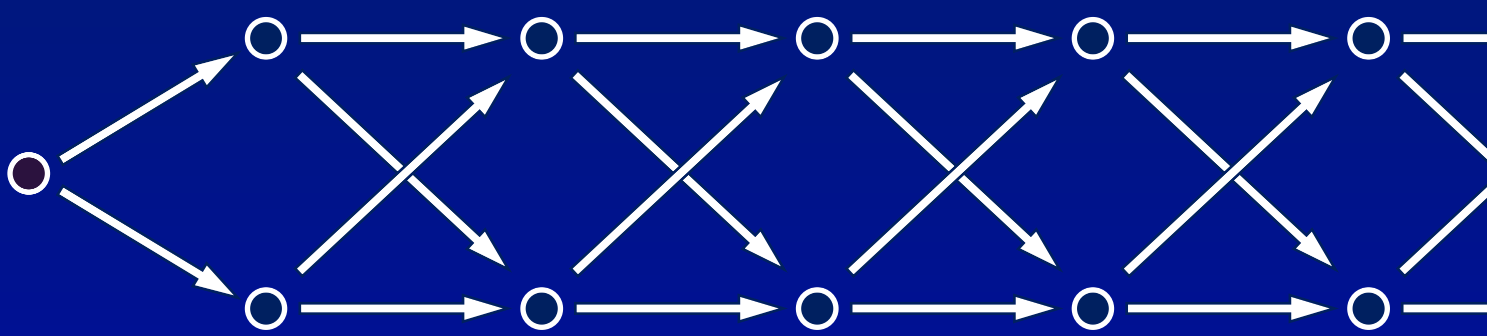
Representations of Infinite-Dimensional Lie Algebras



Dmitry Fuchs

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1

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REPRESENTATIONS OF INFINITE-DIMENSIONAL LIE ALGEBRAS

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0. Introduction

There are important differences between the theories of finite-dimensional and infinite-dimensional Lie algebras.

First, for finite-dimensional Lie algebras, there are strong classification theorems. As a result, if you encounter a finite-dimensional Lie algebra, then, by all probability, you can find full information about it (like name, commonly used notation, description of the root system and representations, etc.) in various books, not speaking of the Internet. On the contrary, if an infinite-dimensional algebra arises on your way, then chances are that it is new. By this reason, a book or a lecture course dedicated to infinite-dimensional Lie algebras is forced to be concentrated on some particular examples (which may be important by some reasons). Here we will deal mainly with Kac-Moody and Virasoro Lie algebras (although there exist other classes of infinite-dimensional Lie algebras with well developed representation theories, for example infinite limits of classical finite-dimensional Lie algebras, or Lie algebras of vector fields).

Second, there exists a very rigid correspondence between finite-dimensional Lie algebras and Lie groups. From some point of view, the theory of finite-dimensional Lie algebras has no independent value: it is just an auxiliary counterpart of the Lie groups theory. These arguments become less convenient in the infinite-dimensional case. To begin with, not every infinite-dimensional Lie algebra corresponds to an infinite-dimensional Lie group. Groups, in particular Lie groups, usually arise as groups of symmetries. But in the infinite-dimensional case, we often consider rather Lie algebra of infinitesimal symmetries than Lie group of symmetries; it is quite common in the quantum physics, for example.

1. Generalities

1.1. Main objects.

1.1.1. Lie algebras we consider. If the other is not stated, our ground field usually will be $\mathbb{C}$ (or, if you prefer, any algebraically closed field of characteristic 0). Actually, almost all Lie algebras that we will consider will be complexifications of real Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. A *representation* of $\mathfrak{g}$ (aka a module over $\mathfrak{g}$ or a $\mathfrak{g}$-module) is a vector space V with a linear map $\rho\colon \mathfrak{g} \to \operatorname{End} V$ such that $\rho([g,h]) = \rho(g)\circ\rho(h) - \rho(h)\circ\rho(g)$. When possible, we abbreviate $[\rho(g)](v)$ to gv.

Regarding the Lie algebra $\mathfrak{g}$ we always assume the existence of a vector space *decomposition* $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ into the sum of three Lie subalgebras of $\mathfrak{g}$ with the following properties: $[\mathfrak{h},\mathfrak{h}] = 0$ (that is, $\mathfrak{h}$ is commutative), $[\mathfrak{h},\mathfrak{n}_\pm] \subset \mathfrak{n}_\pm$ (in all our examples, $= \mathfrak{n}_\pm$); usually, $\dim \mathfrak{h}$ is finite; it is called the *rank* of $\mathfrak{g}$. These assumptions imply that $\mathfrak{h} \oplus \mathfrak{n}_\pm$ is a Lie subalgebra of $\mathfrak{g}$ and that the projections $\mathfrak{h} \oplus \mathfrak{n}_\pm \to \mathfrak{h}$ are Lie homomorphisms. A characteristic example: $\mathfrak{g} = \{\text{matrices}\}, \mathfrak{h} = \{\text{diagonal matrices}\}, \mathfrak{n}_\pm = \{\text{(upper and lower) triangular matrices}\}$.

Later we will impose further conditions on the decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, but for the construction below they are not needed.

1.1.2. Verma modules. Let $\lambda \in \mathfrak{h}^*$. Then there arises a one dimensional $\mathfrak{h}$-module $\mathbb{C}_\lambda = \{\mathbb{C}_\lambda = \mathbb{C}, hz = \lambda(h)\cdot z\}$. The projection $\mathfrak{h} \oplus \mathfrak{n}_+ \to \mathfrak{h}$ makes $\mathbb{C}_\lambda$ a (still one-dimensional) module over $\mathfrak{h} \oplus \mathfrak{n}_+$ ($gz = 0$ for $g \in \mathfrak{n}_+$). By definition, the Verma module (of type λ) is

$$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{h}\oplus\mathfrak{n}_+} \mathbb{C}_\lambda,$$

the induced $\mathfrak{g}$-module (explanations are given in Section 1.1.3 below).

1.1.3. The explanation of Ind.

1.1.3.1. First informal explanation of Ind. Let $\mathfrak{b} \subset \mathfrak{a}$ be a pair of Lie algebras, and let V be a $\mathfrak{b}$-module. How to extend this $\mathfrak{b}$-module structure to a $\mathfrak{a}$-module structure? Indeed, if $v \in V$ and $g \in \mathfrak{a} - \mathfrak{b}$, then where is gv? Mathematicians know how to deal with this situation: just introduce the notation gv and append this gv to V. (This reminds a known Riussian children's poem: *"it is very easy to build a house: just draw it and live in it!"*)

But there arises a difficulty. It is possible that, say, $g, g' \in \mathfrak{a} - \mathfrak{b}$, but $g' - g \in \mathfrak{b}$. So, the difference between appended gv and $g'v$ should be equal to the existing $(g' - g)v$. Dealing with this requires some reasonable factorization. Then, certainly, we will have to apply elements of $\mathfrak{a} - \mathfrak{b}$ to these appended gv, and so on. Fortunately, there exists a construction in algebra, which provides a formalization of the operation of inducing.

1.1.3.2. The enveloping algebra of a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra. Consider the tensor algebra $T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots = \bigoplus_{n=0}^{\infty} \otimes^n \mathfrak{g}$ (with the multiplication $\otimes$; in the definition of $T(\mathfrak{g})$, $\mathfrak{g}$ participates as a vector space, not as a Lie algebra). Let $I \subset T(\mathfrak{g})$ be a two-sided ideal generated by all $g' \otimes g'' - g'' \otimes g' - [g', g''] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset T(\mathfrak{g})$. The quotient $U(\mathfrak{g}) = T(\mathfrak{g})/I$ is the *universal enveloping algebra* of $\mathfrak{g}$. It is a unitary associative algebra.

A matter of notations: the image of $g_1 \otimes \ldots \otimes g_n$ in $U(\mathfrak{g})$ is denoted as $g_1 \ldots g_n$. Obviously, if $\mathfrak{g} \neq 0$, then $\dim U(\mathfrak{g}) = \infty$. There is a way of describing **a basis** in $U(\mathfrak{g})$.

Proposition 1.1. *Let $g_1, g_2, g_3, \ldots$ be a (finite or infinite) ordered basis in $\mathfrak{g}$. Then monomials*

$$g_{i_1} g_{i_2} \ldots g_{i_n} \ (n \geq 0, i_1 \leq i_2 \leq \ldots \leq i_n) \tag{1}$$

form a basis in $U(\mathfrak{g})$.

Proof. The fact that monomials (1) span $U(\mathfrak{g})$, is obvious: we can reorders letters in a monomial $g_{j_1} \ldots g_{j_n}$ at the expense of shorter monomials:

$$\underbrace{g_{j_1} \cdots}\, g_{j_k} g_{j_{k+1}} \underbrace{\cdots g_{j_n}} = \underbrace{g_{j_1} \cdots}\, g_{j_{k+1}} g_{j_k} \underbrace{\cdots g_{j_n}} + \underbrace{g_{j_1} \cdots}\, [g_{j_k}, g_{j_{k+1}}] \underbrace{\cdots g_{j_n}}.$$

The fact that monomials (1) are linearly independent may seem obvious, but actually it is not. This fact is called the Poincaré-Birghoff-Witt theorem (briefly, PBW), and its one-page long proof is contained in all major textbooks on Lie theory. I will not prove it here but will often use it (usually, implicitly).

PBW has a basis-free statement. Let $U_n(\mathfrak{g}) \subset U(\mathfrak{g})$ be a subspace spanned by monomials of length $\leq n$. Then

$$\mathbb{C} \cdot 1 = U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset U_2(\mathfrak{g}) \subset \ldots \subset U(\mathfrak{g})$$

is a multiplicative ($U_i(\mathfrak{g}) U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g})$) filtration. Consider the map $\sigma_n \colon S^n(\mathfrak{g}) \to U_n(\mathfrak{g}) \subset U(\mathfrak{g})$, $\sigma_n(g_1 g_2 \ldots g_n) = \mathrm{Symm}(g_1 g_2 \ldots g_n)$.

Proposition 1.2. (PBW). *The composition*

$$S^n(\mathfrak{g}) \xrightarrow{\sigma_n} U_n(\mathbf{g}) \xrightarrow{\text{proj.}} U_n(\mathbf{g})/U_{n-1}(\mathbf{g})$$

is an isomorphism.

Corollary 1.3. *$U(\mathfrak{g})$ has no zero divisors.*

Proof. If $0 \neq a, 0 \neq b \in U(\mathfrak{g})$ and p, q are maximal integers with $a \in U_p(\mathfrak{g}), b \in U_q(\mathfrak{g})$, then the image of ab in $S^{p+q}\mathfrak{g} = U_{p+q}(\mathfrak{g})/U_{p+q-1}(\mathfrak{g})$ is the product of the (non-zero) images of a and b in $S^p(\mathfrak{g})$ and $S^q(\mathfrak{g})$ and hence $ab \neq 0$.

Proposition 1.4. (Tautology) *A $\mathfrak{g}$-module is the same as a (left) $U(\mathfrak{g})$-module in the sense of usual module theory.*

1.1.3.3. A rigorous definition of Ind. The operation of inducing becomes a habitual operation of a ring extension (the most known example of which is the complexification of real vector spaces):

$$\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V = U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V$$

(here V is a left $U(\mathfrak{b})$-module, $U(\mathfrak{a})$ is a two-sided $U(\mathfrak{a})$-module, hence a left $U(\mathfrak{a})$-module and a right $U(\mathfrak{b})$-module; thus, $U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V$ is a left $U(\mathfrak{a})$-module).

There exists a "dual" operation of "co-inducing,"

$$\operatorname{Coind}_{\mathfrak{a}}^{\mathfrak{b}} V = \operatorname{Hom}_{U(\mathfrak{a})}(U(\mathfrak{b}), V),$$

which turns a *right* $\mathfrak{b}$-module into a *left* $\mathfrak{a}$-module. It is unlikely that we will ever need this operation.

1.1.4. A deviation for the algebra fans. Those who do not belong to the community described in the title may skip this section.

There exist less explicit, but more spectacular, descriptions of universal enveloping algebras and, separately, the operations of inducing and co-inducing in the language of the category theory. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F: \mathcal{D} \to \mathcal{C}$ and $G: \mathcal{C} \to \mathcal{D}$ be functors. Suppose that for every pair of objects $A \in \operatorname{Ob}\mathcal{C})$, $B \in \operatorname{Ob}(\mathcal{D})$ there exist a bijection

$$\iota_{A,B}: \mathcal{C}(FB, A) \to \mathcal{D}(B, GA)$$

natural with respect to A and B (that is, diagrams induced by morphisms of $\mathcal{C}$ and $\mathcal{D}$ are commutative). Then F is called *left adjoint* to G and G is called *right adjoint* to F.

Many famous mathematical constructions (some of them are not obvious) can be described very briefly in the language of adjoint functors. For example, completion of metric spaces is just a functor from the category $\mathcal{M}$ of metric spaces into the category of complete metric spaces $\widehat{\mathcal{M}}$ is just the left adjoint to the embedding $\widehat{\mathcal{M}} \to \mathcal{M}$.

Let $\mathcal{L}ie$ be the category of Lie algebras and $\mathcal{A}ss$ be the category of (unital) associative algebras. There is a natural functor $\mathcal{A}ss \to \mathcal{L}ie$, which assigns to an associative algebra the Lie algebra with the same space and the operation $[A, B] = AB - BA$. The fact is that this functor possesses a right adjoint, and this right adjoint assigns to a Lie algebra its universal enveloping algebra (this is Exercise 6.2).

Another popular notion of the category theory (in some sense, more general than that of adjointness) is the notion of an *initial* (a *terminal*) object of a category. For a fixed $\mathfrak{b}$-module V consider the category of $\mathfrak{a}$-modules, which contain V as a $\mathfrak{b}$-submodule. Then $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V$ is an initial object of this category. Similarly $\operatorname{Coind}_{\mathfrak{a}}^{\mathfrak{b}} V$ is a terminal object of the category of $\mathfrak{a}$-modules equipped with a $\mathfrak{b}$-projection onto V.

1.2. More about Verma modules.

1.2.1. The structure of Verma modules. As a $U(\mathfrak{n}_-)$-module, $M(\lambda)$ is a free module with one generator, which we denote as v_λ or simply v and call a *vacuum vector*.

Notice that $\mathbb{C}v$ is the $\mathfrak{n}_-$-module $\mathbb{C}_\lambda$ (see Section 1.1.2), from which $M(\lambda)$ was obtained by inducing; it has to be contained in $M(\lambda)$ by the construction of inducing (Sections 1.1.3 and 1.1.4). As a $\mathfrak{g}$-module, $M(\lambda)$ has additional properties:

$$hv = \lambda(h) \cdot v \text{ for } h \in \mathfrak{h},\ gv = 0 \text{ for } g \in \mathfrak{n}_+.$$

These properties determine $M(\lambda)$ as a $\mathfrak{g}$-module. Indeed, to compute $g(g_1 \dots g_n v)$ (where $g \in \mathfrak{g}$ and $g_1, \dots g_n \in \mathfrak{n}_-$), we move, using the commutation relations, g from left to the right, and when it reaches v we apply it to v, using the rules above (see the example in Section 1.2.3 below).

An important remark: a non-zero module homomorphism $\varphi\colon M(\mu) \to M(\lambda)$ must be an embedding. Indeed, if $\varphi(v_\mu) = \alpha v_\lambda$, $\alpha \in U(\mathfrak{n}_-)$, then $\varphi(\beta v_\mu) = \beta\alpha v_\lambda$ which is not zero, if $\alpha \neq 0$, $\beta \neq 0$ since $U(\mathfrak{n}_-)$ has no zero divisors (see Section 1.1.3.2).

1.2.2. Why Verma modules are important. Let V be an arbitrary representation of $\mathfrak{g}$. A non-zero vector $w \in V$ is called a *singular vector of type* $\lambda \in \mathfrak{h}^*$, if $\mathfrak{n}_+ w = 0$ and $hw = \lambda(h) \cdot w$ for every $h \in \mathfrak{h}$. Example: v_λ is a singular vector of M_λ of type λ. Natural conditions which we are going to impose on $\mathfrak{g}$ and V will guarantee the existence of singular vector(s) in V (see Proposition 1.8). For a singular vector $w \in V$ of type λ, there arises a module homomorphism $M(\lambda) \to V$, $v \mapsto w$. The image of this homomorphism is a submodule of V, and if V is irreducible, then V becomes an image of $M(\lambda)$, hence the quotient of $M(\lambda)$ over the maximal proper submodule of $M(\lambda)$. In particular, if $M(\lambda)$ is irreducible, then $V = M(\lambda)$. This gives a classification of all irreducible representations of $\mathfrak{g}$ (satisfying the natural conditions promised above); these irreducible representations appear labeled by $\mathfrak{h}^*$ (these labels are called *highest weights*).

This shows the importance of two problems.

Problem (1). For which λ is $M(\lambda)$ reducible?

Problem (2). If $M(\lambda)$ is reducible, then what is its maximal submodule $\neq M(\lambda)$?

Problem (1) has been solved for all major cases, and we will discuss the solution in these lectures. Problem (2) is solved only partially (this part being considerable). We will also discuss some results of this kind.

1.2.3. An example; $\mathfrak{sl}(2)$ as always. The Lie algebra $\mathfrak{sl}(2)$ is 3-dimensional and is spanned by

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};$$

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h;$$

$$\mathfrak{sl}(2) = (\mathbb{C} \cdot f) \oplus (\mathbb{C} \cdot h) \oplus (\mathbb{C} \cdot e) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Let $\lambda \in \mathfrak{h}^* = \mathbb{C}$ (we identify $\lambda \in \mathfrak{h}^*$ with $\lambda(h) \in \mathbb{C}$). The Verma module $M(\lambda)$ has a basis

$\{f^n v \mid n = 0, 1, 2, \ldots\}$. Action of $\mathfrak{sl}(2)$:

$$\begin{aligned}
f(f^n v) &= \underline{f^{n+1}v};\\
h(f^n v) &= hff^{n-1}v = fhf^{n-1}v - 2f^n v = f^2hf^{n-2}v - 4f^n v = \ldots\\
&= f^n hv - 2nf^n v = \underline{(\lambda - 2n)f^n v};\\
e(f^n)v &= fef^{n-1}v + hf^{n-1}v = f^2ef^{n-2}v + fhf^{n-2}v + hf^{n-1}v = \ldots\\
&= f^n ev + f^{n-1}hv + f^{n-2}hfv + \ldots + hf^{n-1}v\\
&= [\lambda + (\lambda - 2) + (\lambda - 4) + \ldots + (\lambda - 2(n-1))]f^{n-1}v\\
&= [n\lambda - n(n-1)]f^{n-1}v = \underline{n(\lambda - (n-1))f^{n-1}v}.
\end{aligned}$$

A submodule L of $M(\lambda)$ must be also an $\mathfrak{h}$-submodule. Since our basis in $M(\lambda)$ consists of eigenvectors of h, L must be spanned by a subset of the basis. Since L must be invariant with respect to f, it must contain, together with a $f^n v$, all $f^m v$ with $m > n$. Thus, L is spanned by $f^n v, f^{n+1}v, f^{n+2}v, \ldots$. The case $n = 0$ is not interesting (then $L = M(\lambda)$). Otherwise, $f^{n-1}v \notin L$, and $ef^n v = n(\lambda - (n-1))f^{n-1}v$ must be zero, that is, $\lambda = n - 1 \Rightarrow \lambda \in \mathbb{Z}_{\geq 0}$.

We see that the Verma module $M(\lambda)$ is reducible if and only if λ is a non-negative integer. Hence, irreducible $\mathfrak{sl}(2)$-modules containing singular vectors are: (1) Verma modules $M(\lambda)$ with $\lambda \notin \mathbf{Z}_{\geq 0}$; (2) quotients $M(n-1)/L$ where n is a positive integer and L is the submodule of $M(n-1)$ constructed above. The latter is n-dimensional, has a basis $v, fv, \ldots, f^{n-1}v$ such that $ev = 0$, $f(f^{n-1}v) = 0$.

It is not hard to prove that every finite-dimensional $\mathfrak{sl}(2)$-module contains a singular vector. (Indeed, if V is a finite-dimensional $\mathfrak{sl}(2)$-module, then $h\colon V \to V$ has an eigenvector, let it be v, $hv = \lambda v$. Then $hev = ehv + 2ev = (\lambda + 2)ev$, and, similarly, $h(e^2v) = (\lambda + 4)e^2v, h(e^3v) = (\lambda + 6)e^2v, \ldots$. Since $h\colon V \to V$ has finitely many eigenvalues, some $e^k v$ should be zero, and the last non-zero $e^k v$ is a singular vector.) Thus, our classification covers, in particular, all finite-dimensional modules, and we get a well known result: for every dimension, there is precisely one isomorphism class of irreducible representations of $\mathfrak{sl}(2)$.

Notice also that there are $\mathfrak{sl}(2)$-modules without singular vectors. The easiest example can be obtained from the formulas above by adding a constant to the coefficient at $ef^n v$. For $\lambda, \mu \in \mathbb{C}$, let $K(\lambda, \mu)$ be the vector space with the basis v_n, $n \in \mathbb{Z}$ and the structure of an $\mathfrak{sl}(2)$-module defined by the formulas

$$\begin{aligned}
hv_n &= (\lambda + 2n)v_n,\\
fv_n &= v_{n-1},\\
ev_n &= (-n(\lambda + n + 1) + \mu)v_{n+1}
\end{aligned}$$

It is possible to make this formulas more symmetric with respect to the parameters by a notation change. Namely, let $J(\alpha, \beta)$ be the vector space with a basis v_n, $n \in \mathbb{Z}$ with

$$\begin{aligned}
hv_n &= (2n + \alpha - \beta)v_n,\\
ev_n &= (\alpha + n)v_{n+1},\\
fv_n &= (\beta - n)v_{n-1}.
\end{aligned}$$

If α is not an integer, then the module $J(\alpha, \beta)$ does not contain singular vectors; if neither α, nor β is an integer, then the $\mathfrak{sl}(2)$-module $J(\alpha, \beta)$ is irreducible. For more properties of the $\mathfrak{sl}(2)$-modules $J(\alpha, \beta)$ and $K(\lambda, \mu)$, see Exercise 6.4.

1.2.4. Final remarks. (1) It is not hard to extend these results to $\mathfrak{sl}(3)$, $\mathfrak{sl}(4)$, etc. We will do much more.

(2) We will show (under some natural conditions imposed on $\mathfrak{g}$) that the module $M(\lambda)$ is reducible if and only if it contains a singular vector not in $\mathbb{C}v$ (equivalently: not of type λ) – see Proposition 1.8 in Section 1.3.3.

1.3. Roots and weights.

It is time to formulate further restrictions on Lie algebra and modules considered.

1.3.1. Roots. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be its finite-dimensional commutative subalgebra (that is, $[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$).

For $0 \neq \alpha \in \mathfrak{h}^*$, we put

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \mid [h, g] = \alpha(h) \cdot g \text{ for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, then we call α a *root* (of $\mathfrak{g}$), call $\mathfrak{g}_\alpha$ a root space, and call non-zero vectors from $\mathfrak{g}_\alpha$ root vectors. The dimension of the space $\mathfrak{g}_\alpha$ is called the *multiplicity* of the root α and is denoted as $\operatorname{mult} \alpha$. The set of all roots is denoted by Δ.

Obviously, if $\alpha, \alpha' \in \Delta$ and $g \in \mathfrak{g}_\alpha, g' \in \mathfrak{g}_{\alpha'}$, then either $[g, g'] = 0$, or $\alpha + \alpha' \in \Delta$ and $[g, g'] \in \mathfrak{g}_{\alpha+\alpha'}$. (Indeed, $[h, [g, g']] = [[h, g], g'] + [g, [h, g']] = (\alpha(h) + \alpha'(h))[g, g']$.)

Our main assumption ("diagonalizability of $\mathfrak{h}$") is that $\dim \mathfrak{g}_\alpha < \infty$ for all $\alpha \in \Delta$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{2}$$

Remark. It follows from the condition (2) that $\mathfrak{h}$ is a *maximal* commutative subalgebra of $\mathfrak{g}$, that is, it is not contained in a bigger commutative subalgebra. Indeed, let $[\hat{h}, \mathfrak{h}] = 0$ and let $\hat{h}$ involve a non-zero component in some $\mathfrak{g}_\alpha$. If $\alpha(h) \neq 0$, then $[\hat{h}, h]$ cannot be 0.

Let $\mathfrak{c}$ be the center of $\mathfrak{g}$ (that is, $\{c \in \mathfrak{g} \mid [c, g] = 0 \text{ for all } g \in \mathfrak{g}\}$). Two things are obvious: first, $\mathfrak{c} \subset \mathfrak{h}$ (follows immediately from (2)); second, any root is zero on $\mathfrak{c}$. In other words, roots are elements of $(\mathfrak{h}/\mathfrak{c})^* \subset \mathfrak{h}^*$. We put $\dim \mathfrak{h} - \dim \mathfrak{c} = n$.

A further assumption says that every $\mathfrak{g}_\alpha$ is contained either in $\mathfrak{n}_+$ or in $\mathfrak{n}_+$. Accordingly, the root α is called *positive* or *negative*, and the set of positive (negative) roots is denoted as Δ_+ (Δ_-). Thus

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha.$$

The two sets Δ_+ and Δ_- are supposed to be *symmetric* to each other: $\{\alpha \in \Delta_+\} \Longleftrightarrow \{-\alpha \in \Delta_-\}$.

A positive root α is called *simple*, if it is not the sum of two positive roots. In all our examples below, there will exist $n = \operatorname{rank} \mathfrak{g}$ simple positive roots, $\alpha_1, \dots, \alpha_n$, and every positive (negative) root is a positive (negative) integral linear combination of simple positive roots.

We will also use notation $\widetilde{\Delta}$ for the set of roots *with multiplicities*, that is, every root α appears in $\widetilde{\Delta}$ $\dim \mathfrak{g}_\alpha$ times. We also will use notations $\widetilde{\Delta}_+, \widetilde{\Delta}_-$ in a similar sense.

More notations: Λ (Λ_+, Λ_-) is the set of all (all non-negative, all non-positive) integral linear combinations of $\alpha_1, \ldots, \alpha_n$.

Obviously, $\mathfrak{c} = \bigcap_{i=1}^{n} \operatorname{Ker} \alpha_i$.

EXAMPLE. Let $\mathfrak{g} = \mathfrak{gl}(n+1) = \{$all complex $(n+1) \times (n+1)$ matrices$\}$ and $\mathfrak{h} = \{$diagonal matrices$\}$. Then $\mathfrak{c} = \{$scalar matrices$\}$ and the notation n matches the same notation above. The roots are α_{ij}, $(i \neq j)$, $\alpha_{ij}(\operatorname{diag}(\lambda_1, \ldots, \lambda_{n+1})) = \lambda_i - \lambda_j$. The space $\mathfrak{g}_{\alpha_{ij}}$ is one-dimensional and consists of all matrices with all entries, except (possibly) the (ij)-entry, are zeroes. We declare the roots $\alpha_i = \alpha_{i,i+1}$, $i = 1, \ldots, n$ simple. Then

$$\alpha_{ij} = \begin{cases} \alpha_i + \ldots + \alpha_{j-1}, & \text{if } i < j \\ -\alpha_j - \ldots - \alpha_{i-1}, & \text{if } i > j. \end{cases}$$

Thus, positive (negative) roots are α_{ij} with $i < j$ $(i > j)$ and $\mathfrak{n}_+$ $(\mathfrak{n}_-)$ is the space of strictly upper (lower) triangular matrices.

1.3.2. Weights. Let $\mathfrak{g}$ be as above and A be a $\mathfrak{g}$-module. For a $\beta \in \mathfrak{h}^*$, we set

$$A_\beta = \{a \in A \mid ha = \beta(h) \cdot a \text{ for all } h \in \mathfrak{h}\},$$

and if $A_\beta \neq 0$, we call β a *weight* of A; the spaces A_β are called *weight spaces*. (Thus, roots of $\mathfrak{g}$ and zero are weights of the *adjoint representation* of $\mathfrak{g}$ in itself.) We will usually assume that $\dim A_\beta < \infty$ for all β and that A satisfies the $\mathfrak{h}$-*diagonalizbility condition*,

$$A = \bigoplus_{\beta \in \{\text{weights}\}} A_\beta. \tag{3}$$

PROPOSITION 1.5. *If a $\mathfrak{g}$-module A satisfies the diagonalizibity condition* (3)*, then every submodule B of A satisfies this condition. More exactly: $B = \bigoplus_\beta (B \cap A_\beta)$.*

Proof. Let $b \in B$. Then $b = a_1 + \ldots + a_k$ where $a_i \in A_{\beta_i}$, all β_i are different. Choose an $h \in \mathfrak{h}$ such that all $\beta_i(h)$ (with $\beta_i \neq 0$) are different from 0. Then for all N,

$$h^N b = h^N a_1 + \ldots + h^N a_k = \beta_1(h)^N a_1 + \ldots + \beta_k(h)^k a_k \in B,$$

which implies $a_i \in B$ for all i (except, possibly, a_i with $a_i \in A_0$, but this a_i is also in B, because $a_1 + \ldots + a_k \in B$).

PROPOSITION 1.6. *If $a \in A_\beta$ and $g \in \mathfrak{g}_\alpha$, then $ga \in A_{\beta+\alpha}$.*

Proof. For $h \in \mathfrak{h}$,

$$h(ga) = g(ha) + [h,g]a = g(\beta(h)a) + (\alpha(h)g)(a) = (\beta(h) + \alpha(h))ga.$$

COROLLARY 1.7. *The set of weights of a Verma module $M(\lambda)$, $\lambda \in \mathfrak{h}^*$ is $\lambda + \Lambda_-$. Moreover, $M(\lambda)_\lambda = \mathbb{C}v_\lambda$.*

Proposition shows that for any weight β of A, the sum $B = \bigoplus_{\gamma\in\beta+\Lambda} A_\gamma$ is a (non-zero) submodule of A. Thus, if A is irreducible then B should be equal to A, which means that all the weights of A are contained in some lattice $\beta + \Lambda \subset \mathfrak{h}^*$.

1.3.3. Modules virtually nilpotent over $\mathfrak{n}_+$. A $\mathfrak{g}$-module A is called virtually nilpotent over $\mathfrak{n}_+$, if for every $a \in A$ there exists a $k \geq 0$ such that $g_1 \dots g_k a = 0$ for all $g_1, \dots, g_k \in \mathfrak{n}_+$. For example, finite-dimensional modules are virtually nilpotent over $\mathfrak{n}_+$.

PROPOSITION 1.8. *Verma modules are virtually nilpotent over $\mathfrak{n}_+$.*

Proof. This follows from Corollary 1.7. Indeed, for a $\mu \in \Lambda_-$, there exists a k such that $\mu + \lambda_1 + \dots + \lambda_k \notin \Lambda_-$ for any non-zero $\lambda_1, \dots, \lambda_k \in \Lambda_+$. Hence, if $a \in M(\lambda)_{\lambda+\mu}$, then $g_1 \dots g_k a = 0$ for any $g_1, \dots, g_k \in \mathfrak{n}_+$.

PROPOSITION 1.9. *Let A be a virtually nilpotent over $\mathfrak{n}_+$ $\mathfrak{g}$-module with at least one non-zero weight space A_β (for example, satisfying the diagonalizibility condition).Then A contains a singular vector. Moreover, if $A_\beta \neq 0$, then there is a singular vector of type $\beta + \alpha$ with $\alpha \in \Lambda_+$.*

Proof. Take a non-zero $a \in A_\beta$ and choose a $k \geq 0$ as in the definition of the virtual nilpotency. We choose this k minimal possible, so if $k > 0$, then there exist $g_1, \dots, g_{k-1} \in \mathfrak{n}_+$ such that $g_1 \dots g_{k-1} a \neq 0$ (if $k = 0$, then a is a singular vector itself). Without loss of generality, we may assume that every g_i $(1 \leq i \leq k-1)$ belongs to some $\mathfrak{g}_{\gamma_i}$ with $\gamma_i \in \Delta_+$. Then $b = g_1 \dots g_{k-1} a \in A_{\beta+\alpha}$, $\alpha = \gamma_1 + \dots + \gamma_{k-1} \in \Lambda_+$ and $gb = 0$ for every $g \in \mathfrak{n}_+$. Hence, b is a singular vector of type $\beta + \alpha$.

Now, let us prove a proposition promised in Section 1.2.4.

PROPOSITION 1.10. *A Verma module $M(\lambda)$ is reducible if and only if it contains a singular vector not in $\mathbb{C}v_\lambda$.*

Proof. If $M(\lambda)$ contains a singular vector w of type $\neq \lambda$, that is of type $\lambda + \alpha$, $0 \neq \alpha \in \Lambda_-$, then there arises a non-zero homomorphism (hence an embedding, this is not important now) $M(\lambda + \alpha) \to m(\lambda)$ whose image is a non-zero submodule of $M(\lambda)$ which is contained contained in $\bigoplus_{\beta\in\lambda+\alpha+\Lambda_-} M(\lambda)_\beta$ and hence does not contain v_λ; hence $M(\lambda)$ is reducible. Conversely, if $M(\lambda)$ contains a proper submodule L, then L must be virtually nilpotent over $\mathfrak{n}_+$ (since $M(\lambda)$ is) and hence contain a singular vector. This singular vector cannot be proportional to v_λ, because v_λ generates the whole $M(\lambda)$.

2. Kac-Moody algebras, definition, examples, and rough classification

2.1. Definition.

There are several more or less equivalent definitions of Kac-Moody algebras and several possible levels of generality. Our definition is close to that from the original work of Kac [1], and our level of generality is somewhat above average. We begin with a "simplified" definition, which will be slightly modified below, in Section 2.3.2. We will show there that this modification will almost not affect the material preceding it.

2.1.1. Generators and relations. We assume fixed some $n \times n$ matrix $A = \|a_{ij}\|$ which will be called the *Cartan matrix*. The diagonal entries of A are all equal to 2, all the non-diagonal entries are non-positive integers. The matrix will be assumed *irreducible* in the sense that it cannot be made block-diagonal (with non-zero blocks) by applying the same permutation to rows and columns. Also we suppose that the matrix is *symmetrizable* in the sense that there exist non-singular integral diagonal matrix D such that $A = DA^{\mathrm{sym}}$ where the matrix $A^{\mathrm{sym}} = \|a_{ij}^{\mathrm{sym}}\|, a_{ij}^{\mathrm{sym}} = \dfrac{a_{ij}}{d_i}$ is symmetric. In other words, there are non-zero integers $d_1, \dots, d_n$ such that $d_j a_{ij} = d_i a_{ji}$ for all i, j. An additional assumption that d_i are positive integers with $\gcd(d_1, \dots, d_n) = 1$ (which we impose) makes these numbers unique. The symmetrizability condition implies that the entries a_{ij}, a_{ji} are zeroes or non-zeroes simultaneously.

Our Lie algebra will have $3n$ (linearly independent) generators

$$e_1, \dots, e_n; h_1, \dots h_n; f_1, \dots, f_n$$

(these notations may remind of our description of $\mathfrak{sl}(2)$; no wonder, the whole construction below is a sort of generalization of that description), satisfying the following relations.

(1) $[h_i, h_j] = 0$;
(2) $[h_i, e_j] = a_{ij} e_j$;
(3) $[h_i, f_j] = -a_{ij} f_j$;
(4) $[e_i, f_j] = \begin{cases} 0, & \text{if } i \neq j, \\ h_i, & \text{if } i = j; \end{cases}$
(5) $\underbrace{[e_i, [\dots, [e_i}_{k}, e_j] \dots] = 0$, if $k > -a_{ij}$;
(6) similar to (5) for f's.

The Lie algebra $\mathfrak{g}$ is called a Kac-Moody algebra and is sometimes denoted as $\mathfrak{g}(A)$. The number n is called the *rank* of $\mathfrak{g}(A)$.

REMARK. In the most general definition of Kac-Moody algebras the Cartan matrix is just arbitrary complex $n \times n$ matrix; sometimes it is supposed to be symmetrizable. Main difficulty arising in this general approach is that relations (5) and (6) must be reformulated in a form which is far from being explicit.

2.1.2. Decomposition $\mathfrak{g}(\mathbf{A}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Subalgebras of $\mathfrak{g}(A)$ generated, respectively, by $f_1, \dots, f_n$, by $h_1, \dots, h_n$, and by $e_1, \dots, e_n$ are denoted by $\mathfrak{n}_-, \mathfrak{h}$, and $\mathfrak{n}+$.

PROPOSITION 2.1. *The Lie algebra $\mathfrak{g}(A)$ is the direct sum of Lie subalgebras $\mathfrak{n}_-, \mathfrak{h}$, and $\mathfrak{n}_+$, and all the properties stated in Section 1.1.1 hold.* (Certainly, $\mathfrak{h}$ is just the n-dimensional commutative Lie algebra spanned by $h_1, \dots, h_n$.)

Proof. It follows from the Jacobi identity that if a Lie algebra $\mathfrak{g}$ is *generated* by a system $\{g_1, g_2, \dots, g_n\}$ then it is spanned by monomials of the form $[g_{i_1}, [g_{i_2}, [\dots, g_{i_k}] \dots]]]$, for which we use a shorter notation $[g_{i_1}, g_{i_2}, \dots, g_{i_k}]$ (for example, $[[a, b], [c, d]] = [a, b, c, d] - [b, a, c, d]$. The Jacobi identity takes the form

$$[h, g_1, g_2, \dots, g_n] = [[h, g_1], g_2, \dots, g_n] + [g_1, [h, g_2], g_3, \dots, g_n] + \dots + [g_1, \dots, g_{n-1}, [h, g_n]].$$

From this identity and relations (1) - (4) above, we obtain

$$[h_i, e_{j_1}, \dots, e_{j_k}] = (a_{ij_1} + \dots + a_{ij_k})[e_{j_1}, \dots, e_{j_k}],$$

similar for $[h_i, f_{j_1}, \ldots, f_{j_k}]$,

$$[f_i, e_{j_1}, \ldots, e_{j_k}] = \delta_{ij_1}[h_i, e_{j_2}, \ldots, e_{j_k}] + \delta_{ij_2}[e_{j_1}, h_i, e_{j_3}, \ldots, e_{j_k}] + \ldots + \delta_{ij_k}[e_{j_1}, \ldots, e_{j_{k-1}, h_i}],$$

and similar for $[e_i, f_{j_1}, \ldots, f_{j_k}]$. Using these equalities, we can transform (moving from the right to the left) any monomial $[g_1, \ldots, g_k]$, where every g_i is one of the generators $e_1, \ldots, e_n, h_1, \ldots h_n, f_1, \ldots, f_n$ to the sum of elements of $\mathfrak{n}_-$, $\mathfrak{h}$ and $\mathfrak{n}_+$.

2.1.3. Center of $\mathfrak{g}$(A). We do not assume that the matrix A is non-degenerate; moreover, it will be degenerate in the most important for us cases. In the degenerate case, the algebra $\mathfrak{g}(A)$ has a non-zero center $\mathfrak{c}(A)$: it consists of linear combinations $\sum_i k_i h_i$ for which $\sum_i k_i a_{ij} = 0$ for all j.

2.2 Examples.

2.2.1. $\mathfrak{sl}$(3). Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Since the matrix A is non-degenerate, the Lie algebra $\mathfrak{g}(A)$ is generated by $e_1, e_2, h_1, h_2, f_1, f_2$ while $\mathfrak{n}_+$ is generated by e_1, e_2 with relations $[e_1, [e_1, e_2]] = 0$, $[e_2, [e_2, e_1]] = 0$. This implies .that $\mathfrak{n}_+$ is spanned by $e_1, e_2, [e_1, e_2]$, and hence $\dim \mathfrak{n}_+ \leq 3$; similarly, $\dim \mathfrak{n}_- \leq 3$, and hence $\dim \mathfrak{g}(A) \leq 8$. Let us show that actually $\dim \mathfrak{g}(A) = 8$ and $\mathfrak{g}(A) \cong \mathbf{sl}(3)$. For this purpose, we identify in $\mathfrak{sl}(3)$ six elements which we will denote again by $e_1, e_2, h_1, h_2, f_1, f_2$:

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$f_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

An immediate check shows that these elements present a system of generators for $\mathfrak{sl}(3)$ and satisfy all relations imposed in Section 2.1 (with A as above). Hence, if we assign to $e_1, e_2, h_1, h_2, f_1, f_2 \in \mathfrak{g}(A)$ the elements of $\mathfrak{sl}(3)$ bearing the same notations, we will obtain a homomorphism $\mathfrak{g}(A) \to \mathfrak{sl}(3)$ which is onto since it covers all the generators and is 1–1 since $\dim \mathfrak{sl}(3) = 8$.

2.2.2. $\mathfrak{so}$(5). Let $A = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$. Obviously, A is symmetrizable (with $d_1 = 1$, $d_2 = 2$). Also A is non-degenerate, hence $\mathfrak{g}(A)$ is generated by $e_1, e_2, h_1, h_2, f_1, f_2$, and $\mathfrak{n}_+$ is generated by e_1, e_2. The relations now are $[e_1, [e_1, e_2]] = 0$, $[e_2, [e_2, [e_2, e_1]]] = 0$ which implies that $\mathfrak{n}_+$ is spanned by $e_1, e_2, [e_1, e_2]$, and $[e_2, [e_2, e_1]]$ (we notice that $, [e_1, [e_2, [e_2, e_1]]] = [[e_1, e_2], [e_2, e_1]] + [e_2, [e_1, [e_2, e_1]]] = 0$). Hence, $\dim \mathfrak{n}_+ \leq 4$ and $\dim \mathfrak{g}(A) \leq 10$.

Actually, the Lie algebra $\mathfrak{g}(A)$ is isomorphic to the Lie algebra $\mathfrak{so}(5, \mathbb{C})$ of (complex) skew-symmetric 5×5 matrices. To prove this, we need to specify in $\mathfrak{so}(5, \mathbb{C})$ 6 matrices, which we again denote by $e_1, e_2, h_1, h_2, f_1, f_2$. We use the notation $D_{k,\ell}$ $(1 \leq k < \ell \leq 5)$ for the skew symmetric 5×5 matrix $\|a_{ij}\|$ with $a_{k\ell} = -a_{\ell k} = 1$ and $a_{ij} = 0$ if $\{i, j\} \neq \{k, \ell\}$. We put

$$e_1 = D_{15} + iD_{25},\ e_2 = D_{35} + iD_{45},\ h_1 = 2iD_{12},\ h_2 = 2iD_{34},\ f_1 = D_{15} - iD_{25},\ f_2 = D_{35} - iD_{45}.$$

It is easy to check that these $e_1, e_2, h_1, h_2, f_1, f_2$ generate $\mathfrak{so}(5)$ and satisfy all the relations of $\mathfrak{g}(A)$. This provides a homomorphism of $\mathfrak{g}(A)$ onto $\mathfrak{so}(5)$ and this is an isomorphism, since $\dim \mathfrak{g}(A) \leq 10 = \dim \mathfrak{so}(5)$. (We leave the details to the reader as Exercise 6.6.)

2.2.3. $\mathbf{G_2}$. Let $A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. Again, A is symmetrizable (with $d_1 = 1$, $d_2 = 3$) and non-degenerate, $\mathfrak{g}(A)$ is generated by $e_1, e_2, h_1, h_2, f_1, f_2$, and $\mathfrak{n}_+$ is generated by e_1, e_2 with the relations $[e_1, [e_1, e_2]] = 0$, $[e_2, [e_2, [e_2, [e_2, e_1]]]] = 0$. It is not hard to show that $\mathfrak{n}_+$ is spanned in this case by 6 vectors:

$$e_1, e_2, [e_1, e_2], [e_2, [e_2, e_1]], [e_2, [e_2, [e_2, e_1]]], [e_1, [e_2, [e_2, [e_2, e_1]]]].$$

Hence, $\dim \mathfrak{n}_+ \leq 6$ and $\dim \mathfrak{g}(A) \leq 14$.

It is not hard to prove (Exercise 6.7) that actually $\dim \mathfrak{g}(A) = 14$ and $\mathfrak{g}(A)$ is the exceptional Lie algebra G_2. To do this, it is even not necessary to know what G_2 is. Using elementary tools from the Lie theory one can prove that there exists (up to an isomorphism) only four simple (non-commutative) complex Lie algebras of dimension ≤ 14: $\mathfrak{sl}(2), \mathfrak{sl}(3), \mathfrak{so}(5)$, and one more Lie algebra of dimension 14 (which is denoted as G_2). Hence this 14-dimensional algebra is $\mathfrak{g}(A)$.

2.2.4. $\mathbf{A_1^1}$. Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. This matrix is degenerate, thus $\mathfrak{g}(A)$ has a center, and this center is 1-dimensional and spanned by $h_1 + h_2$. Hence, the Lie algebra $\mathfrak{g}(A)/\mathfrak{c}(A)$ is generated by e_1, e_2, h, f_1, f_2 where h is represented by h_1 or $-h_2$. For the reader's convenience sake, we list all the defining relations for these generators:

$$[e_1, f_1] = h,\ [e_2, f_2] = -h,\ [e_1, f_2] = [e_2, f_1] = 0,$$

$$[h, e_1] = 2e_1,\ [h, e_2] = -2e_2,\ [h, f_1] = -2f_1, [h, f_2] = 2f_2,$$

$$[e_1, [e_1, [e_1, e_2]]] = 0, [e_2, [e_2, [e_2, e_1]]] = 0, [f_1, [f_1, [f_1, f_2]]] = 0, [f_2, [f_2, [f_2, f_1]]] = 0$$

We will construct an infinite-dimensional Lie algebra which is isomorphic to $\mathfrak{g}(A)/\mathfrak{c}(A)$. This is the Lie algebra $\mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$; the elements of this Lie algebra are 2×2 matrices $\begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & -p_{11}(t) \end{bmatrix}$ whose entries are complex polynomials in one variable t with negative powers allowed (differently, polynomial functions in $\mathbb{C}^* = \mathbb{C} - \{0\}$). We will single out 5 elements of this Lie algebra:

$$e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, f_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & t^{-1} \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that these matrices generate $\mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$ and satisfy all the above relations. This gives a homomorphism $\mathfrak{g}_{00}(A) \to \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}])$ which is onto since it cover all generators.

Actually, this homomorphism is an isomorphism (it is Exercise 6.8). The Kac-Moody algebra of this example has a canonical notation A_1^1.

2.2.5. $\mathbf{A_2^2}$. Let $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. As in Section 2.2.4, the matrix A is degenerate, and there is a one-dimensional center $\mathfrak{c}(A)$; it is spanned by $2h_1 + h_2$. As before, the Lie algebra $\mathfrak{g}(A)/\mathfrak{c}(A)$ is generated by e_1, e_2, h, f_1, f_2 where h is represented by $2h_1$ or $-h_2$, and the relations are

$$[e_1, f_1] = h/2,\ [e_2, f_2] = -h,\ [e_1, f_2] = [e_2, f_1] = 0,$$

$$[h, e_1] = 4e_1,\ [h, e_2] = -2e_2,\ [h, f_1] = -4f_1, [h, f_2] = 2f_2,$$

$$[e_1, [e_1, e_2]\,] = 0, [e_2, [e_2, [e_2, [e_2, [e_2, e_1]\,]\,]\,]\,] = 0,$$
$$[f_1, [f_1, f_2]\,] = 0, [f_2, [f_2, [f_2, [f_2, [f_2, f_1]\,]\,]\,]\,] = 0$$

Here is a construction of an infinite-dimensional Lie algebra isomorphic to $\mathfrak{g}(A)/\mathfrak{c}(A)$. This is the Lie algebra of 3×3 matrices $\|p_{ij}(t)\|$ whose entries belong to $\mathbb{C}[t, t^{-1}]$ with the following additional properties:

$$p_{11}(t) + p_{22}(t) + p_{33}(t) = 0;\ p_{ij}(t) = -p_{ji}(-t).$$

In other words, elements of this Lie algebras are finite sums $A_k t^k$ where A_k are traceless 3×3 matrices and in addition to that every matrix A_k is symmetric, if k is odd, and skew-symmetric, if k is even. The commutator is given by the usual formula. Let

$$e_1 = \begin{bmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & -1 \end{bmatrix} \cdot t,\ f_1 = \begin{bmatrix} -1 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{bmatrix} \cdot t^{-1},$$

$$e_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & i \\ 0 & -i & 0 \end{bmatrix},\ h = \begin{bmatrix} 0 & 0 & 2i \\ 0 & 0 & 0 \\ -2i & 0 & 0 \end{bmatrix},\ f_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & i \\ 0 & -i & 0 \end{bmatrix}.$$

Again, it is not hard to check that these $e_1, \dots, f_2$ generate our Lie algebra and satisfy the above relations, so there arises a homomorphism of $\mathfrak{g}(A)/\mathfrak{c}$ onto Lie algebra constructed.

Actually, this is an isomorphism (Exercise 6.9).

The canonical notation for the Lie algebra of this example is A_2^2.

2.3. Roots.

2.3.1. Grading. Kac-Moody algebras of rank n have a natural n-grading (that is, a grading by n integers): $\mathfrak{g}(A) = \sum_{k_1,\dots,k_n} \mathfrak{g}(A)_{k_1,\dots,k_n}$. Namely, we assign degrees to generators:

$$\deg h_i = (0, \dots, 0), \deg e_j = (0, \dots, 0, \underset{j}{1}, 0 \dots, 0), \deg f_j = (0, \dots, 0, \underset{j}{-1}, 0 \dots, 0))$$

This gives rise to our grading: $\mathfrak{g}_{0,\dots,0} = \mathfrak{h}$, monomials $[e_{j_1}, \dots, e_{j_k}]$ with $\sum \deg e_{j_i} = (k_1, \dots, k_n)$ span $\mathfrak{g}(A)_{k_1,\dots,k_n}$, while monomials $[f_{j_1}, \dots, f_{j_k}]$ with $\sum \deg f_{j_i} = (k_1, \dots, k_n)$

span $\mathfrak{g}(A)_{k_1,\dots,k_n}$. Thus $\mathfrak{n}_+$ is the sum of $\mathfrak{g}(A)_{k_1,\dots,k_n}$ with all $k_1,\dots,k_n$ non-negative (and not all zero), and $\mathfrak{n}_-$ is the sum of $\mathfrak{g}(A)_{k_1,\dots,k_n}$ with all $k_1,\dots,k_n$ non-positive. If the set $k_1,\dots,k_n$ contains both positive and negative numbers, then $\mathfrak{g}(A)_{k_1,\dots,k_n}=0$.

According to formulas in Sections 2.1.1 and 2.1.2, for every $g\in\mathfrak{g}(A)_{k_1,\dots,k_n}$, $[h_i,g]=\left(\sum_j a_{ij}k_j\right)g$. Thus, according to the definition in Section 1.3.3, every $g\in\mathfrak{g}(A)_{k_1,\dots,k_n}$ is a root vector corresponding to the root $h_i\mapsto\sum a_{ij}k_j$. However, an inconvenience appears in the case, when $\mathfrak{g}(A)$ has a nontrivial center: vectors in different spaces $\mathfrak{g}(A)_{k_1,\dots,k_n}$, $\mathfrak{g}(A)_{k'_1,\dots,k'_n}$ can correspond to the same root (this happens when $\sum_j(k_j-k'_j)a_{ij}=0$ for all i). Thus, if we follow the definition of Section 1.3.1, we have to admit that the root spaces may be not the individual spaces $\mathfrak{g}(A)_{k_1,\dots,k_n}$, but the sums of two or more such spaces.

2.3.2. A modification of the construction of $\mathfrak{g}(\mathbf{A})$. To avoid this inconvenience, we have to modify the definition of the Kac-Moody algebra. Our modification will not affect the parts $\mathfrak{n}_+$ and $\mathfrak{n}_-$, but will expand the part $\mathfrak{h}$. The new $\mathfrak{h}$ will be a direct sum of the old $\mathfrak{h}$ and a copy of $\mathfrak{c}(A)$. Thus, the new $\mathfrak{h}$ we contain linearly independent $h_1,\dots,h_n$, and also vectors $c_{\ell_1,\dots,\ell_n}$ for every sequence $\ell_1,\dots,\ell_n$ such that $\sum_i\ell_ia_{ij}=0$ for all j. *From now on, we use the notation $\mathfrak{h}$ for this expanded space.* The relation (1) in Section 2.1.1 is expanded to the statement that all the commutators in (expanded) $\mathfrak{h}$ are zeroes, the relations (2) - (6) remain unchanged, and two new relations appear:

(7) $[c_{\ell_1,\dots,\ell_n},e_j]=\ell_je_j$;
(8) $[c_{\ell_1,\dots,\ell_n},f_j]=-\ell_jf_j$.

Hence, for $g\in\mathfrak{g}(A)_{k_1,\dots,k_n}$, $[c_{\ell_1,\dots,\ell_n},g]=\left(\sum_j k_i\ell_j\right)g$.

Notice that *in the case of non-degenerate matrix A our modification does not change anything.*

We introduce *simple positive roots* $\alpha_1,\dots,\alpha_n\in\mathfrak{h}^*$ by the formulas

$$\alpha_j(h_i)=a_{ij},\alpha_j(c_{\ell_1,\dots,\ell_n})=\ell_j.$$

Obviously, $\alpha_1,\dots,\alpha_n$ are linearly independent.

Poroposition 2.1. *Roots of $\mathfrak{g}(A)$ are precisely non-zero integral linear combinations $k_1\alpha_1+\dots+k_n\alpha_n$ such that $\mathfrak{g}(A)_{k_1,\dots,k_n}\neq0$. Moreover, $\mathfrak{g}(A)_{k_1,\dots,k_n}$ is the root space of the root $k_1\alpha_1+\dots+k_n\alpha_n$.*

Proof. Indeed, for $\alpha=k_1\alpha_1+\dots+k_n\alpha_n$ and $g\in\mathfrak{g}(A)_{k_1,\dots,k_n}$, $[h_i,g]=\sum_j a_{ij}k_j=\alpha(h_i)g$ and $[c_{\ell_1,\dots,\ell_n},g]=\left(\sum_j k_j\ell_j\right)g=\alpha(c_{\ell_1,\dots,\ell_n})g$.

Sometimes, in particular, in Section 2.3.2, we will abbreviate the notation $\alpha=k_1\alpha_1+\dots+k_n\alpha_n$ to $k_1,\dots,k_n$.

2.3.3. Root systems for the examples in Section 2.2. Constructions of the five Kac-Moody algebras of rank 2 in Section 2.2 contain the explicit description of bases, which can be regarded as descriptions of system of roots in the sense of the previous section. Here they are:

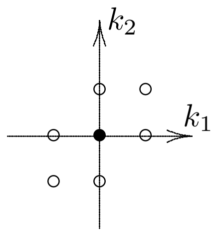
Roots of $\mathfrak{sl}(3)$: $(0,1),(1,0),(1,1)$; $(0,-1),(-1,0),(-1,-1)$.

Roots of $\mathfrak{so}(5)$: $(0,1),(1,0),(1,1),(1,2)$; $(0,-1),(-1,0),(-1,-1),(-1,-2)$.

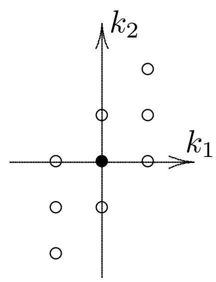
Roots of G_2: $(0,1),(1,0),(1,1),(1,2),(1,3),(2,3),(0,-1),(-1,0),(-1,-1),(-1,-2),(-1,-3),(-2,-3)$.

Roots of A_1^1: $(0,1)$ and $(k,k-1),(k,k),(k,k+1)$ for $k=1,2,\ldots$; $(0,-1)$ and $(k,k+1),(k,k),(k,k-1)$ for $k=-1,-2,\ldots$;

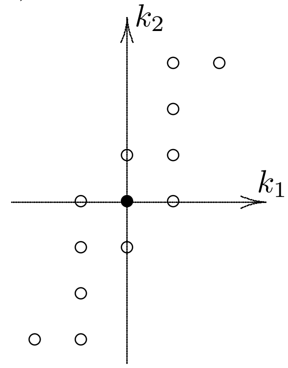
Roots of A_2^2: $(0,1)$, $(k,2k-1),(k,2k),(k,2k+1)$ for $k=1,2,3,4,5,6,\ldots$, and $(k,2k-2),(k,2k+2)$ for $k=2,4,6,\ldots$;$(0,-1)$, $(k,2k+1),(k,2k),(k,2k-1)$ for $k=-1,-2,-3,-4,-5,-6,\ldots$, and $(k,2k+2),(k,2k-2)$ for $k=-2,-4,-6,\ldots$



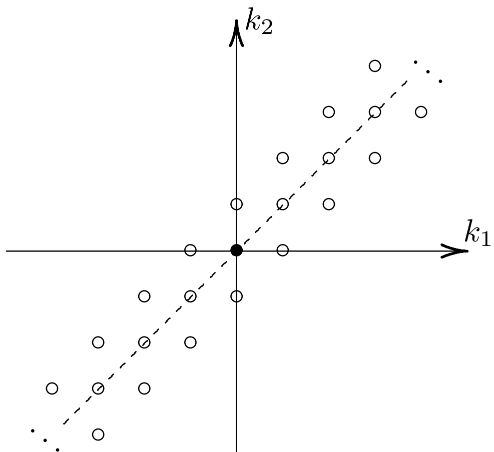
$\mathfrak{sl}(3)$



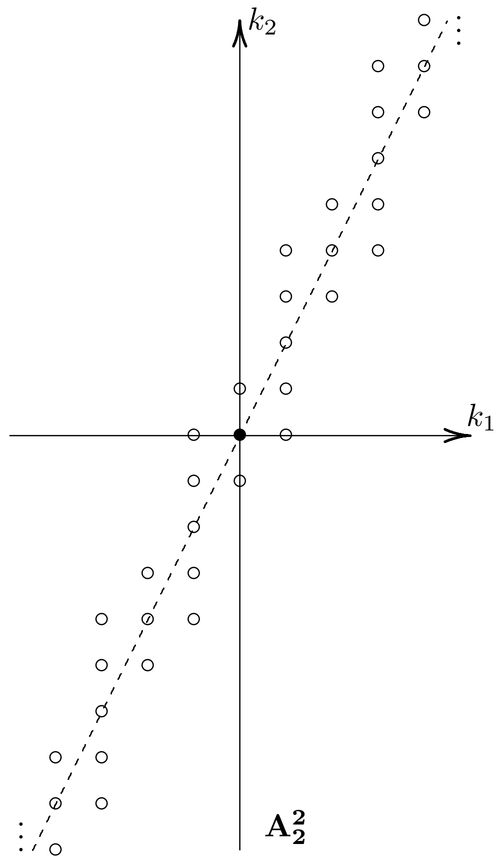
$\mathfrak{so}(5)$



G_2



$\mathbf{A_1^1}$



$\mathbf{A_2^2}$

These root systems are displayed on the diagrams above. Roots are marked with light dots. All root spaces have dimension 1. The origin $(0,0)$ is not a root, and we mark it with a heavy dot.

(The meaning of the flashed symmetry lines will be explained later.)

2.4. A rough classification of Kac-Moody algebras.

Roughly, the set of Kac-Moody Lie algebras may be divided into three big classes.

CLASS 1. If the matrix A^{sym} is positive definite (and only in this case), the algebra $\mathfrak{g}(A)$ is finite-dimensional. This class of Kac-Moody Lie algebras is the same as the class of complex simple Lie algebras, and their Cartan matrices are listed in all major reference books on Lie theory (Bourbaki gives a reliable information). For example, the Cartan matrix of $\mathfrak{sl}(n+1)$ is the $n \times n$ matrix

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Our examples in Sections 2.2.1 - 2.2.3 belong to this class.

CLASS 2. If the matrix A^{sym} is "almost positive definite," which means that one eigenvalue is zero (thus, the matrix is singular) and all other eigenvalues are positive, then the algebra $\mathfrak{g}(A)$ is called an *affine Lie algebra*.

The class of affine Lie algebras is, in turn, divided into two subclasses (of which the second is conveniently omitted in the Wikipedia articles on Kac-Moody algebras as well as on Affine Lie algebras).

SUBCLASS 2.1. For a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$, the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is $\mathfrak{g}(\widetilde{A})/\mathfrak{c}$ where $\widetilde{A}$ is obtained from A by adding one row and one column (as the last row and column) so that $\det \widetilde{A} = 0$ and $\widetilde{A}$ satisfies all the requirements for a Cartan matrix (it is not difficult to prove that these conditions determine $\widetilde{A}$ uniquely). For example, if $\mathfrak{g} = \mathfrak{sl}(n)$, $n > 2$, then $\widetilde{A}$ is the $n \times n$ matrix

$$\begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{bmatrix}.$$

The algebra A_1^1 belongs to this class.

SUBCLASS 2.2. Let again $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and $\tau \colon \mathfrak{g} \to \mathfrak{g}$ be a non-trivial automorphism of a finite order: $\tau^r = \mathrm{id}$. Example: $A \mapsto -A^{\mathrm{t}}$ for $\mathfrak{sl}(n)$, $n \geq 3$. (By the way, all such automorphism are classified, and with one exception, $r = 2$; the exception is an automorphism of order 3 of $\mathfrak{o}(8)$.) Then

$$\mathfrak{g} = \bigoplus_{s=0}^{r-1} \mathfrak{g}_s \text{ where } \mathfrak{g}_s = \{g \in \mathfrak{g} \mid \tau(g) = \exp(2\pi i s/r) g\}$$

(this is a vector space decomposition; of the summands, only $\mathfrak{g}_0$ is a subalgebra, all the rest are modules over $\mathfrak{g}_0$). Then

$$\bigoplus_{k=-\infty}^{\infty} \mathfrak{g}_{k \bmod r} t^k$$

is a Lie subalgebra of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, and this Lie algebra is a $\mathfrak{g}_{00}(A)$ for some Cartan matrix A of the type considered. The algebra A_2^2 belongs to this class.

REMARKS. (1) In some work, the term *Kac-Moody algebra* is used in a restricted sense: the authors mean *affine algebra*. Sometimes, this term is used even in a more narrow sense: it is used as a name of the algebra A_1^1.

(2) There are only two affine algebras of rank 2: A_1^1 and A_2^2. We can add that there are precisely 6 affine algebras of rank 3. We leave the proof to the reader, (it is Exercise 6.10); here we restrict ourselves to the list of their Cartan matrices:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -3 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix}$$

CLASS 3: all the rest. The following is true for all $\mathfrak{g}(A)$ in this class: the dimensions

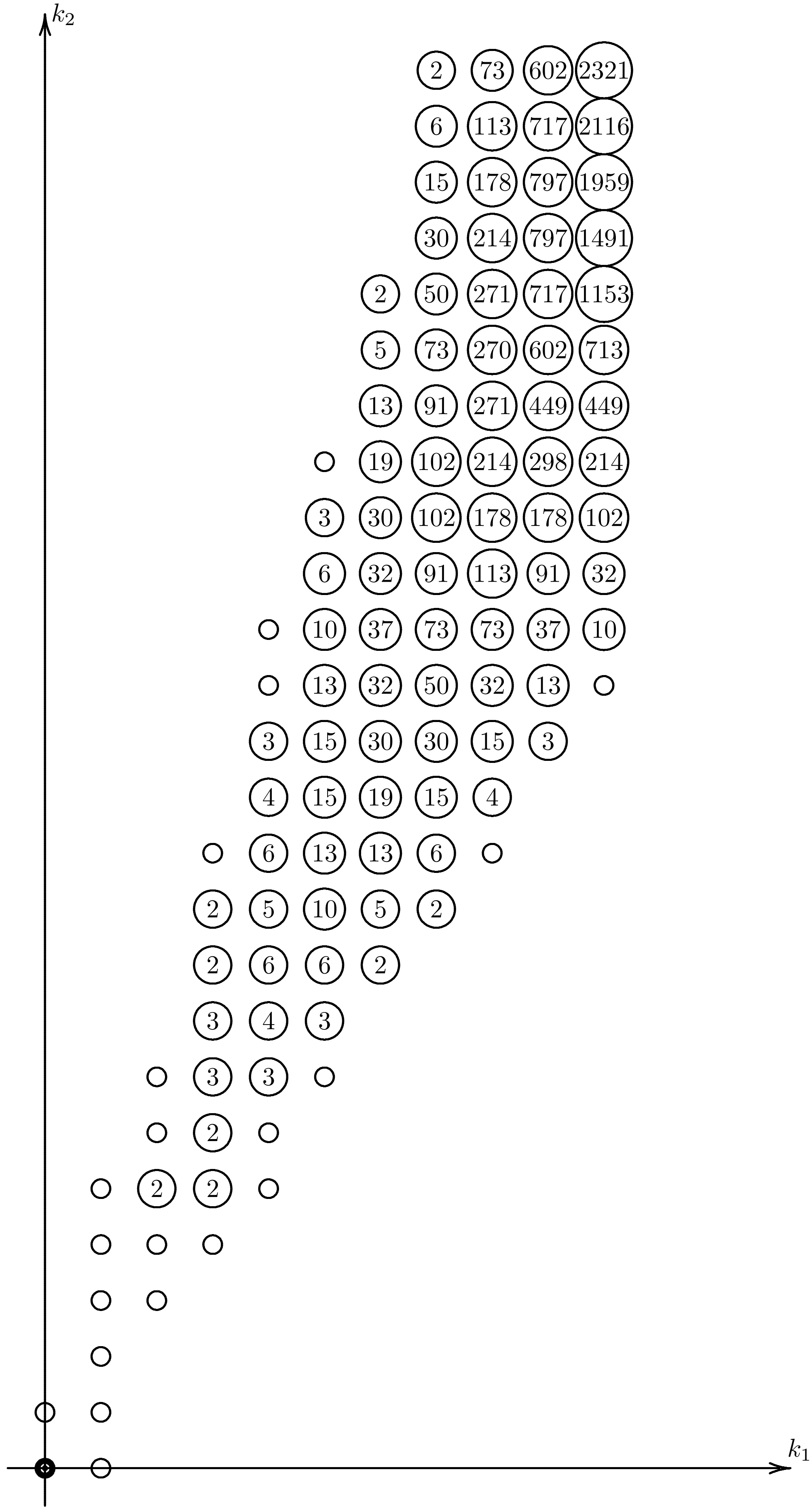
$$D_k = \sum_{k_1+\ldots+k_n=k} \dim \mathfrak{g}(A)_{k_1\alpha_1+\ldots+k_n\alpha}$$

grow exponentially as $k \to \pm\infty$. Traditionally, these Kac-Moody algebras are regarded as less appealing; the situation may be changing, however. For your entertainment, I show on the next page the "root diagram" for $\mathfrak{g}(A)$ with $A = \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix}$; on the diagram, the circles correspond to the roots $k_1\alpha_1 + k_2\alpha_2$, the numbers in the circles are dimensions of the corresponding root spaces.

3. The determinant formula.

3.1 The Shapovalov form.

3.1.1 How much does $M(\lambda)$ depends on λ? Let $M(\lambda)$ be a Verma module over the Kac-Moody Lie algebra $\mathfrak{g}(A)$ where $\lambda \in \mathfrak{h}^*$, and let $\lambda_i = \lambda(h_i)$. As a vector space, $M(\lambda)$ does not depend on λ at all: it is the same as $U(\mathfrak{n}_-)$. Moreover, basically, $M(\lambda)$ depends only on the n complex numbers $\lambda_1, \ldots, \lambda_n$; more precisely, if $\lambda(h_i) = \lambda'(h_i)$ for all i, then the actions of f_i, h_i, and e_i in $M(\lambda)$ and $M(\lambda')$ are the same, and all the difference between $M(\lambda)$ and $M(\lambda')$ lies in the action of the elements, added to $\mathfrak{h}$ in Section 2.3.2. Indeed, both modules are spanned by $g_1 \ldots g_N v$ where $g_i \in \mathfrak{n}_-$. To apply a $g \in \mathfrak{n}_+$ to $g_1 \ldots g_N v$ we push g through $g_1 \ldots g_N$, and at each step we either switch g with a g_i or replace both by the commutator $[g, g_i]$. This process depends on λ only at the last moment, when g, or its commutator with some g_i's reaches v.


k_2
k_1

At this step, we take the $\mathfrak{h}$-component of the survivor and apply it to v; but this component arises from commutators of elements of $\mathfrak{n}_+$ and $\mathfrak{n}_-$, and it is a linear combination of h_i's.

The additional structure in $M(\lambda)$, which is generated by the action of the expanded $\mathfrak{g}(A)$ is the decomposition into weight spaces. In particular, $\mathbb{C}v_\lambda = M(\lambda)_\lambda$ and, for $\eta \in \Delta_+$, $U(\mathfrak{n}_-)_{-\eta}v_\lambda = M(\lambda)_{\lambda-\eta}$.

The arguments above show that the problems of reducibility, of the description of submodules, etc. are equivalent for the modules $M(\lambda)$ and $M(\lambda')$ with $\lambda(h_i) = \lambda'(h_i)$ for all i.

In particular, the reducible modules $M(\lambda)$ correspond to λ's from a subset of the space $\mathbb{C}^n(\lambda_1, \ldots, \lambda_n)$; our goal is to describe this subset.

3.1.2. The construction of the Shapovalov form. We will need for this construction two things: a projection $\beta\colon U(\mathfrak{g}(A)) \to U(\mathfrak{h})$ (the latter is the symmetric, that is, polynomial, algebra of $\mathfrak{h}$) and an involution $\sigma\colon \mathfrak{g}(A) \to \mathfrak{g}(A)$. The projection β is determined by the obvious canonical isomorphism $U(\mathfrak{g}(A)) = U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{h}) \otimes_{\mathbb{C}} U(\mathfrak{n}_+)$ and (also canonically defined "augmentations" $U(\mathfrak{n}_\pm) \to \mathbb{C}$. The involution σ is defined by the relations $\sigma(e_i) = f_i, \sigma(f_i) = e_i, \sigma \mid_{\mathfrak{h}} = \text{id}$. It is an *anti-automorphism*: $\sigma[g, g'] = -[\sigma(g), \sigma(g')] = [\sigma(g'), \sigma(g)]$. Obviously, for any $\eta = (k_1, \ldots, k_n)$, $\sigma(U(\mathfrak{g}(A))_\eta) = U(\mathfrak{g}(A))_{-\eta}$. Notice in addition that $\beta \circ \sigma = \beta$.

Now, for $x, y \in U(\mathfrak{g}(A))$ set $F(x, y) = \beta(\sigma(x)y)$. This F is a bilinear (obviously) symmetric $(F(y, x) = \beta(\sigma(y)x) = \beta(\sigma(\sigma(y)x)) = \beta(\sigma(x)\sigma(\sigma(y))) = \beta(\sigma(x)y) = F(x, y))$ form on $U(\mathfrak{g}(A))$ with values in $U(\mathfrak{h})$. It is called the *Shapovalov form*. It will be instrumental in studying Verma modules.

Notice that if $x \in U(\mathfrak{n}_-)_{-\eta}, y \in U(\mathfrak{n}_-)_{-\eta'}$, and $\eta \neq \eta'$, then $F(x, y) = 0$. By this reason, the Shapovalov form F is determined by its restrictions to the root spaces $U(\mathfrak{n}_-)_{-\eta}$. We denote this restrictions as F_η.

3.1.3. Why is the Shapovalov form so important? Consider a Verma module $M(\lambda)$ and suppose that it is *reducible*, that is, there is a submodule $0 \neq A \subsetneq M(\lambda)$. An important remark: $A \neq M(\lambda) \Rightarrow A \cap \mathbb{C}v_\lambda = 0$. We already know (Section 3.1.1) that A is *graded*, that is,

$$A = \bigoplus_{\eta \in \Delta_+} A_{\lambda-\eta}, \text{ where } A_{\lambda-\eta} = A \cap U(\mathfrak{n}_-)_{-\eta}v_\lambda.$$

Suppose now that for some non-zero $x \in \mathfrak{n}_-$, now $xv_\lambda \in A_{\lambda-\eta}$. Take an arbitrary $y \in U(\mathfrak{n}_-)_{-\eta}$ and consider $\sigma(y)xv_\lambda$. On one hand, this should be an element of A (since A is a submodule). On the other hand, $\sigma(y)xv_\lambda$ should be an element of $M(\lambda)_{\lambda-\eta+\eta} = M(\lambda)_0 = \mathbb{C}v_\lambda$. Because of our "important remark" above, this must be zero: $\sigma(y)xv_\lambda = 0$. But $\sigma(y)xv_\lambda = \lambda(F_\eta(y, x))v_\lambda$, thus $\lambda(F_\eta(y, x)) = 0$, and since this holds for an arbitrary $y \in U(\mathfrak{n}_-)_{-\eta}$, this means that $x \in \text{Ker}(\lambda \circ F_\eta)$, in particular that the complex-valued symmetric bilinear form $\lambda \circ F_\eta$ is degenerate, has a zero determinant. We get a strong necessary condition of a reducibility of $M(\lambda)$: if $M(\lambda)$ is reducible, then $\det(\lambda \circ F_\eta) = 0$ for some $\eta \in \Delta_+$. Moreover, this condition is also sufficient, because of the following obvious statement:

$$\sum_{\eta\in\Delta_+}\mathrm{Ker}(\lambda\circ F_\eta)v_\lambda$$

is a submodule of $M(\lambda)$ (not containing v_λ).

Thus, the reducibility problem for Verma modules is reduced to the computation of (zeroes of) $\det(\lambda\circ F_\eta)=\lambda(\det F_\eta)$. And in fact, this computation was completed more than 30 years ago. The result of this computation is the *determinant formula*, which we will explore in Sections 3.2 and 3.3. Now, for a better understanding of the Shapovalov construction, we will make this computation in a couple of easiest cases.

3.1.4. First examples of the computation of $\lambda\circ F_\eta$. Let $\eta=\alpha_i$, $x=y=e_i$. Then

$$x\sigma(y)v_\lambda=e_if_iv_\lambda=f_i\underbrace{e_iv_\lambda}_{0}+h_iv_\lambda=\lambda_i.$$

Thus, $\lambda\circ F_{\alpha_i}(e_i,e_i)=\lambda_i$.

Let now $\eta=\alpha_i+\alpha_j\ (i\neq j),\ x=e_ie_j, y=e_je_i$. Then

$$x\sigma(y)v_\lambda=e_ie_jf_if_jv_\lambda=e_if_i\underbrace{e_jf_jv_\lambda}_{\lambda_jv_\lambda}=\lambda_j\underbrace{e_if_iv_\lambda}_{\lambda_iv_\lambda}=\lambda_i\lambda_jv_\lambda.$$

Thus, $\lambda\circ F_{\alpha_i+\alpha_j}(e_ie_j,e_je_i)=\lambda_i\lambda_j$. Now let $x=y=e_ie_j$. We have:

$$\begin{aligned}x\sigma(y)v_\lambda=e_ie_jf_jf_iv_\lambda&=e_i(f_je_j+h_j)f_iv_\lambda\\&=e_if_jf_i\underbrace{e_jv_\lambda}_{0}+e_if_i\underbrace{h_jv_\lambda}_{\lambda_jv_\lambda}-a_{ji}\underbrace{e_if_iv_\lambda}_{\lambda_iv_\lambda}=\lambda_i(\lambda_j-a_{ji})v_\lambda.\end{aligned}$$

Thus, $\lambda\circ F_{\alpha_i+\alpha_j}(e_ie_j,e_ie_j)=\lambda_i(\lambda_j-a_{ji})$, and, by $i\leftrightarrow j$, $\lambda\circ F_{\alpha_i+\alpha_j}(e_je_i,e_je_i)=\lambda_j(\lambda_i-a_{ij})$. From this,

$$\det(\lambda\circ F_{\alpha_i+\alpha_j})=\begin{vmatrix}\lambda_i(\lambda_j-a_{ji}) & \lambda_i\lambda_j\\ \lambda_i\lambda_j & \lambda_j(\lambda_i-a_{ij})\end{vmatrix}=\lambda_i\lambda_j(a_{ij}a_{ji}-a_{ij}\lambda_i-a_{ji}\lambda_j).$$

3.2. The determinant formula; the statement and examples.

3.2.1. The statement. The statement below contains several new notations. They will be explained immediately after the statement.

THEOREM 3.1. (Shapovalov [2] for the finite-dimensional case, Kac and Kazhdan [3] for the general case.) *Up to a non-zero factor,*

$$\det F_\eta=\prod_{\alpha\in\Delta_+}\prod_{m=1}^{\infty}\left(h_\alpha+\rho(h_\alpha)-\frac{m\langle\alpha,\alpha\rangle}{2}\right)^{P(\eta-m\alpha)\cdot\mathrm{mult}(\alpha)}.$$

(The non-zero factor appears because the determinant of a quadratic form depends on the choice of a basis; we will never mention it below.)

Explanation of the notation. (1) For a positive root $\alpha = k_1\alpha_1 + \ldots + k_n\alpha_n \in \Delta_+$, we set $h_\alpha = \sum_{i=1}^{n} \frac{k_i}{d_i} h_i$ (the integer d_i are diagonal entry of a matrix D that diagonalizes A, see Section 2.1.1). (2) ρ is an element of $\mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all i; thus $\rho(h_\alpha) = \sum_{i=1}^{n} \frac{k_i}{d_i}$. (3) We define an inner product in the space spanned by α_i's by the formula $\langle \alpha_i, \alpha_j \rangle = a_{ij}^{\mathrm{sym}} = \frac{a_{ij}}{d_i}$. In particular, $\langle \alpha, \alpha \rangle = \sum_{i,j} \frac{a_{ij}}{d_i} k_i k_j$. (4) The *Kostant partition function* $P \colon \Lambda \to \mathbb{Z}_{\geq 0}$ is defined by the formula $P(\zeta) = \dim U(\mathfrak{n}_+)_\zeta$. In particular, $P(0) = 1$, $P(\zeta) = 0$, if $\zeta \notin \Lambda_+$. (5) $\mathrm{mult}(\alpha)$ is the multiplicity of the root α, that is, $\dim \mathfrak{g}(A)_\alpha$.

It should be noted, that for a fixed η the product in the formula has finitely many factors, since $\eta - m\alpha \in \Lambda_+$ only for finitely many pairs m, α.

For a specific $\lambda \in \mathfrak{h}^*$, we can compute $\det(\lambda \circ F_\eta)$ in terms of $\lambda_i = \lambda(h_i)$:

$$\begin{aligned}\det(\lambda \circ F_\eta) &= \prod_{\alpha \in \Delta_+} \prod_{m=1}^{\infty} \left(\lambda(h_\alpha) + \rho(h_\alpha) - \frac{m\langle \alpha, \alpha \rangle}{2} \right)^{P(\eta - m\alpha)\cdot \mathrm{mult}(\alpha)} \\ &= \prod_{\alpha \in \Delta_+} \prod_{m=1}^{\infty} \left(\sum_{i=1}^{n} \frac{k_i}{d_i}(\lambda_i + 1) - \frac{m\langle \alpha, \alpha \rangle}{2} \right)^{P(\eta - m\alpha)\cdot \mathrm{mult}(\alpha)}\end{aligned}$$

With the results of Section 3.1.3, this means that reducible Verma modules $M(\lambda)$ correspond to points $(\lambda_1, \ldots, \lambda_n)$ belonging to a countable union of hyperplanes

$$\sum_{i=1}^{n} \frac{k_i}{d_i}(\lambda_i + 1) = \frac{m\langle \alpha, \alpha \rangle}{2}$$

taken for all positive roots α and positive integers m (for such a pair, we certainly can find an η such that $P(\eta - m\alpha) > 0$; take, for example, $\eta = m\alpha$).

Before proving the determinant formula, let us consider some examples.

3.2.2. Examples.

3.2.2.1. $\mathfrak{sl}(\mathbf{n+1})$. In this case, the rank is n and the non-zero entries of the Cartan matrix are

$$a_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i - j| = 1. \end{cases}$$

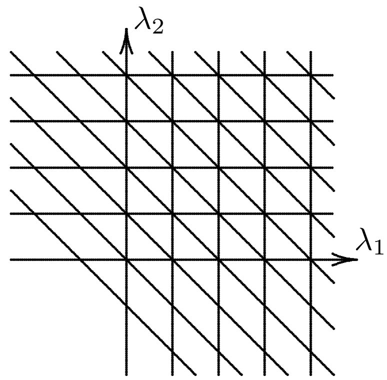
The positive roots are $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j,\ 1 \leq i \leq j \leq n$, (see Section 2.3.4, Class 1) and $\langle \alpha_{ij}, \alpha_{ij} \rangle = 2$ for all i, j. Indeed, $d_1 = \ldots = d_n = 1$ (since the Cartan matrix is symmetric) and

$$\begin{aligned}&\langle \alpha_i + \alpha_{i+1} + \ldots + \alpha_j, \alpha_j + \alpha_{i+1} + \ldots + \alpha_j \rangle \\ &\quad = \langle \alpha_i, \alpha_i \rangle + \ldots + \langle \alpha_j, \alpha_j \rangle + 2\langle \alpha_i, \alpha_{i+1} \rangle + \ldots + 2\langle \alpha_{j-1}, \alpha_j \rangle \\ &\quad = \underbrace{2 + \ldots + 2}_{j-i+1} - \underbrace{2 - \ldots - 2}_{j-i} = 2\end{aligned}$$

the equations of the hyperplanes of reducible Verma modules are $\lambda_i+\ldots+\lambda_j+(j-i+1)=m$, that is, $\lambda_i+\ldots+\lambda_j\in\mathbb{Z}_{\geq i-j}$. In the case $n=1$, this corresponds to the result of Section 1.2.3. For $n=2$, these hyperplanes are the lines

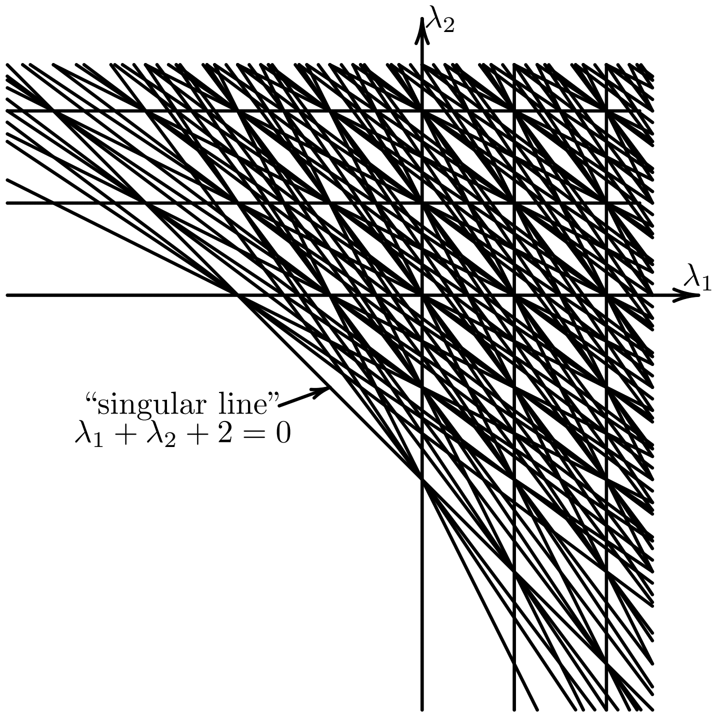
$$\lambda_1=0,1,2,3,\ldots;\lambda_2=0,1,2,3,\ldots;\lambda_1+\lambda_2=-1,0,1,2,\ldots$$

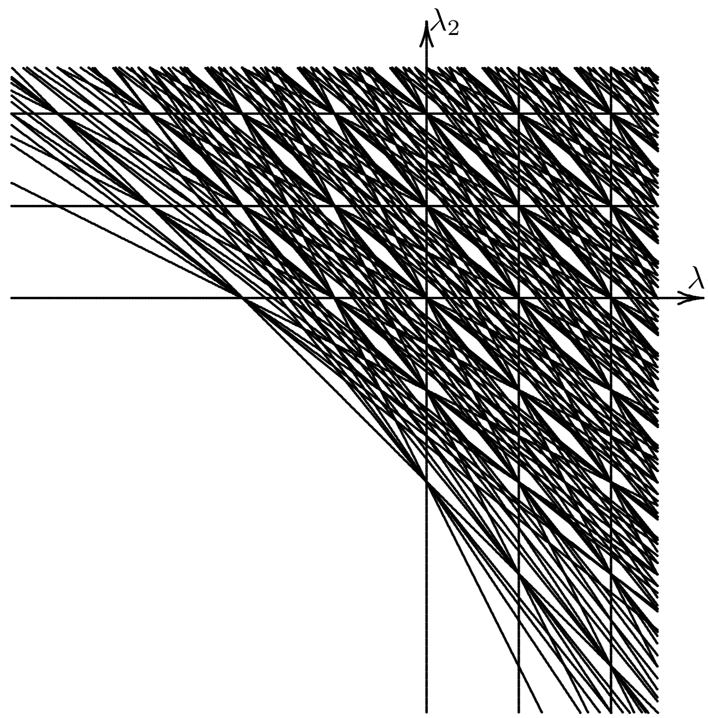
(See the picture below.)



3.2.2.2. $\mathbf{A_1^1}$. In this case, $n=2, a_{11}=a_{22}=2, a_{12}=a_{21}=-2, d_1=d_2=1$. The positive roots are $k\alpha_1+(k-1)\alpha_2,\ (k-1)\alpha_1+k\alpha_2,\ k\alpha_1+k\alpha_2\ (k>0)$ (see Section 2.3.3). Independently of k, the inner square of the first two is 2, the inner square of the third is 0 (A is singular!). The equations of the lines corresponding to reducible Verma modules are ($m\in\mathbb{Z}_{>0}$)

$$\begin{aligned} k\lambda_1+(k-1)\lambda_2+2k-1-m&=0,\\ (k-1)\lambda_1+k\lambda_2+2k-1-m&=0,\\ \lambda_1+\lambda_2+2&=0\end{aligned}$$





Notice that the lines corresponding to the roots $k\alpha_1 + k\alpha_2$ are all the same, independently of k and m: $\lambda_1 + \lambda_2 + 2 = 0$ (we mark this line as "singular" at the picture). The other lines fill densely the half-plane $\lambda_1 + \lambda_2 > -2$ and form a nowhere dense set in the complementary half-plane. The two drawings on the previous page show these lines for $k \leq 3$ and $k \leq 5$.

REMARK. The singular line appears by the following reason. The root $\alpha = k_1\alpha_1 + \ldots + k_n\alpha_n$ has a zero inner square, $\langle \alpha, \alpha \rangle = 0$, if and only if $k_1 h_1 + \ldots + k_n h_n$ belongs to the center $\mathfrak{c}$. (This is Exercise 6.11.) Moreover, all elements of Λ_+ proportional to α are roots with zero inner squares, and the hyperplanes $\sum_{i=1}^{n} \frac{k_i}{d_i}(\lambda_i + 1) = \frac{m\langle \alpha, \alpha \rangle}{2}$ (see Section 3.2.1) are the same for all these roots and all m. Our "singular line" provides an example of such "hyperplane."

3.3. Proof of the determinant formula.

The main ingredients of the proof (and the understanding) of the determinant formula are:

(1) invariant (Killing) form;
(2) the Casimir operator.

3.3.1 Invariant form. In the classical finite-dimensional theory, the notion of the Killing form is one of the most important. The fact is, that for a simple finite-dimensional Lie algebra $\mathfrak{g}$ there exists a unique, up to a non-zero multiple, non-degenerate symmetric invariant bilinear form $\langle\ ,\ \rangle$; the invariance means $\langle [g, h], k \rangle = \langle g, [h, k] \rangle$ for all $g, h, k \in \mathfrak{g}$. The most common construction is $\langle g, h \rangle = \mathrm{tr}(\mathrm{ad}\, g \circ \mathrm{ad}\, h)$ where $\mathrm{ad}\, g \colon \mathfrak{g} \to \mathfrak{g}$ is $h \mapsto [g, h]$ (the uniqueness can be easily deduced from the simplicity of $\mathfrak{g}$). In the infinite-dimensional case, however, this construction does not work (no trace!). Still the notion of the Killing form is naturally extended to the Kac-Moody case.

Let $A, \mathfrak{g}(A), \mathfrak{h}, \ldots$ mean the same as before.

PROPOSITION 3.2. *There exists a non-degenerate symmetric invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}(A)$ with the following additional properties*:

(1) *the form is non degenerate on $\mathfrak{h}$ and on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ for every $\alpha \in \Delta_+$*;
(2) $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$, *if $\alpha + \beta \neq 0$*;
(3) $\langle h_i, h \rangle = d_i \alpha_i(h)$ *for all $h \in \mathfrak{h}, i = 1, \ldots, n \Leftrightarrow \langle h_\alpha, h \rangle = \alpha(h)$ for all $\alpha \in \Lambda, h \in \mathfrak{h}$.*

Before proving the proposition, we list two corollaries with brief comments.

COROLLARY 3.3. $\langle h_i, h_j \rangle = d_i a_{ji} = d_j a_{ij}$; $\langle h_{\alpha_i}, h_{\alpha_j} \rangle = \dfrac{a_{ij}}{d_i} = a_{ij}^{\mathrm{sym}}$.

Since $\langle\ ,\ \rangle$ is non-degenerate on $\mathfrak{h}$, it gives rise to an inner product on $\mathfrak{h}^*$: if $\eta', \eta'' \in \mathfrak{h}^*$ and $\eta'(h) = \langle h', h \rangle$, $\eta''(h) = \langle h'', h \rangle$ for any $h \in \mathfrak{h}$, then $\langle \eta', \eta'' \rangle = \langle h', h" \rangle = \eta'(h") = \eta"(h')$.

COROLLARY 3.4. $\langle \eta, \alpha \rangle = \eta(h_\alpha)$ *for every $\eta \in \mathfrak{h}^*, \alpha \in \Lambda$.*

In particular, $\langle \alpha_i, \alpha_j \rangle = \langle h_{\alpha_i}, h_{\alpha_j} \rangle = a_{ij}^{\mathrm{sym}}$. This corresponds to the definition given in Section 3.2.1.

COROLLARY 3.5. *If $g \in \mathfrak{g}_\alpha, g' \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Delta_+$, then $[g, g'] = \langle g, g' \rangle h_\alpha$.*

This shows that the space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ has dimension 1.

Proof of Corollary 3.5. For $h \in \mathfrak{h}$, $\langle [g,g'],h\rangle = \langle g,[g',h]\rangle = \alpha(h)\langle g,g'\rangle = \langle \langle g,g'\rangle h_\alpha, h\rangle$ which means that $[g,g'] = \langle g,g'\rangle h_\alpha$, since $\langle\ ,\ \rangle|_{\mathfrak{h}}$ is non-degenerate.

Proof of Proposition 3.2. First, we define $\langle\ ,\ \rangle$ on $\mathfrak{h}$. Let $\mathfrak{h}_0 \subset \mathfrak{h}$ be the subpace of $\mathfrak{h}$ with the basis $h_1,\dots,h_n$; choose a complimentary subspace: $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{k}$. Condition (3) of Proposition determines $\langle\ ,\ \rangle$ on $\mathfrak{h}_0 \otimes \mathfrak{h}$. In addition to that, put $\langle\ ,\ \rangle = 0$ on $\mathfrak{k}$; this determines the form on $\mathfrak{h}$, and it remains to check that it is non-degenerate. Let $\langle \overline{h}, h\rangle = 0$ for all $h \in \mathfrak{h}$. Then $\langle h_{\alpha_i}, \overline{h}\rangle = \alpha_i(\overline{h}) = 0$ for all i, and hence $\overline{h} \in \mathfrak{c} \subset \mathfrak{h}_0$, $\overline{h} = \sum_i u_i h_{\alpha_i}, u_i \in \mathbb{C}$. Hence $\langle \overline{h}, h\rangle = \sum_i u_i \langle h_{\alpha_i}, h\rangle = \sum_i \alpha_i(h) = 0$ for all $h \in \mathfrak{h}$, hence $\sum_i u_i\alpha_i = 0$, hence $u_i = 0$ for all i, and hence $\overline{h} = 0$.

Let $\mathfrak{g}_N = \bigoplus\limits_{k_1+\ldots+k_n=N} \mathfrak{g}_{k_1\alpha_1+\ldots+k_n\alpha_n}$ and let $\mathfrak{g}(N) = \bigoplus\limits_{-N\le M\le N} \mathfrak{g}_M$. First, we extend our definition of $\langle\ ,\ \rangle$ from $\mathfrak{g}_0 = \mathfrak{h}$ to $\mathfrak{g}(1)$ by taking $\langle e_i, f_j\rangle = d_i\delta_{ij}$ (this must be so by Corollary 3.5). To check the invariance, at this stage, we need only to check that $\langle [e_i,h], f_j\rangle = \langle e_i, [h,f_j]\rangle$. But $[e_i,h_k] = -a_{ki}e_i, [h_k,f_j] = -a_{kj}f_j, [e_i, c_{\ell_1,\ldots,\ell_n}] = \ell_i e_i, [c_{\ell_1,\ldots,\ell_n}, f_j] = -\ell_j f_j$ (we use notations from Section 2.3.2). If $j \neq i$, then $\langle e_i, f_j\rangle = 0$, and hence $\langle [e_i,h], f_j\rangle = \langle e_i, [h,f_j]\rangle = 0$. If $j = i$, then $\langle [e_i,h], f_j\rangle = \langle e_i,[h,f_j]\rangle = -a_{k,i}d_i$ for $h = h_k$ and $\langle [e_i,h], f_j\rangle = \langle e_i,[h,f_j]\rangle = -\ell_i d_i$ for $h = c_{\ell_1,\ldots,\ell_n}$.

Now let us show how to extend the definition of $\langle\ ,\ \rangle$ from $\mathfrak{g}(N-1)$ $(N>1)$ to $\mathfrak{g}(N)$. We only need consider the case when the factors are $x \in \mathfrak{g}_{-N}$ and $y \in \mathfrak{g}_N$. We can write $y = \sum_i [u_i, v_i]$ where each u_i and each v_i lies in some $\mathfrak{g}_M$ with $0 < M < N$. We put:

$$\langle x, y\rangle = \sum_i \langle [x,u_i], v_i\rangle,$$

and the only thing that requires a proof is that this sum does not depend on the choice of a presentation $y = \sum_i [u_i,v_i]$. To do this, let us choose a similar presentation for x: $x = \sum_j [u'_j, v'_j]$. Using the invariance of $\langle\ ,\ \rangle$ on $\mathfrak{g}(N-1)$ and the definition of the Lie algebra, we can write:

$$\begin{aligned}\sum_i \langle [x,u_i],v_i\rangle &= \sum_{i,j}\langle [[u'_j,v'_j],u_i],v_i\rangle = \sum_{i,j}\langle [u'_j,[v'_j,u_i]],v_i\rangle - \sum_{i,j}[v'_j,[u'_j,u_i]],v_i\rangle \\ &= -\sum_{i,j}\langle [[v'_j,u_i],u'_j],v_i\rangle + \sum_{i,j}\langle [[u'_j,u_i],v'_j],v_i\rangle \\ &= \sum_{i,j}\langle [u'_j,[v'_j,u_i]],v_i\rangle + \sum_{i,j}\langle [u'_j,u_i],[v'_j,v_i]\rangle \\ &= \sum_{i,j}\langle u'_j, [[v'_j,u_i],v_i] + [u_i,[v'_j,v_i]]\rangle = \sum_{i,j}\langle u'_j,[v'_j,[u_i,v_i]]\rangle = \sum_j \langle u'_j,[v'_j,y]\rangle\end{aligned}$$

We see that the equality $\sum_i \langle [x,u_i],v_i\rangle = \sum_j \langle u'_j,[v'_j,y]\rangle$ holds for any presentations $y = \sum_i [u_i,v_i]$, $x = \sum_j [u'_j,v'_j]$; hence, neither of these expressions depends on either of these presentations.

3.3.2. Casimir operator. Another important notion of the finite-dimensional theory is the Casimir operator. Algebraically speaking, for a (finite-dimensional, simple) Lie algebra $\mathfrak{g}$, we determine the center of the universal enveloping algebra $U(\mathfrak{g})$. If $\gamma \in U(\mathfrak{g})$ is a central element, then for any $\mathfrak{g}$- (that is, $U(\mathfrak{g})$-) module M, the transformation $\gamma\colon M \to M$ is a *module* homomorphism. Moreover, any module homomorphism $f\colon M \to N$ commutes with γ : $f(\gamma x) = \gamma f(x)$. It turns out that the center of $U(\mathfrak{g})$ is quite ample: it is, actually, isomorphic to a polynomial algebra of $n = \operatorname{rank}\mathfrak{g}$ variables. This center is important, among other things, in geometry and topology (more specifically, in the theory of characteristic classes). But there is one central element which is instrumental in representation theory; the transformation corresponding to this element is called the *Casimir operator*.

The construction of it is very simple: if $g_1, \dots, g_N$ is a basis in $\mathfrak{g}$ and $g^1, \dots, g^N$ is a dual basis (with respect to the Killing form), then $\gamma = \sum_i g_i g^i$. For example, $h^2+2ef+2fe$ is a central element of $U(\mathfrak{sl}(2))$; you can transform it into, say $2h + h^2 + 4fe$.

How to generalize it to the infinite-dimensional case? Immediately, we encounter a difficulty: the sum will be infinite. Still, let us try. Let $\{u_i\}$ and $\{u^i\}$ ($i = 1, \dots, \dim\mathfrak{h}$) be dual (with respect to $\langle\,,\,\rangle$) bases of $\mathfrak{h}$; and $\{e^i_\alpha\}$ and $\{e^i_{-\alpha}\}$ ($i = 1, \dots, \dim\mathfrak{g}_\alpha = \operatorname{mult}\alpha$) be dual bases of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ where α is a positive root. Then the expression $\sum_\alpha \sum_i e^i_{-\alpha} e^i_\alpha$ is infinite, but not hopeless: it can be applied to any $\mathfrak{g}(A)$-module M , which is virtually nilpotent over $\mathfrak{n}_+$: for every element of this module only finitely many summands of this infinite sum will produce non-zero images. (Informally, we are generalizing to the infinite-dimensional case rather the expression $2h+h^2+4fe$ than $h^2+2ef+2fe$ – see the discussion of the case of $\mathfrak{sl}(2)$ several lines above.)

To pass to a precise definition, we need one more notation: define $\rho^* \in \mathfrak{h}$ by the requirement $\langle\rho^*, h\rangle = \rho(h)$ (see Section 3.2.1 for the definition of ρ; thus, $\langle\rho^*, h_i\rangle = 1$ and $\langle\rho^*, c_{\ell_1\dots\ell_n}\rangle = 0$). Put

$$\Omega = 2\rho^* + \sum_{i=1}^{\dim\mathfrak{h}} u^i u_i + 2\sum_{\alpha\in\Delta_+}\sum_{i=1}^{\operatorname{mult}\alpha} e^i_{-\alpha} e^i_\alpha.$$

Recall that although Ω cannot be considered as an element of $U(\mathfrak{g})$, it determines an operator in an arbitrary virtually $\mathfrak{n}_+$-module, in particular, a Verma module, over $\mathfrak{g}$.

Proposition 3.6. *Ω commutes with the action of $U(\mathfrak{g}(A))$.*

Proof. We need to prove that Ω commutes with any generator of $\mathfrak{g}$.

Commuting with $\mathfrak{h}$ is obvious. Indeed, ρ^* and u_i, u^i belong to $\mathfrak{h}$ themselves, so they commute with any $h \in \mathfrak{h}$. Furthermore,

$$[h, e^i_{-\alpha} e^i_\alpha] = [h, e^i_{-\alpha}] e^i_\alpha + e^i_{-\alpha}[h, e^i_\alpha] = (-\alpha(h) + \alpha(h)) e^i - \alpha e^i_\alpha = 0.$$

Let us prove now that Ω commutes with e_j (the case of f_j is absolutely similar, and we will not consider it.) First,

$$[\rho^*, e_j] = \alpha_j(\rho^*) e_j = \langle\rho^*, h_{\alpha_j}\rangle e_j = \rho(h_{\alpha_j}) e_j$$

(the first of the equality above follows from the definition of a root, since $e_j \in \mathfrak{g}_{\alpha_j}$; the second equality follows from Part (3) of Proposition 3.2; and the last equality follows from the definition of ρ^*). Second,

$$\left[\sum_i u^i u_i, e_j\right] = \sum_i (u^i[u_i, e_j] + [u^i, e_j]u_i) = \sum_i (\alpha_j(u_i)u^i e_j + \alpha_j(u^i)e_j u_i)$$
$$= \underbrace{\sum_i \alpha_j(u_i)u^i}_{h_{\alpha_j}} e_j + \underbrace{\sum_i \alpha_j(u^i)u_i}_{h_{\alpha_j}} e_j - \underbrace{\sum_i \alpha_j(u^i)\alpha_j(u_i)}_{\langle\alpha_j,\alpha_j\rangle} e_j$$
$$= \big(2h_{\alpha_j} - \langle\alpha_j, \alpha_j\rangle\big) e_j$$

(we used the fact that α and h_α correspond to each other with respect to the isomorphism $\mathfrak{h} \leftrightarrow \mathfrak{h}^*$ induces by $\langle\,,\,\rangle$; this is implied by Corollary 3.4).

It remains to compute the commutator $\left[\sum_\alpha \sum_i e^i_{-\alpha} e^i_\alpha, e_j\right]$. For this, we need the following

Lemma. *Let $\alpha, \beta \in \Delta$ and $z \in \mathfrak{g}_{\beta-\alpha}$. Then, in $U(\mathfrak{g}(A))$,*

$$\sum_{i=1}^{\operatorname{mult}\alpha} e^i_{-\alpha}[z, e^i_\alpha] = \sum_{i=1}^{\operatorname{mult}\beta} [e^i_{-\beta}, z]e^i_\beta.$$

Proof of Lemma. We will prove more: in $\mathfrak{g}(A)_{-\alpha} \otimes \mathfrak{g}(A)_\beta$,

$$\sum_{i=1}^{\operatorname{mult}\alpha} e^i_{-\alpha} \otimes [z, e^i_\alpha] = \sum_{i=1}^{\operatorname{mult}\beta} [e^i_{-\beta}, z] \otimes e^i_\beta.$$

Take arbitrary $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\beta}$. Then

$$\left\langle \sum_i e^i_{-\alpha} \otimes [z, e^i_\alpha], e \otimes f \right\rangle = \sum_i \langle e^i_{-\alpha}, e\rangle\langle [z, e^i_\alpha], f\rangle = \sum_i \langle e^i_{-\alpha}, e\rangle\langle e^i_\alpha, [f, z]\rangle$$
$$= \left\langle \sum_i \langle e^i_{-\alpha}, e\rangle e^i_\alpha, [f, z] \right\rangle = \langle e, [f, z]\rangle.$$

In a very similar way,

$$\left\langle \sum_i [e^i_{-\beta}, z] \otimes e^i_\beta, e \otimes f \right\rangle = \langle [z, e], f\rangle.$$

Hence, the two sides of our equality have the same inner product with an arbitrary $e \otimes f$, hence, they are equal.

Now, let us turn to $\sum_\alpha \sum_i \left[e^i_{-\alpha} e^i_\alpha, e_j\right]$. We have:

$$\sum_{\alpha\in\Delta_+} \sum_{i=1}^{\operatorname{mult}\alpha} \left[e^i_{-\alpha} e^i_\alpha, e_j\right] = \sum_{\alpha\in\Delta_+} \sum_{i=1}^{\operatorname{mult}\alpha} e^i_{-\alpha}\left[e^i_\alpha, e_j\right] + \sum_{\beta\in\Delta_+} \sum_{i=1}^{\operatorname{mult}\beta} \left[e^i_{-\beta}, e_j\right] e^i_\beta$$

But $\left[e^i_\alpha, e_j\right] = 0$, whenever $\alpha + \alpha_j$ is not a root. So, for the first sum, we may assume that $\alpha+\alpha_j$ is a root. But then our Lemma for $z = e_j$ shows that the summand in the first sum, which corresponds to α cancels with the summand in the second sum, which corresponds to $\beta = \alpha + \alpha_j$. This cancellations kills the first sum altogether; what remains in the second sum? Again, $\left[e^i_{-\beta}, e_j\right] = 0$, if $-\beta + \alpha_j$ is not a root **or zero**; the latter happens if $\beta = \alpha_j$. And the summand corresponding to this β is all that survives the cancellation in the second sum. The space $\mathfrak{g}_{\alpha_j}$ is one-dimensional, for it basis we can take $e^1_{\alpha_j} = e_j$. What is the "dual basis" in $\mathfrak{g}_{-\alpha_j}$? Since $\langle f_j, e_j\rangle = d_j$, we have to take $e^1_{-\alpha_j} = \dfrac{1}{d_j} f_j$. And this is our final result:

$$\sum_{\alpha\in\Delta_+} \sum_{i=1}^{\operatorname{mult}\alpha} \left[e^i_{-\alpha} e^i_\alpha, e_j\right] = [e^1_{-\alpha_j}, e_j] e^1_{\alpha_j} = -\frac{1}{d_j} h_j e_j = -h_{\alpha_j} e_j.$$

It remains to combine all the computations:

$$\begin{aligned}[\Omega, e_j] &= 2[\rho^*, e_j] + \sum_i [u^i u_i, e_j] + \sum_\alpha \sum_i [e^i_{-\alpha} e^i \alpha, e_j] \\ &= 2\rho(h_{\alpha_j}) e_j + (2h_{\alpha_j} - \langle \alpha_j, \alpha_j\rangle) e_j - 2h_{\alpha_j} e_j = 0,\end{aligned}$$

since $\langle \alpha_j, \alpha_j \rangle = a^{\mathrm{sym}}_{ij} = \dfrac{2}{d_j} = 2\rho(h_{\alpha_j})$.

This completes the proof of Proposition 3.6, and with that, constructing the Casimir operator.

3.3.3. Casimir operator and singular vectors.

Proposition 3.7. *Let a virtually $\mathfrak{n}_+$-nilpotent module contain a singular vector w of the type $\lambda \in \mathfrak{h}^*$. Then $\Omega w = \langle \lambda + 2\rho, \lambda\rangle w$.*

Proof. Since w is a singular vector of the type λ, $ew = 0$ for every $e \in \mathfrak{n}_+$ and $hw = \lambda(h) w$ fort every $h \in \mathfrak{h}$. Hence

$$\Omega w = (2\rho^* + \sum u^i u_i) w = (2 \underbrace{\lambda(\rho^*)}_{\langle\rho,\lambda\rangle} + \underbrace{\sum \lambda(u^i)\lambda(u_i)}_{\langle\lambda,\lambda\rangle}) w = \langle \lambda + 2\rho, \lambda\rangle w.$$

Corollary 3.8. *In the Verma module $M(\lambda)$, the operator Ω acts as the multiplication by $\langle\lambda + 2\rho, \lambda\rangle$.*

Indeed, $\Omega v_\lambda = \langle \lambda + 2\rho, \lambda\rangle v_\lambda$, and v_λ generates the whole module.

Let us now return to $\det \lambda \circ F_\eta$. This determinant equal to zero if and only if $M(\lambda)$ has a proper submodule with a non-zero intersection with $M(\lambda)_{\lambda-\eta}$, which means, in turn, that $M(\lambda)$ contains a singular vector w 0f the type $\lambda - \beta$ where $\beta \in \Lambda_+$ and $\eta - \beta \in \Lambda_+ \cup 0$. Thus, on one side $\Omega w = \langle \lambda + 2\rho, \lambda\rangle w$ (since $w \in M(\lambda)$) and, on the other side, $\Omega W = \langle \lambda - \beta + 2\rho, \lambda - \beta\rangle w$ by Proposition 3.7. We arrive at the equality

$$\langle \lambda + 2\rho, \lambda\rangle = \langle \lambda - \beta + 2\rho, \lambda - \beta\rangle,$$

which, after cancellations and division by 2, becomes

$$\langle \lambda + \rho, \beta \rangle = \frac{1}{2} \langle \beta, \beta \rangle.$$

In other words, this means that $\det \lambda \circ F_\eta = 0$ if and only if there exists a $\beta \in \Lambda_+$ such that $\eta - \beta \in \Lambda_+ \cup 0$ and $\langle \lambda + \rho, \beta \rangle = \frac{1}{2} \langle \beta, \beta \rangle$.

COROLLARY 3.9. $\det \lambda \circ F_\eta$ is the product of a certain amount of expressions $\langle \lambda + \rho, \beta \rangle - \frac{1}{2} \langle \beta, \beta \rangle$ (some of them may be repeated several times).

It remains to specify those β (with their multipliciies), which actually appear in $\lambda \circ \det F_\eta$. For this purpose, we compute the "leading term" of $\det F_\eta$.

The theorem we are proving that the factors of $\det \lambda \circ F_\eta$ are precisely the products $m\alpha$ where m is a positive integers and $\alpha \in \Delta_+$. This is what we expect to obtain.

3.3.4. The leading term of the determinant. Since all the entries of the matrix of the Shapovalov form F_η, and hence $\det F_\eta$, belong to $\mathbb{C}[h_1, \ldots, h_n]$, we can speak of the highest total degree ("leading") term of $\det F_\eta$.

PROPOSITION 3.10. (Shapovalov in the finite-dimensional case; but the proof in general case is the same.) *Up to a constant non-zero factor, the leading term of* $\det F_\eta$ *is*

$$\prod_{\alpha \in \Delta_+} \prod_{m=1}^{\infty} h_\alpha^{\operatorname{mult}(\alpha) P(\eta - m\alpha)}.$$

Remark. This shows that every linear factor that occurs in $\det F_\eta$, must be of the form $h_\alpha +$ a constant for some positive root α; this shows, in turn, that for every factor of $\det F_\eta$ detected in Section 3.3.3, β must be proportional to some positive root α.

Proof of Proposition 3.10. Order in some way the positive roots of $\mathfrak{g}(A)$: $\Delta_+ = \{\beta_1, \beta_2, \beta_3, \ldots\}$ and choose a basis $b_i^1, \ldots, b_i^{d_i}$ in $\mathfrak{g}_{\beta_i}$ (where $d_i = \dim \mathfrak{g}_{\beta_i}$). This gives us a basis in $U(\mathfrak{n}_+)_\eta$ in which every vector has a form $b_{i_1}^{p_1} \ldots b_{i_s}^{p_s}$ where $\beta_{i_1} + \ldots + \beta_s = \eta$, $i_1 \leq \ldots \leq i_s$, $1 \leq p_u \leq d_{i_u}$ and $p_u \leq p_{u+1}$ if $i_u = i_{u+1}$. For the basis in $(\mathfrak{g})_{-\beta_i}$ dual to our basis in $\mathfrak{g}_{\beta_i}$ we use the notation $c_i^1, \ldots, c_i^{d_i}$ (with the same d_i as before). Thus, $[b_i^p, c_i^q] = \delta_{pq} h_{\beta_i}$.

Let us now compute the matrix entry of the form F_η corresponding to the vectors $b_{i_1}^{p_1} \ldots b_{i_s}^{p_s}, b_{j_1}^{q_1} \ldots b_{j_t}^{q_t}$. For this purpose, we take $b_{i_1}^{p_1} \ldots b_{i_s}^{p_s} c_{j_1}^{q_1} \ldots c_{j_t}^{q_t} \in U(\mathfrak{g})$ and push $b_{i_s}^{p_s}$ through $c_{j_1}^{q_1} \ldots c_{j_t}^{q_t}$. First, $b_{i_s}^{p_s}$ either goes through $c_{j_1}^{q_1}$, or both $b_{i_s}^{p_s}$ and $c_{j_1}^{q_1}$ are replaced by the commutator $[b_{i_s}^{p_s}, c_{j_1}^{q_1}]$. The latter is either a root vector in $\mathfrak{n}_-$, or an element of $\mathfrak{h}$, or a root vector in $\mathfrak{n}_+$. In the first case we stop here, in the other cases we push $b_{i_s}^{p_s}$ or what we got, through $c_{j_2}^{q_2}$; and so on, to $c_{j_t}^{q_t}$. Possible results of the whole travel: (1) no h appears, $b_{i_s}^{p_s}$ disappears (so s decreases by one) and the number t of root vectors in $\mathfrak{n}_-$ stays the same or decreases; (2) one h appears (at the end of the word), s decreases by one, and t decreases by one or more. Then we push in the same way $b_{i_{s-1}}^{p_{s-1}}$, and so on.

The final result will be the sum of an amount of terms, of which only those make contributions into the value of F_η, which are pure elements of $U(\mathfrak{h})$ (do not contain anything from $\mathfrak{n}_\pm$). Also, as we will see, we will be interested only in the cases with $t \leq s$. In these cases, the maximal number of h's is s, and it arises only if $t = s$ and $b_{j_1}^{q_1} \dots b_{j_t}^{q_t}$ is the same as $b_{i_1}^{p_1} \dots b_{i_s}^{p_s}$. The "diagonal" matrix entry will be $h_{\beta_{i_1}} \dots h_{\beta_{i_s}}$.

We refer to s as the *length* of the basis vector $b_{i_1}^{p_1} \dots b_{i_s}^{p_s}$. If $\eta = k_1\alpha_1 + \dots + k_n\alpha_n$, then the lengths of the basis vectors vary form $k = k_1 + \dots + k_n$ to 1 (which occurs only if η itself is a root). We sort the basis vectors by their lengths (in the decreasing order). Then the matrix of F_η falls into blocks:

	k	$k-1$	$k-2$	
k	F_{kk}	$F_{k,k-1}$	$F_{k,k-2}$	$\vdots$
$k-1$	$F_{k-1,k}$	$F_{k-1,k-1}$	$F_{k-1,k-2}$	$\vdots$
$k-2$	$F_{k-2,k}$	$F_{k-2,k-1}$	$F_{k-2,k-2}$	$\vdots$
	$\dots$	$\dots$	$\dots$	

We want to detect the highest degree term(s) of the determinant of this matrix. All the diagonal entries of the block F_{kk} have degree k, and all the other entries of the whole matrix have degrees $< k$. So, the diagonal entries of F_{kk} must be taken for the term of the highest degrees. Because of this, no entries from the blocks F_{ik}, $i < k$ can participate in our term of the determinant. Throughout the blocks $F_{k-1,j}$ $j \leq k-1$, only the diagonal entries of $F_{k-1,k-1}$ have degree $k-1$, all the other entries have degrees $< k-1$. Thus, we have to include these diagonal entries into the highest degree term of the determinant. Continuing in this way, we find that we have to include into the highest degree term the diagonal entries of $F_{k-2,k-2}$, of $F_{k-3,k-3}$, and so on.

We arrive at the following result. *The highest degree term of* $\det F_\eta$ *is the product of the diagonal entries of the matrix of* F_η. From this, we can derive the explicit form of this highest degree term:

$$\prod_{s=1}^{k} \prod_{\substack{\beta_{i_1}^{p_1} \dots \beta_{i_s}^{p_s} \in \\ \text{basis of } U(\mathfrak{n}_+)_\eta}} (h_{\beta_{i_1}} \dots h_{\beta_{i_s}}). \tag{4}$$

To complete the proof of Proposition 3.10, we need to calculate the number of h_α's in the product (4). For $m = 0, 1, 2, \dots$ and $u = 1, 2, \dots, \operatorname{mult}\alpha$, denote by $\mathcal{P}_m^u$ the set of the elements of our basis of $U(\mathfrak{n}_+)_\eta$ which contain precisely m factors e_α^u and set $P_m^u = \operatorname{card}\mathcal{P}_m^u$. It is clear that P_m^u does not depend on u and that for any u there is a $1-1$ correspondence between $\mathcal{P}_m^u \cup \mathcal{P}_{m+1}^u \cup \mathcal{P}_{m+2}^u \cup \dots$ and the standard basis of $U(\mathfrak{n}_+)_{\eta - m\alpha}$ (just remove from every element of $\mathcal{P}_m^u \cup \mathcal{P}_{m+1}^u \cup \mathcal{P}_{m+2}^u \cup \dots$ m factors e_α^u, and we will obtain the basis of

$U(\mathfrak{n}_+)_{\eta-m\alpha})$. Thus, $P(\eta - m\alpha) = \sum_{j=m}^{\infty} P_j^u$. It is clear also that the number of occurrences of e_α^u throughout the whole basis of $U(\mathfrak{n}_+)_\eta$ is

$$\sum_{m=1}^{\infty} mP_m^u = \sum_{m=1}^{\infty}\sum_{j=m}^{\infty} P_j^u = \sum_{m=1}^{\infty} P(\eta - m\alpha).$$

Since the result of this computation does not depend of u, the total amount of $e_\alpha^1, \dots, e_\alpha^{\mathrm{mult}\,\alpha}$ throughout the whole basis of $U(\mathfrak{n}_+$, which is the same as the amount of factors h_α in the product (2) is $\sum_{\alpha\in\Delta_+}\sum_{m=1}^{\infty} \mathrm{mult}(\alpha)P(\eta - m\alpha)$.

This completes the proof of Proposition 3.10, and, hence, the proof of Theorem 3.1.

Now we pass to our last subject concerning the Kac-Moody algebras.

3.4. Structure of Verma modules over Kac-Moody algebras.

I will present here, without proof, one more result from the Kac-Kazhdan paper [3]. It concerns the structure of the Verma module $M(\lambda)$ over the Kac-Moody algebra $\mathfrak{g}(A)$. Recall that $M(\lambda)$ contains the *maximal submodule* $I(\lambda) \not\ni v_\lambda$ (which is the sum of such submodules of $M(\lambda)$), and the quotient $L(\lambda) = M(\lambda)/I(\lambda)$ is irreducible; moreover, this construction provides *all irreducible virtually* $\mathfrak{n}_+$*-nilpotent* $\mathfrak{g}(A)$*-modules*.

We use the term *subquotient* of a module A all modules of the form B/C where $B \supset C$ are submodules of A.

THEOREM 3.11 [3]. *The module $L(\mu)$ is (isomorphic to) an irreducible subquotient of $M(\lambda)$ if and only if there exists a sequence $\beta_1, \dots, \beta_k$ of positive roots of $\mathfrak{g}(A)$ and a sequence of positive integers $n_1, \dots, n_k$ such that $\lambda - \mu = n_1\beta_1 + \dots + n_k\beta_k$ and*

$$(\lambda + \rho - n_1\beta_1 - \dots - n_{i-1}\beta_{i-1})(h_{\beta_i}) = \frac{1}{2}n_,\langle\beta_i, \beta_i\rangle,\ i = 1, \dots, k.$$

A stronger version of this theorem is due to I. Bernstein, I. Gelfand, and S. Gelfand [4].

The proof requires an additional tool from the Lie theory, the so-called *Jantzen filtration*, and this is why I refrain from proving it here.

3.5 Explicit formulas for singular vectors.

Suppose that a $\lambda \in \mathfrak{h}^*$ satisfies the equation

$$\left(\langle\lambda + \rho, \alpha\rangle + \frac{m\langle\alpha, \alpha\rangle}{2}\right) = 0$$

for just one pair α, m. In this case, there exists a singular vector in $M(\lambda)_{\lambda - m\alpha}$, and it is often useful to have a more or less explicit description of this vector*. However, this description almost never exists. Below we discuss a rather enigmatic formula from the article of Malikov, Feigin, and myself [5].

3.5.1. An example: the case of A_1^1. (See Section 2.2.4.) Consider a Verma module $M(\lambda_1, \lambda_2)$ where λ_1 and λ_2 satisfy the equation $k\lambda_1 + (k-1)\lambda_2 + 2k - 1 - m = 0$; a parametric equation of the line determined by this equation is

$$\lambda_1 = m - 1 - (k-1)t, \ \lambda_2 = -m - 1 + kt.$$

Proposition 3.12. *For λ as above,*

$$f_1^{m+(k-1)t} f_2^{m+(k-2)t} f_1^{m+(k-3)t} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t} v_\lambda \tag{5}$$

is annihilated by e_1 and e_2.

Remark. Strictly speaking, this expression makes sense only if all the exponents $m + jt$, $|j| \leq k-1$, are non-negative integers. However, we will be able to extend the applicability of Proposition 3.12 to all positive integers m, k and all $t \in \mathbb{C}$.

Lemma 1. *For any g, h in any Lie algebra the following holds:*

$$gh^s = h^s g + \sum_{u \geq 1} \binom{s}{u} h^{s-u} [...[\,[g, \underbrace{h], h], \dots h}_{u}].$$

Proof of Lemma 1. Induction with respect to s:

$$\begin{aligned} gh^{s+1} &= gh^s h = h^s gh + \sum_{u \geq 1} \binom{s}{u} h^{s-u} [...[\,[g, \underbrace{h], h], \dots h}_{u}] h \\ &= h^{s+1} g + h^s [g, h] + \sum_{u \geq 1} \binom{s}{u} h^{s+1-u} [...[\,[g, \underbrace{h], h], \dots h}_{u}] \\ &\qquad + \sum_{u \geq 1} \binom{s}{u} h^{s-u} [...[\,[g, \underbrace{h], h], \dots h}_{u+1}] \end{aligned}$$

---

* Imagine, for example, that some $\mathfrak{g}$-module has a singular vector w of type λ. Then w is annihilated by any element of $\mathfrak{n}_+$ and the action of $\mathfrak{h}$ on w is determined by λ. But is it possible that $gw = 0$ for some non-zero $g \in \mathfrak{n}_-$, or even $\in U(\mathfrak{n}_-)$? We know that if $M(\lambda)$ is irreducible then it is not possible at all, and if $M(\lambda)$ is reducible, then it is possible only if gw is contained in the maximal proper submodule of $M(\lambda)$. Singular vector in $M(\lambda)_{\lambda-\eta}$, $0 \neq \eta \in \Lambda_+$ generates a submodule in $M(\lambda)$ isomorphic to $M(\lambda - \eta)$. Thus, a knowledge of singular vectors in $M(\lambda)$ is important for studying all representations of $\mathfrak{g}$, not necessarily Verma modules.

$$= h^{s+1}g + \binom{s}{0} h^{s+1-1}[g,h] + \sum_{u\geq 1}\binom{s}{u} h^{s+1-u}[\ldots[\,[g,\underbrace{h],h],\ldots h}_{u}]$$

$$+ \sum_{u\geq 1}\binom{s}{u} h^{s+1-(u+1)}[\ldots[\,[g,\underbrace{h],h],\ldots h}_{u+1}]$$

$$= h^{s+1}g + \sum_{u\geq 1}\left[\binom{s}{u} + \binom{s}{u-1}\right] h^{s+1-u}[\ldots[\,[g,\underbrace{h],h],\ldots h}_{u}]$$

$$= h^{s+1}g + \sum_{u\geq 1}\binom{s+1}{u} h^{s+1-u}[\ldots[\,[g,\underbrace{h],h],\ldots h}_{u}]$$

A similar formula exists for switching g^s with h^v:

Lemma 2.

$$g^s h^t = \sum_{u,v\geq 0}\binom{s}{u}\binom{t}{v} h^{t-v} g^{s-u} Q_{uv}(g,h) \tag{6}$$

where $Q_{uv}(g,h)$ is a polynomial of commutators of g and h (not depending of s and t). These polynomials satisfy the conditions $Q_{00}(g,h) = 1$, $Q_{u0}(g,h) = 0$, if $u > 0$, $Q_{0v}(g,h) = 0$, if $v > 0$, and, for $u, v > 0$,

$$Q_{uv}(g,h) = [Q_{u,v-1}(g,h),h] + \sum_{w\geq 1}\binom{u}{w}[\underbrace{g\ldots,[g,[g}_{w},h]\,]\ldots]Q_{u-w,v-1}(g,h).$$

These properties determine $Q_{uv}(g,h)$ uniquely.

The proof is straightforward (although it requires a lengthy computation); we leave it to the reader. As well, we leave to the reader the proof of the following corollary of Lemma 2.

Corollary 3.13. *If $[\underbrace{g\ldots,[g,[g}_{p+1},h]\,]\ldots] = 0$ and $u > pv$, then $Q_{u,v}(g,h) = 0$. Similarly, if $[\underbrace{h\ldots,[h,[h}_{q+1},g]\,]\ldots] = 0$ and $v > qu$, then $Q_{u,v}(g,h) = 0$.*

It is clear that the polynomial $Q_{uv}(g,h)$ has the degree u in g and the degree v in h. A simple computation (based on Lemma 2) shows that

$$Q_{u1}(g,h) = [\underbrace{g\ldots,[g,[g}_{u},h]\,]\ldots],\ Q_{1v}(g,h) = [\underbrace{h\ldots,[h,[h}_{v},g]\,]\ldots]$$

$$Q_{22}(g,h) = [\,[g,[g,h]\,],h] + 2[g,h]^2.$$

A beginning of the formula (5) for $u \leq 2, v \leq 2$ looks as follows

$$g^s h^t = h^t g^s + st h^{t-1} g^{s-1}[g,h] + \binom{s}{2} t h^{t-1} g^{s-2}[g,[g,h]\,]$$

$$+ s\binom{t}{2} h^{t-2} g^s[\,[g,h],h] + \binom{s}{2}\binom{t}{2} h^{t-2} g^{s-2}([\,[g,[g,h]\,],h] + 2[g,h]^2) + \ldots$$

REMARK. It is obvious that the sum in the right hand side of the formula (6) is finite ($u \le s, v \le t$), if s and t are positive integers. It may be less obvious, but is also true that the sum is finite If s is a positive integer and $\underbrace{[g\dots,[g,[g}_{p+1},h]]\dots] = 0$: in this case, according to Corollary 3.13, $Q_{u,v}(g,h) = 0$ if $v > pu$; hence, non-zero summands in (6) may appear only if $u \le s$ and $v \le 2u (\le 2s)$. (Certainly, in this remark you can simultaneously swap $u \leftrightarrow v$, $g \leftrightarrow h$.)

Our goal is to legalize considering monomials $g_1^{\alpha_1} g_2^{\alpha_2} \dots g_s^{\alpha_s}$ where $g_1, g_2, \dots, g_s$ are elements of some Lie algebra (some of which can be the same) and $\alpha_1, \alpha_2, \dots, \alpha_s$ are complex numbers. We will try to transform such an expression to an element of the universal enveloping algebra by means of formulas like (6). It may work, if the total degree of every g_i in this monomial is a positive integer. Then we can get an expression where all exponents are integers. However, even in this case, we can arrive at monomials with negative exponents; moreover, it may happen that the result will be rather an infinite series, like this: $\sum_{u_1,u_2,\dots,u_t \in \mathbb{Z}_{\ge 0}} g_1^{k_1-u_1} \dots g_t^{k_t-u_t} R_{u_1,\dots,u_t}(g_1,\dots,g_t)$ where $R_{u_1,\dots,u_t}$ are commutator polynomials. We hope that all undesirable terms will vanish (because, maybe, of relations in our Lie algebra), and if it happens, we say that our monomial $g_1^{\alpha_1} g_2^{\alpha_2} \dots g_s^{\alpha_s}$ "*makes sense*". We will see that this is not impossible, and the monomial (5) is an important example of a monomial that does make sense.

Proof of Proposition 3.12. The computations below will be based on the following formulas (which are obvious, but also follow from Lemma 1):

$$e_i f_j^\alpha = \begin{cases} f_i^\alpha e_i + \alpha f_i^{\alpha-1} h_i - \alpha(\alpha-1) f_i^{\alpha-1}, & \text{if } j = i \\ f_j^\alpha e_i, & \text{if } j \ne i \end{cases}; h_i f_j^\alpha = \begin{cases} f_i^\alpha h_i - 2\alpha f_i^\alpha, & \text{if } j = i, \\ f_j^\alpha h_i + 2\alpha f_j^\alpha, & \text{if } j \ne i. \end{cases} \quad (7)$$

We want to prove that $e_1 f_1^{m+(k-1)t} f_2^{m+(k-2)t} f_1^{m+(k-3)t} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t} v_\lambda = 0$. For this purpose we take the e_1 through all the f's, using the formulas above. There are the following possibilities. First, our e_1 can reach safely the right end; in this case we get an expression ending with $e_1 v_\lambda$ which is zero; done with this possibility. Second, e_1 can interact interact with some $f_1^{m+(k-s)t}$. Then e_1 turns into h_1, $f_1^{m+(k-s)t}$ turns into $f_1^{m+(k-s)t-1}$, and there arises a coefficient $(m + (k - s)t)$. After that, the newborn h_1 continues moving right, and the following things can happen. First, this h_1 can interact once more with the same $f_1^{m+(k-s)t-1}$; then h_1 disappears and there arises an additional factor $-(m + (k - s)t - 1)$ (see formulas (7)). Second, ur h_1 can interact with some $f_{1 \text{ or } 2}^{m+(k-s')t}$; then again h_1 disappears and there arises the additional factor $\pm 2(m + (k - s)t)(m + (k - s')t)$, where the sign is $+$, if s' is odd and $-$, if s' is even (see formulas (7) again). Third, our h_1 can safely reach v_λ and interact with this v_λ and produces an additional factor λ_1.

Let us combine now all the above computations. What we obtain, is the monomial $f_1^{m+(k-1)t} f_2^{m-(k-2)t} \dots f_1^{m-(k-s)t-1} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t} v_\lambda$ times $m-(k-s)t$ and times the following sum:

$$
\begin{array}{l}
-(m+(k-s)t-1) \\
\left.\begin{array}{ll}
\quad -2(m-(k-(s+1)t)+2(m-(k-(s+2)t) & [=2t] \\
\quad -2(m-(k-(s+3)t)+2(m-(k-(s+4)t) & [=2t] \\
\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\quad -2(m-(k-(s-(2k-2)t)) & \\
\qquad\qquad +2(m-(k-(s-2k-1))t) & [=2t]
\end{array}\right\} 2t\frac{(2k-1)-(s+1)+1}{2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\lambda_1
\end{array}
$$

Thus, the coefficient at $f_1^{m+(k-1)t} f_2^{n-(k-2)t} \dots f_1^{m-(k-s)t-1} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t} v_\lambda$ in $e_1 f_1^{m+(k-1)t} f_2^{m+(k-2)t} f_1^{m+(k-3)t} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t} v_\lambda$ is $\lambda_1 - m + (k-1)t + 1$ which is zero by our condition on λ's.

Hence $e_1 f_1^{m+(k-1)t} f_2^{m+(k-2)t} f_1^{m+(k-3)t} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t} v_\lambda = 0$. Similar facts for e_2, and f_1, f_2 are proved similarly. This completes the proof of Proposition 3.12.

We can now formulate the first main result concerning A_1^1.

THEOREM 3.14 [5]. *The expression*

$$F(m,k,t) = f_1^{m+(k-1)t} f_2^{m+(k-2)t} f_1^{m+(k-3)t} \dots f_2^{m-(k-2)t} f_1^{m-(k-1)t}$$

makes sense for any $m, k \in \mathbb{Z}_{>0}$ and any $t \in \mathbb{C}$. If $\lambda_1 = m-1-(k-1)t$, $\lambda_2 = -m-1+kt$, then $F(m,k,t)v_\lambda$ is a singular vector of $M(\lambda)$ of weight $\lambda - m(k\alpha_1 + (k-1)\alpha_2)$. All these assertions remain true, if we do a swap $\lambda_1 \leftrightarrow \lambda_2$, $f_1 \leftrightarrow f_2$, $\alpha_1 \leftrightarrow \alpha_2$.

The part of this theorem, which remains to be proved, is the making sense assertion. In the article cited above, we deduce it from some known constructions in the representation theory that for given m, k the expression makes sense for infinitely many values of t, which, certainly, implies the result. Still, it is true that the fact can be proven by direct computations (making use of Remark after Corollary 3.13). To encourage the reader to make these direct computations, I show such a computation in the first non-trivial case, namely for $k = 2$:

$$
\begin{aligned}
f_1^{m+t} f_2^m f_1^{m-t} &= f_1^{m+t} f_1^{m-t} f_2^m + \sum_{u,v>0} \binom{m}{u}\binom{m+t}{v} f_1^{m+t} f_1^{m-t-v} f_2^{m-u} Q_{uv}(f_2, f_1) \\
&= f_1^{2m} f_2^m + \sum_{u,v>0} \binom{m}{u}\binom{m+t}{v} f_1^{2m-v} f_2^{m-u} Q_{uv}(f_2, f_1).
\end{aligned}
$$

But $\binom{m}{u} = 0$, if $u > m$, and $Q_{uv}(f_2, f_1) = 0$, if $v > 2u$ (see Corollary 3.13), hence if $u \le m, v > 2m$. Thus, our expression contains f_1 and f_2 only to non-negative integral powers.

For example,

$$f_1^{1+t} f_2 f_1^{1-t} = f_1^2 f_2 + (1-t) f_1 [f_2, f_1] - \frac{(1-t)t}{2} [\,[f_2, f_1], f_1].$$

Finally, consider the case of λ satisfying the equation $\lambda_1+\lambda_2+2=0$. In this case, there should be a singular vector in $M(\lambda)_{\lambda-(\alpha_1+\alpha_2)}$, and this vector generates a submodule of $M(\lambda)$ isomorphic to $M(\lambda-(\alpha_1+\alpha_2))$. But $(\lambda-(\alpha_1+\alpha_2))(h_1)=\lambda_1-(2-2)=\lambda_1$, and, similarly, $(\lambda-(\alpha_1+\alpha_2))(h_2)=\lambda_2$. We see that the submodule is isomorphic to the whole module $M(\lambda)$. Then it contains a singular vector of its own, the latter generates a submodule which is again isomorphic to $M(\lambda)$, and so on. Below, we give explicit formulas for many singular vectors, and, actually, as it was shown by Malikov [6], our construction gives all of them.

Let us introduce notations for some elements $\mathfrak{n}_-$:

$$b_1=f_1,\ b_2=f_2,\ b_3=[f_1,b_2],\ b_4=-[f_1,b_3],\ b_5=[f_2,b_3],$$

and, by induction,

$$b_{3k}=[f_1,b_{3k-1}],\ b_{3k+1}=-[f_1,b_{3k}],\ b_{3k+2}=[f_2,b_{3k}].$$

(Notice that the projection $\mathfrak{g}(A)\to\mathfrak{sl}(2)\otimes\mathbb{C}[t,t^{-1}]$ takes b_{3k-1},b_{3k},b_{3k+1} into $e\otimes t^{-k}, h\otimes t^{-k}, f\otimes t^{-k}$.)

These $b_1,b_2,b_3,\ldots$ form a basis in $\mathfrak{n}_-$, and they satisfy the relations $[b_i,b_j]=a_{ij}b_{i+j}$ where $a_{ij}=0$ or ±1 and $a_{ij}\equiv j-i \bmod 3$. (This is Exercise 6.12.)

LEMMA. (1) $[h_1,b_j]=\alpha_j b_j$ *where* $\alpha_j=\begin{cases}0, & \text{if } j\equiv 0 \bmod 3,\\ -2, & \text{if } j\equiv 1 \bmod 3,\\ 2, & \text{if } j\equiv 2 \bmod 3;\end{cases}$ $[h_2b_j]=-[h_1,b_j]$ *for all* j.

(2) $[e_1,b_1]=h_1$*; for* $j\neq 1$, $[e_1,f_j]=\beta_j b_{j-1}$ *where* $\beta_j=\begin{cases}2, & \text{if } j\equiv 0 \bmod 3,\\ -2, & \text{if } j\equiv 1 \bmod 3,\\ 0, & \text{if } j\equiv 2 \bmod 3.\end{cases}$

(3) $[e_2,b_2]=h_2$*; for* $j\neq 2$, $[e_2,b_j]=\gamma_j b_{j-2}$ *where* $\gamma_j=\begin{cases}-2, & \text{if } j\equiv 0 \bmod 3,\\ 0, & \text{if } j\equiv 1 \bmod 3,\\ 2, & \text{if } j\equiv 2 \bmod 3.\end{cases}$

Proof: a direct computation by induction.

THEOREM 3.15. [5]. *Let* $\lambda_1+\lambda_2+2=0$*, and let*

$$B_k=b_1b_{3k-1}+b_2b_{3k-2}+\ldots+b_{3k-1}b_1-(\lambda_1+1)b_{3k}.$$

Then, for any $k_1,\ldots,k_q$,

$$B_{k_q}B_{k_{q-1}}\ldots B_{k_1}v_\lambda$$

is a singular vector in $M(\lambda)_{\lambda-(k_1+\ldots+k_q)(\alpha_1+\alpha_2)}$*. Moreover, the operators* B_k *commute with each other, and the singular vectors shown above are linearly independent.*

Proof. The relation $B_kB_{k'}=B_{k'}B_k$ is checked directly, the linear independence of the singular vectors shown follows from PBW. For the main statement, it is sufficient to check that $e_1B_kv_\lambda=e_2B_kv_\lambda=0$; it is also checked directly, but let us do it.

$$\begin{aligned} e_1b_1b_{3k-1}&=h_1b_{3k-1}+b_1b_{3k-1}e_1=b_{3k-1}(h_1+2)+b_1b_{3k-1}e_1,\\ e_1b_{3k-1}b_1&=b_{3k-1}h_1+b_{3k-1}b_1e_1,\\ e_1b_{3k}&=2b_{3k-1}+b_{3k}e_1,\end{aligned}$$

and for $j \neq 1, 3k-1$,

$$e_1 b_j b_{3k-j} = \left\{ \begin{array}{lr} 2b_{j-1}b_{3k-j}, & \text{if } j \equiv 1 \bmod 3 \\ \qquad\qquad - 2b_j b_{3k-j-1}, & \text{if } j \equiv 0 \bmod 3 \\ 2b_{j-1}b_{3k-j} + 2b_j b_{3k-j-1}, & \text{if } j \equiv 2 \bmod 3 \end{array} \right\} + b_j b_{3k-j} e_1$$

The summation shows that

$$\begin{aligned} e_1 B_k v_\lambda &= [b_{3k-1}(h_1+2) + b_{3k-1}h_1 - 2(\lambda_1+1))b_{3k-1} + B_k e_1]v_\lambda \\ &= (\lambda_1 + 2 + \lambda_1 - 2(\lambda_1+1))b_{3k-1}v_\lambda = 0. \end{aligned}$$

Similarly, $e_2 B_k v_\lambda = 0$.

3.5.2. The general case. Now we will generalize the results of Section 3.4.1 to the case of arbitrary Kac-Moody algebras. The proofs are similar to proofs in Section 3.4.1, and we will not repeat them.

3.5.2.1. The Weyl group. The notion of the Weyl group is one of the main notions of the classical Like theory. It is important that it is naturally extended to the Kac-Moody theory, but we did not need it up to now. Now, it is time to introduce it.

Let $\lambda, \mu \in \mathfrak{h}^*$ and $\langle \mu, \mu \rangle \neq 0$. Put

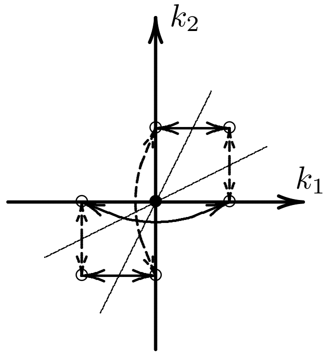
$$s_\mu(\lambda) = \lambda - \frac{2\langle \mu, \lambda \rangle}{\langle \mu, \mu \rangle}\mu.$$

We get a linear transformation $s_\mu\colon \mathfrak{h}^* \to \mathfrak{h}^*$ with the following properties: $s_\mu^2 = \mathrm{id}$, $s_\mu(\mu) = -\mu$, $s_\mu(\lambda) = \lambda$, if $\langle \mu, \lambda \rangle = 0$; in other words, s_μ is a reflection of $\mathfrak{h}^*$ in the orthogonal complement $\{\lambda \mid \langle \mu, \lambda \rangle = 0\}$ of μ in the direction of μ. Put $s_i = s_{\alpha_i}$ and let W be the group of transformations of $\mathfrak{h}^*$ generated by $s_1, \dots, s_n$. This group is called the *Weyl group* of $\mathfrak{g}(A)$.

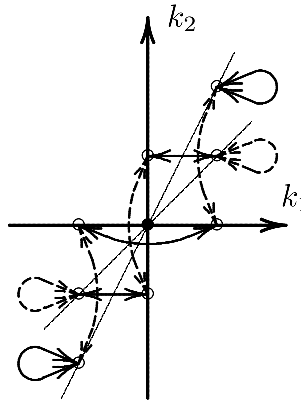
Notice that $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. In particular, $\mathbf{A} = \mathrm{span}(\alpha_1, \dots, \alpha_n)$ is W-invariant, and s_i's are identities on the orthogonal complement of $\mathbf{A}$; thus, W may be regarded as the group of transformations of $\mathbf{A}$. It is clear also that the action of W preserves the inner product $\langle\, , \rangle$.

It is true also that the action of W *always take roots into roots*; this is Exercise 6.13.

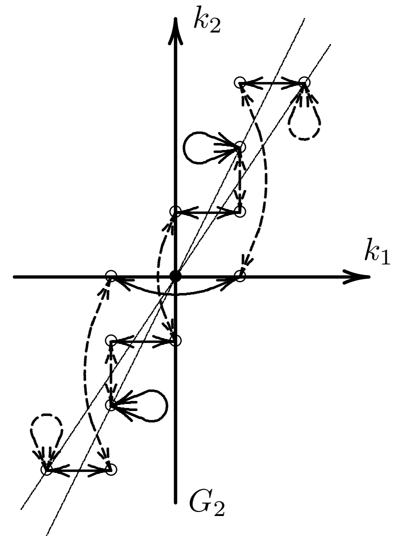
EXAMPLES. (1) The picture below shows the action of the Weyl groups in the sets of roots of the Lie algebras $\mathfrak{sl}(3), \mathfrak{so}(5)$, and G_2; solid arrows show the action of s_1, and dashed arrows show the action of s_2.



$\mathfrak{sl}(3)$



$\mathfrak{so}(5)$



G_2

Our next example contains a description of the Weyl group for $\mathfrak{sl}(n+1)$, which, certainly may be applied to $\mathfrak{sl}(3)$. As to the two other example above, the description of the Weyl groups and their orbits in the sets of roots is left to the reader.

(2) Let $\mathfrak{g}(A) = \mathfrak{sl}(n+1)$. In this case $\alpha_1, \ldots, \alpha_n$ is a basis of $\mathfrak{h}^*$, and the action of reflections s_i is the following:

$$s_i(\alpha_j) = \begin{cases} -\alpha_j, & \text{if } j = i, \\ \alpha_j + \alpha_i, & \text{if } j = i \pm 1, \\ \alpha_j & \text{otherwise.} \end{cases}$$

Expand $\mathfrak{h}^*$ by adding one more basis vector, α_{n+1}, put $s_i(\alpha_{n+1}) = \alpha_{n+1}$ for all i; then make a basis change $\alpha_1 = \beta_1 - \beta_2, \ldots, \alpha_n = \beta_n - \beta_{n+1}, \alpha_{n+1} = \beta_1 + \ldots + \beta_{n+1}$. The transformation s_i becomes a transposition $\beta_i \leftrightarrow \beta_{i+1}$. Thus, W is the symmetric group $S(n+1)$.

(3) Let $\mathfrak{g}(A) = A_1^1$. Then the action of the Weyl group is

$$s_1(\alpha_1) = -\alpha_1, \qquad s_1(\alpha_2) = 2\alpha_1 + \alpha_2,$$
$$s_2(\alpha_1) = \alpha_1 + 2\alpha_2, \qquad s_2(\alpha_2) = -\alpha_2.$$

Return to the diagram (A_1^1) in Section 2.3.3; we see that s_1 and s_2 are reflections in the flashed line, in the horizontal and vertical direction respectively. (Similar holds for $\mathfrak{g}(A) = A_2^2$: see diagram (A_2^2).) As a group, W (for A_1^1 and A_2^2, as well, actually, as for all infinite-dimensional Kac-Moody algebras of rank 2) is a free product of two copies of $\mathbb{Z}/2\mathbb{Z}$; its elements are

$$s_{1 \text{ or } 2} \ldots s_1 s_2 s_1 s_2 s_1 s_2 \ldots s_{1 \text{ or } 2}.$$

Returning to the case of A_1^1, put $S_k = \underbrace{s_1 s_2 s_1 \ldots s_1}_{2k-1}$; it is easy to check that

$$S_k(\alpha_1) = -(2k-1)\alpha_1 - (2k-2)\alpha_2, \; S_k(\alpha_2) = 2k\alpha_1 + (2k-1)\alpha_2$$

and to deduce from that that $S = s_\alpha$ where $\alpha = k\alpha_1 + (k-1)\alpha_2$. You may wish to repeat all this with the interchange $1 \leftrightarrow 2$.

Similar statements hold for A_2^2: see Exercise 6.14.

We will need also transformations $s_\mu^\rho\colon \mathfrak{h}^* \to \mathfrak{h}^*$ given by $s_\mu^\rho(\lambda) = s_\mu(\lambda+\rho) - \rho$ (we assume, as before, that a $\rho \in \mathfrak{h}^*$ with $\rho(h_i) = 1$ for all i is fixed). It is the reflection in the direction of μ in the (affine) hyperplane parallel to the orthogonal complement to μ and passing through $-\rho$. In general, this is not a linear transformation: $s_\mu^\rho(0)$ does not need to be zero (but $s_\mu^\rho(-\rho) = -\rho$).

Below, we will use the notation φ^ρ for all tranformations $\varphi\colon \mathfrak{h} \to \mathfrak{h}$. Namely, $\varphi^\rho(\lambda) = \varphi(\rho+\lambda) - \rho$.

3.5.2.2. Real and imaginary roots. The statements made in connection with Example (2) have, actually, far-reaching generalizations. Call a root α of $\mathfrak{g}(A)$ *real*, if the only root proportional to α is $-\alpha$. Roots, which are not real, are called *imaginary*. It is a common knowledge that *all roots of finite-dimensional simple (and semisimple) Lie algebras are real* (so the notions of real and imaginary roots do not belong to the classical Lie theory). But for the Kac-Moody algebras these notions are important. From the root diagrams for A_1^1 and A_2^2, we see that roots on the flashed lines are imaginary and all the other roots are real. Similar situation occurs for all *affine algebras* (Section 2.3.3). As to more general case, I have neither desire nor opportunity to discuss the details (I can recommend the book "Infinite-dimensional Lie algebras" by Victor Kac [4], especially the last, 3-rd, edition.) But I provide some important properties of real and imaginary roots in a proposition below.

Proposition 3.16. (1) *If α is an imaginary root of $\mathfrak{g}(A)$, then $m\alpha$ is also a root for all non-zero integers m.*

(2) *Real roots are precisely those from the Weyl group orbits of the simple roots $\alpha_1, \ldots, \alpha_n$; in particular, all real roots have positive inner squares.*

(3) *Imaginary roots have non-positive inner squares.*

(4) *The set of elements of W which are reflections in hyperplanes coincides with the set of reflections s_α for all real roots of α of $\mathfrak{g}(A)$.*

For affine algebras, it is true that every imaginary root has zero inner square, and they all belong to the hyperplane $d_1\alpha_1 + \ldots + d_n\alpha_n$. For A_1^1 and A_2^2 it is visible from our diagrams, and, frankly, the proof for all affine algebras is not too hard. As to the other Kac-Moody algebras, I will illustrate the proposition for the case of $\mathfrak{g}\begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix}$ (see the diagram on the next page). The diagram repeats that on Page 20, but I drop the dimensions of the root spaces. Also on the diagram below I restrict myself to the $\mathfrak{n}_+$-domain (the $\mathfrak{n}_-$-domain is symmetric).

The roots on the axes are α_1 and α_2. Their Weyl group orbits of these roots marked with the letters A and B (the two generators of the Weyl group are the reflection of all rows and the reflection of all columns). We see that there are only 11 real roots within the picture: α_1, α_2, $\alpha_1+\alpha_2$, $\alpha_1+4\alpha_2$, $\alpha_1+5\alpha_2$, $3\alpha_1+4\alpha_2$, $3\alpha_1+11\alpha_2$, $4\alpha_1+5\alpha_2$, $4\alpha_1+15\alpha_2$, $8\alpha_1+11\alpha_2$, $11\alpha_1+15\alpha_2$. The inner products $\langle\alpha_1,\alpha_j\rangle$ are entries a_{ij}^{sym} of the symmetrized Cartan matrix. Thus, for the example above, $\langle\alpha_1,\alpha_1\rangle = 2, \langle\alpha_1,\alpha_2\rangle = -1$, and $\langle \alpha_2, \alpha_2 \rangle = \dfrac{2}{5}$. From this, $\langle k_1\alpha_1 + k_2\alpha_2, k_1\alpha_1 + k_2\alpha_2 \rangle = 2k_1^2 - 2k_1k_2 + \dfrac{2}{5}k_2^2$, and the root $k_1\alpha_1 + k_2\alpha_2$ is *imaginary*, if $0.2764 \approx \dfrac{5-\sqrt{5}}{5} < \dfrac{k_1}{k_2} < \dfrac{5+\sqrt{5}}{5} \approx 0.7236$. The main impression is that for the "hyperbolic" Kac-Moody algebras real roots are rather rare. All the roots between the two slight lines are imaginary.

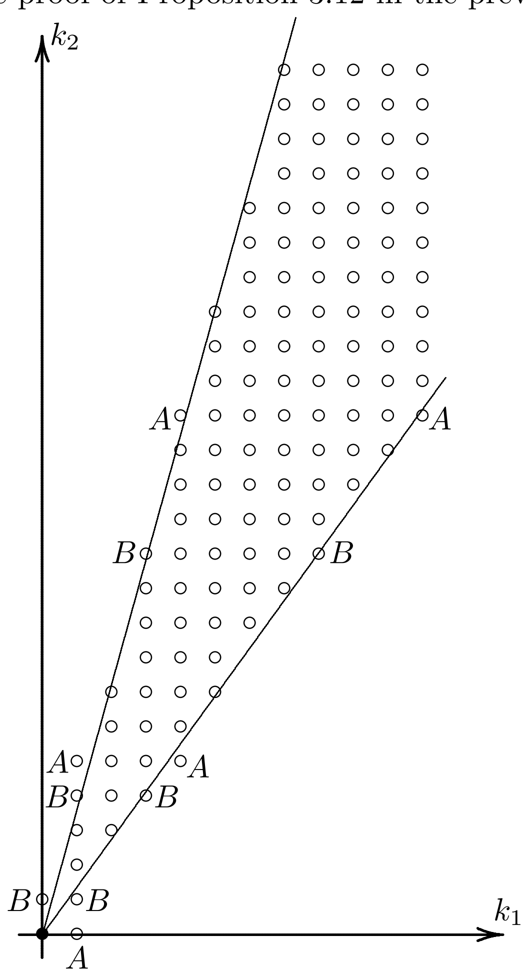
3.5.2.3. Formulas for singular vectors. Take some $\lambda \in \mathfrak{h}^*$ and some sequence $i_1, i_2, \dots, i_N$ of integers between 1 and n. Put $\lambda^0 = \lambda$, $\lambda^1 = s^\rho_{i_1}(\lambda^0), \dots, \lambda^N = s^\rho_{i_N}(\lambda^{N-1})$. The vector $\overrightarrow{\lambda^{m-1}\lambda^m}$ is collinear with α_{i_m}; define $\gamma_m \in \mathbb{C}$ by the condition $\overrightarrow{\lambda^{m-1}\lambda^m} = \gamma_m\alpha_{i_m}$.

PROPOSITION 3.17.

$$f_{i_N}^{-\gamma_N} \dots f_{i_1}^{-\gamma_1} v_\lambda \tag{8}$$

is a singular vector in $M(\lambda)$. (We do not state here that the expression above "makes sense"; we assert only that it is annihilated by each e_j.)

Proof is similar to the proof of Proposition 3.12 in the previous section.



The expression (8) can make sense only it has an integral positive degree, that is, if $-\gamma_1\alpha_1 - \dots - \gamma_N\alpha_N \in \Lambda_+$. This degree is equal to $\overrightarrow{\lambda^N\lambda^0}$, that is, $w^\rho(\lambda) - \lambda$ where

$w = s_{i_N} \dots s_{i_1}$. For an $\eta \in \Lambda_+$, the equation $w^\rho(\lambda) - \lambda = \eta$ determines in $\mathfrak{h}^*$ an empty set, or an affine plane parallel to the plane of fixed points of w. The most essential case is when this plane has maximal possible dimension, that is if it is a hyperplane, that is, if w is a reflection, that is, if $w = s_\alpha$ for some real root α. But in this case $w^\rho(\lambda) - \lambda$ is proportional to α. We arrive at the conclusion that in this case we must have $s_\alpha^\rho(\lambda) - \lambda = m\alpha$ with a positive integral m, which is the same as the Kac-Kazhdan equation $\lambda(h_\alpha) + \rho(h_\alpha) = \dfrac{m\langle\alpha, \alpha\rangle}{2}$.

THEOREM 3.18 [5]. *Let α be a positive* **real** *root of $\mathfrak{g}(A)$, let $s_\alpha = s_{i_N} \dots s_{i_1}$ be a shortest possible presentation of s_α as a product of s_i's, let $\lambda \in \mathfrak{h}^*$ satisfy the Kac-Kazhdan equation $s_\alpha^\rho(\lambda) - \lambda = m\alpha$ where m is a positive integer. Let $\gamma_1, \dots, \gamma_N \in \mathbb{C}$ be defined as above. Then*

$$f_{i_N}^{-\gamma_N} \dots f_{i_1}^{-\gamma_1} v_\lambda$$

makes sense and is a singular vector in $M(\lambda)_{\lambda - m\alpha}$.

For imaginary roots α with negative $\langle\alpha, \alpha\rangle$ the method of [5] gives the following result. Let α and β are two such roots, and let $w\alpha = \beta$ for some $w = s_{i_N} \dots s_{j_1} \in W$. For a $\lambda \in \mathfrak{n}_-$, we put $\mu = \lambda - n\alpha$ and define sequences $\gamma_1, \dots \gamma_N; \delta_1, \dots, \delta_N$ by the formulas:

$$\lambda^0 = \lambda,\ \lambda^j = s_{i_j}^\rho(\lambda^{j-1}),\ \overrightarrow{\lambda^{j-1}\lambda^j} = \gamma_j \alpha_{i_j};$$
$$\mu^0 = \mu,\ \mu^j = s_{i_j}^\rho(\mu^{j-1}),\ \overrightarrow{\mu^{j-1}\mu^j} = \delta_j \alpha_{i_j}.$$

Suppose that we know a polynomial function $F_{m\alpha}$ on $\{\lambda \in \mathfrak{h}^* \mid s_\alpha^\rho(\lambda) - \lambda = m\alpha\}$ such that $F_{m\alpha}(\lambda)v_\lambda$ is a singular vector in $M(\lambda)_{\lambda - m\alpha}$. Then the following holds:

THEOREM 3.19 [5].

$$f_{i_1}^{\delta_1} \dots f_{i_N}^{\delta_N} F_{m\alpha}(\lambda) f_{i_N}^{-\gamma_N} f_{i_1}^{-\gamma_1} v_\mu$$

makes sense and is a singular vector in $M(w(\lambda + \rho) - \rho)_{w(\lambda+\rho)-\rho-m\beta}$.

In other words, if we already have some formula for singular vectors for some imaginary root α with negative $\langle\alpha, \alpha\rangle$, then we can derive from it formulas for all roots from the W-orbit of α.

Notice, in conclusion, that Theorem 3.15 also can be generalized at least to Kac-Moody algebras $\mathfrak{g}(A)$ from the Subclass 2.1 in Section 2.4, that is, to central extensions of the algebra $\mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}_0$ is a simple finite-dimensional Lie algebra. It is true (we know it for A_1^1) that all imaginary roots of the algebra $\mathfrak{g}(A)$ have zero inner squares.

Let $\{u_i\}$ be a basis in $\mathfrak{g}_0$ and $\{u^i\}$ be the dual (with respect to the Killing form) basis. For a positive integer n, set

$$T_n = \sum_{0<s<n} \sum_i (u_i \otimes t^{-s})(u^j \otimes t^{s-n}).$$

This is an element of $U(\mathfrak{n}_-(A)$.

Let now $\lambda \in \mathfrak{h}(A)^*$ satisfies the equation $(\lambda + \rho)(h_\alpha) = 0$ (this is a generalization of the equation $\lambda_1 + \lambda_2 = -2$ for A_1^1).

Theorem 3.20 [5] *For any positive* $k_1, \dots, k_N$,

$$T_{k_N} \dots T_{k_1} v_\lambda$$

is a singular vector in $M_(\lambda)$; *it does not depend on the order of* $k_1, \dots, k_N$.

Notice that we do not state that this formula gives *all* singular vectors of $M(\lambda)$.

For finite-dimensional Kac-Moody algebras the formulas above may be transformed to a traditional form without negative and non-real exponent. The results do not look attractive. Still for $\mathfrak{sl}(3)$ there are rather compact formulas for singular vectors (see Exercise 6.15).

With this results, we finish studying representations of Kac-Moody algebras and pass to the Virasoro algebra.

4. Virasoro algebra and its representations.

Although there is no commonly recognized classification of infinite-dimensional Lie algebras, there are several important classes of them: limits of classical Lie algebras (several versions of $\mathfrak{gl}(\infty), \mathfrak{o}(\infty)$, and so on); Cartan algebras of formal (or polynomial) vector fields; Kac-Moody algebras; current algebras; algebras of vector fields on manifolds; and some others. The Virasoro algebra does not belong to any of these classes, and constitutes a kind os separate class, consisting only of it (although, sometimes there appear some "higher Virasoro algebras," but they exist in a kind of shadow of the Virasoro algebra. This makes the Virasoro algebra an extremely interesting mathematical object, but still more it is known by its famous applications in Physics (the string theory, the conformal fields theory, etc.) It should be mentioned that the Virasoro algebra is named after the outstanding Argentine physicist Miguel Virasoro (1940-2021).

The Virasoro algebra is a *central extension* of the *Witt algebra*. So we begin with a description of the latter.

4.1. The Witt algebra.

4.1.1. Definition. The Witt algebra is a Lie algebra with the basis $\{e_i \mid i \in \mathbb{Z}\}$ and the commutator $[e_i, e_j] = (j-i)e_{i+j}$. (Compare with the descriptions of $A_1^1/\mathfrak{c}$ and $A_2^2/\mathfrak{c}$ in terms of b_i's in Section 3.1.5 and Exercise 6.12.) It may be considered as the Lie algebra of polynomial vector fields in $\mathbb{C}^*$: $e_i = x^{i+1}\dfrac{d}{dx}$. Notation: $\mathfrak{Witt}$.

Proposition 4.1. *The Lie algebra* $\mathfrak{Witt}$ *is simple.*

Proof. Let $I \subset \mathfrak{Witt}$ be an ideal, and let a non-trivial linear combination $g = \sum_{j=1}^N m_j e_{k_j}$ belong to I. The successive commutators $[e_0, g], [e_0, [e_0, g]], \dots$ are $\sum k_j^n e_{k_j}$, $n = 1, 2, \dots$, and the all belong to I. This shows that all e_{k_j} belong to I, and hence I is spanned by $e_i \mid i \in A$ for some $A \subset \mathbb{Z}$; but since $[e_i, e_j] = (j-i)e_{i+j},\ j \in A$ implies $i + j \in A$ for all $i \neq j$ which shows that $A = \mathbb{Z}$ or 0).

Corollary 4.2. $\mathfrak{Witt}$ *has no non-trivial finite-dimensional representations.*

Indeed, the kernel of a non-zero homomorphism $\mathfrak{Witt} \to \mathfrak{gl}(n)$ would have been a proper ideal of $\mathfrak{Witt}$..

4.1.2. $\mathfrak{Witt}$-modules $\mathcal{F}_{\lambda\mu}$. The identification of e_i with $x^{i+1}\dfrac{d}{dx}$ makes $\mathbb{C}[x, x^{-1}]$ a $\mathfrak{Witt}$-module. If we denote $x^j \in \mathbb{C}[x, x^{-1}]$ by f_j, then $e_i f_j = j f_{i+j}$. The other major example of a representation of $\mathfrak{Witt}$ is the adjoint representation. If we denote e_j considered as a vector in the $\mathfrak{Witt}$-module $\mathfrak{Witt}$ by f_j, then the module structure will be described by the formula $e_i f_j = (j-i)f_{i+j}$. We obtain a generalization of both these representations, if we fix two complex numbers, λ and μ, and define $\mathcal{F}_{\lambda\mu}$ as a space with the basis $f_j, j \in \mathbb{Z}$ and the module structure $e_i f_j = (j + \mu - \lambda(i+1))f_{i+j}$ (an easy computation shows that $(e_i e_j - e_j e_i)f_k = (j-i)e_{i+j}f_k = [e_i, e_j]f_k$.

An analytic description of $\mathcal{F}_{\lambda\mu}$ is as follows: the space is the space of "forms" $x^\mu p(x)(dx)^{-\lambda}, p \in \mathbb{C}[x, x^{-1}]$, $f_j = x^{\mu+j}(dx)^{-\lambda}, e_i = x^{i+1}\dfrac{d}{dx}$ (as above). In particular, $\mathbb{C}[x, x^{-1}] = \mathcal{F}_{00}, \mathfrak{Witt} = \mathcal{F}_{1,1}$.

REMARKS. (1) The Witt algebra has a natural $\mathbb{Z}$-grading, $\deg e_i = i$. The module $\mathcal{F}_{\lambda\mu}$ is also graded: $\deg f_j = j$.

(2) There is a $\mathfrak{Witt}$-module isomorphism $\mathcal{F}_{\lambda,\mu+1} \to \mathcal{F}_{\lambda\mu}$ of degree 1: $f_j \mapsto f_{j+1}$

See also Exercises 6.16 and 6.17.

4.1.3. Dual, contragredient, and reverse modules. Let $M = \oplus_{k\in\mathbb{Z}} M_k$ be a graded $\mathfrak{Witt}$-module ($e_i(M_k) \subset M_{k+1}$) with all M_k finite-dimensional. We associate with M three graded modules: the dual module $M^* = \oplus_{k\in\mathbb{Z}}(M^*)_k$, the contragredient module $\overline{M} = \oplus_{k\in\mathbb{Z}}\overline{M}_k$, and the reverse module $M^\circ = \oplus_{k\in\mathbb{Z}} M_k^\circ$ in the following way:

$$(M^*)_k = (M_{-k})^*,\ [(M^*)_k \xrightarrow{e_i} (M^*)_{k+i}] = [(M_{-k-i} \xrightarrow{-e_i} M_{-k}]^*;$$

$$\overline{M}_k = (M_k)^*,\ [\overline{M}_k \xrightarrow{e_i} \overline{M}_{k+i}] = [M_{k+i} \xrightarrow{e_{-i}} M_k]^*;$$

$$M_k^\circ = M_{-k},\ [M_k^\circ \xrightarrow{e_i} M_{k+i}^\circ] = [M_{-k} \xrightarrow{-e_{-i}} M_{-k-i}].$$

It is clear that each of these operation repeated twice is the identity, and the composition of any two of these operations is the third one (dual to contragredient is reverse, etc.).

Let us apply these operations to the modules $\mathcal{F}_{\lambda\mu}$ of Section 4.1.2.

PROPOSITION 4.3. $\mathcal{F}^*_{\lambda\mu} = \mathcal{F}_{-1-\lambda,-1-\mu}, \overline{\mathcal{F}}_{\lambda\mu} = \mathcal{F}_{-1-\lambda,\mu-2\lambda-1}, \mathcal{F}^\circ_{\lambda\mu} = \mathcal{F}_{\lambda,-\mu+2\lambda}$.

Proof. Let $\{f_j^*\}$ be dual to $\{f_j\}$: $f_j^*(f_k) = \delta_{jk}$. Then in $\mathcal{F}^*_{\lambda\mu}, \overline{\mathcal{F}}_{\lambda\mu}, \mathcal{F}^\circ_{\lambda\mu}$ respectively

$$[e_i(f^*_{-k})](f_{-k-i}) = (f^*_{-k})(-e_i f_{-k-i}) = -(\mu-k-i-\lambda(i+1)) = ((-1-\mu)+k-(-1-\lambda)(i+1));$$

$$[e_i f_k^*](f_{k+i}) = f_k^*(e_{-i}f_{k+i}) = \mu + k + i - \lambda(-i-1) = (\mu - 2\lambda - 1) + k - (-1-\lambda)(i+1);$$

$$e_i(f_{-k}) = -e_{-i}f_{-k} = -(\mu - k - \lambda(-i+1))f_{-k-i} = ((-\mu+2\lambda) + k - \lambda(i+1))f_{-k-i}.$$

4.2. Construction of the Virasoro algebra.

The Virasoro algebra is a central extension of the Witt algebra. We have encountered central extensions before: for example, the Kac-Moody algebra A_1^1 is a central extension of the Lie algebra $\mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}]$. But we avoided considering the general operation of central extension of Lie algebras. But we cannot avoid it anymore.

4.2.1. Central extensions. A (one-dimensional) central extension of a Lie algebra $\mathfrak{g}$ is a Lie algebra $\widetilde{\mathfrak{g}}$ given with Lie algebra homomorphisms

$$\mathbb{C} \xrightarrow{i} \widetilde{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \tag{9}$$

($\mathbb{C}$ is regarded as a commutative Lie algebra) such that i is one-to-one, p is onto, $\ker(p) = i(\mathbb{C})$ and $i(\mathbb{C})$ is contained in the center of $\widetilde{\mathfrak{g}}$. There exists a convenient classification of central extensions of a given Lie algebra. Since p is onto, there exists a linear map (not necessarily a Lie algebra homomorphism) $q: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ such that $p \circ q = \mathrm{id}$. For $g, h \in \mathfrak{g}$, $[q(g), q(h)]$ does not need to be equal to $q[g,h]$; but $p([q(g), q(h)] - q[g,h]) = [p \circ q(g), p \circ q(h)] - p \circ q[g,h] = [g,h] - [g,h] = 0$. Hence, $[q(g), q(h)] - q[g,h] = i(c(g,h))$ for a unique $c(g,h)$. This function $c: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ (together with q and the Lie algebra structure of $\mathfrak{g}$) fully determines the central extension (9): for $\widetilde{g}, \widetilde{h} \in \widetilde{\mathfrak{g}}$,

$$[\widetilde{g}, \widetilde{h}] = q[p(\widetilde{g}), p(\widetilde{h})] - i(c(p(\widetilde{g}), p(\widetilde{h})]). \tag{10}$$

(*Proof*: $q[p(\widetilde{g}), p(\widetilde{h})] = [q \circ p(\widetilde{g}), q \circ p(\widetilde{h})] + i(c(p(\widetilde{g}), p(\widetilde{h})))$; furthermore, since $p(q \circ p(\widetilde{g}) - \widetilde{g}) = 0$, $x = q \circ p(\widetilde{g}) - \widetilde{g} \in i(\mathbb{C}) \subset \mathrm{center}(\widetilde{\mathfrak{g}})$ and, similarly, $y = q \circ p(\widetilde{h}) - \widetilde{h} \in \mathrm{center}(\widetilde{\mathfrak{g}})$; hence, $[q \circ p(\widetilde{g}), q \circ p(\widetilde{h})] = [\widetilde{g} + x, \widetilde{h} + y] = [\widetilde{g}, \widetilde{h}]$.)

We can take formula (10) (for linear maps i, p, q, c with properties listed above) for the definition of a Lie algebra $\widetilde{\mathfrak{g}}$, but we need to be sure that the axioms from the definition of a Lie algebra (anti-commutativity and the Jacobi identity) hold. Translated into the language of c, these mean that

$$c(g,h) = -c(h,g),\ c([g,h],k) + c([h,k],g) + c([k,g],h) = 0;$$

a bilinear function $c: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ with these properties is called a *cocycle* (or a 2-*cocycle*) of $\mathfrak{g}$.

Thus, a cocycle determines a central extension, and every central extension is determined by a cocycle. We need only to find out how much a central extension determined a cocycle. The transition from (7) to c involves a choice of q. For a different choice of q, some q' with $p \circ q' = \mathrm{id}$, we have $p \circ (q' - q) = 0$, that is, $q' - q = i \circ b$ for a $b: \mathfrak{g} \to \mathbb{C}$. For the corresponding cocycle c',

$$\begin{aligned} i \circ c'(p,q) &= [q'(g), q'(p)] - q'[g,h] = [q(g) + i \circ (b), q(h) + i \circ (h)] - (q + i \circ b)[g,h] \\ &= [q(g), q(h)] - (q + i \circ b)[g,h] = i \circ (c(g,h) - b[g,h]), \end{aligned}$$

that is, $c'(g,h) = c(g,h) - b[g,h]$. Cocycles c', c with this property are called *cohomologous*, $c' \sim c$. The vector space $\mathrm{cocycles}(\mathfrak{g})/\sim$ is called the (2-*dimensional*) *cohomology* of $\mathfrak{g}$ and is denoted as $H^2(\mathfrak{g})$. The main result of the construction above is *a one-to-one correspondence between the equivalence classes of central extensions of* $\mathfrak{g}$ *and* $H^2(\mathfrak{g})$.

Remarks. (1) The zero cocycle corresponds to the *trivial extension* $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ (a Lie algebra isomorphism).

(2) The multiplication of a cocycle by a non-zero constant corresponds to the multiplication of i by the same constant, that is, essentially, does not change the central extension.

In conclusion, we will explain that the notion of a central extension arises naturally in the representation theory. A *projective representation* of a Lie algebra $\mathfrak{g}$ in a vector space V is a linear map $\rho: \mathfrak{g} \to \mathrm{End}(V)$ such that for any $g, h \in \mathfrak{g}$, $\rho[g,h] - (\rho(g) \circ \rho(h) - \rho(h) \circ \rho(g) = c(g,h) \cdot \mathrm{id}$ for some $c(g,h) \in \mathbb{C}$. It is easy to check that this c must be a cocycle. Moreover, cohomologous cocycles c', c, $c'(g,h) = c(g,h) - b[g,h]$ correspond to "equivalent" projective representations ρ', ρ, $\rho'(g) = \rho + b(g)\cdot$. Thus, (equivalence classes of) projective representations of $\mathfrak{g}$ is the same as representations of central extensions of g with $\rho(i(a)) = a \cdot \mathrm{id}$.

4.2.2. A deviation: back to the affine algebras.

4.2.2.1. The cocycle. We noticed in Section 3.4 that the simplest infinite-dimensional Kac-Moody algebras, like A_1^1, are central extensions of "current algebras" $\mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}]$, which may be regarded as polynomials on $\mathbb{C}^*$ with value in (a finite-dimensional simple) complex Lie algebra $\mathfrak{g}_0$. The cocycle which gives rise to this central extension is $c(g \otimes t^m, h \otimes t^n) = \delta_{m+n,0}\langle g, h\rangle$, where $\langle\, ,\, \rangle$ is the Killing form on $\mathfrak{g}_0$. A real (and more analytic) version of this central extension is the following. The current algebra is the Lie algebra of, say, $\mathcal{C}^\infty$, functions on the circle S^1 with values in a real simple Lie algebra $\mathfrak{g}_0$ with the commutator $[\varphi, \psi](\theta) = [\varphi(\theta), \psi(\theta)]$, and the cocycle, which determines the central extension, is $c(\varphi, \psi) = \int_{S^1} \langle \varphi(\theta), \psi(\theta)\rangle d\theta$.

4.2.2.2. From central extension of Lie algebras to central extensions of Lie groups. The construction of central extensions (and the language of cocycles) exists in many different contexts. In algebra, the central extension of a group G by an Abelian group A is defined as a group $\widehat{G}$ given with an embedding of A into G as a central subgroup and with an isomorphism $\widehat{G}/A \cong G$.

There exists the following classification of such central extensions. They correspond to *cocycles* $c: G \times G \to A$ satisfying the rule $c(g, hk) = c(gh, k)$. Two cocycles are called *cohomologous*, if their difference has the form $c(g,h) = d(gh)$ for some function $d: G \to A$. Central extensions, which correspond to cohomologous cocycles are isomorphic.

In the Lie theory, the commonly considered situation is the following. G and $\widehat{G}$ are Lie groups, A is $\mathbb{R}$ (or S^1) and all homomorphisms are Lie homomorphisms. It is known that finite-dimensional simple Lie groups, as well as Lie algebras, do not have non-trivial central extensions; still, there are important examples. The simplest example is the so called Heisenberg algebra: it is the central extension of the two-dimensional commutative algebra with basis a, b. It is extended by one more basis element c with $[a,b] = c, [c,a] = 0, [c,b] = 0$.

The correspondence between Lie algebras and Lie groups, which exists in the classical finite-dimensional theory, exists also for central extensions. In particular, the Lie group, which corresponds to the Heisenberg algebra, is the group of real unipotent 3×3 matrices $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$; its center is $\{a = b = 0\} \cong \mathbf{R}$; the quotient over the center is the Abelian group $\mathbb{R}^2$.

4.2.2.3. The case of the current algebras. The "Lie group," which corresponds to the Lie algebra $\mathcal{C}^\infty(S^1, \mathfrak{g}_0)$ is the "current group" $\mathcal{C}^\infty(S^1, G_0)$, where G_0 is the Lie group, which corresponds to the Lie algebra $\mathfrak{g}_0$. So, what we are looking for, must be a central extension of the current group. But here we arrive at a huge disappointment: there exists a theorem that states that current groups have no central extension, either by $\mathbb{R}$, or by S^1.

But there exists a way out of this hopeless situation. The central extension (by S^1) of the current group $\mathcal{C}^\infty(S^1, G_0)$ is a group structure not on $\mathcal{C}^\infty(S^1, G_0) \times S^1$, but rather on the (infinite-dimensional) manifold, whose projection onto $\mathcal{C}^\infty(S^1, G_0)$ is a non-trivial fibration. Just for fun, I will describe this "manifold" in the case when G_0 is $SU(2)$, which topologically is the sphere S^3. The current group, topologically, is $\mathcal{C}^\infty(S^1, S^3)$. Consider the set of pairs (f, F), where $F\colon D^2 \to S^3$ and $f = F|_{S^1}$. We say that $(f, F_1) \sim (f, F_2)$, if F_1 and F_2 "cobound" zero volume (certainly, this volume is defined modulo the volume of S^3). Thus, the "manifold" of equivalence classes is fibered over $\mathcal{C}^\infty(S^1, S^3)$, and the fiber is S^1. The group structure in out extended "manifold" is induced by the group structure in $S^3 = SU(2)$.

4.2.3. Definition of the Virasoro algebra.

PROPOSITION 4.4. (1) *The formula*

$$c(e_i, e_j) = \frac{1}{12}\delta_{-i,j}(j^3 - j)$$

determines a cocycle of the Witt algebra.

(2) *This cocycle is not cohomologous to zero.*

Proof. (1) We need to check that $c([e_i, e_j], e_k) + c([e_j, e_k], e_i) + c([e_k, e_i], e_j) = 0$, and we can assume that $i+j+k = 0$ (otherwise the expression in the left hand side is $0+0+0$). In this case

$$\begin{aligned}
12(c([e_i, e_j], e_k) &+ c([e_j, e_k], e_i) + c([e_k, e_i], e_j)) \\
&= (j-i)(k^3-k) + (k-j)(i^3-i) + (i-k)(j^3-j) \\
&= [(j-i)k^3 + (k-j)i^3 + (i-k)j^3] - [(j-i)k + (k-j)i + (i-k)j] \\
&= (i+j+k)[(j-i)k^2 + (k-j)i^2 + (i-k)j^2] - 0 = 0.
\end{aligned}$$

(2) If $c(e_i, e_j) = b([e_i, e_j])$, then $c(e_{-1}, e_1) = 2b(e_0)$ and $c(e_{-2}, e_2) = 4b(e_0)$ which is impossible since $c(e_{-1}, e_1) = 0$ and $c(e_{-2}, e_2) = \frac{1}{2}$.

The central extension of the Witt algebra determined by the cocycle c is called the Virasoro algebra and is denoted as $\mathfrak{Vir}$. Thus, $\mathfrak{Vir}$ has the basis $\ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots; z$ and the commutator relations

$$[z, e_i] = 0, [e_i, e_j] = (j-i)e_{i+j} + \frac{1}{12}\delta_{-i,j}(j^3 - j)z.$$

Actually, the cocycle c represents the unique, up to a constant factor, 2-dimensional cohomology class of $\mathfrak{Witt}$ (this is Exercise 6.19).

HISTORICAL NOTES. The cocycle (proportional to) c first appeared (in the context of the Lie theory over fields of finite characteristic) in mid-60's (Block [6]). It was rediscovered in 1968 by Israel Gelfand and myself [7]; we, actually computed not just H^2, but the full cohomology H^* of $\mathfrak{Witt}$ (not defined in these lectures). Then the Virasoro algebra was rediscovered in the early 70's by physicists, for whom it was the algebra of infinitesimal symmetries in some important quantum field theories, in particular, in the string theory. The factor $\frac{1}{12}$ (which, actually, makes many formulas better looking) and, which is much more important, some basic constructions considered below (Fock spaces, vertex operators, etc.) are contributions of physicists.

4.3. Verma modules over the Virasoro algebra.

4.3.1. Generators and grading. The Virasoro algebra has a natural $\mathbb{Z}$-grading: $\deg e_i = i$, $\deg z = 0$. We split it in our usual way: $\mathfrak{Vir} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_-$ is spanned by e_i with $i > 0$ and $\mathfrak{n}_+$ is spanned by e_i with $i < 0$ (we accept this strange agreement to make our notations compatible with those of some key works on representations of the Virasoro algebra), $\mathfrak{h}$ is spanned by e_0 and z, and $\mathfrak{n}_+$ is spanned by e_i with $i < 0$. There arise Verma modules $M(\lambda)$ labeled by $\lambda \in \mathfrak{h}^*$; we will use for $M(\lambda)$ the notation $M(h,c)$ where $h = \lambda(e_0)$ and $c = \lambda(z)$. The module $M(h,c)$ has a basis $e_{i_1} \ldots e_{i_q} v,\ i_1 \leq \ldots \leq i_q$ arranged into "levels" $M(h,c)_k, k \geq 0$:

$$\begin{array}{cl}
v & \text{level } 0\\
e_1 v & \text{level } 1\\
e_1^2 v \quad e_2 v & \text{level } 2\\
e_1^3 v \quad e_1 e_2 v \quad e_3 v & \text{level } 3\\
e_1^4 v \quad e_1^2 e_2 v \quad e_1 e_3 v \quad e_2^2 v \quad e_4 v & \text{level } 4\\
\cdots\cdots\cdots\cdots\cdots\cdots &
\end{array}$$

It is clear from this diagram that $\dim M(h,c)_k = \mathbf{p}(k)$, the number of partitions of k. By the construction of the Verma modules, $e_{-i}(v) = 0$ for all $i > 0$ and $e_0(v) = hv, z(v) = cv$. The relations $[z, e_k] = 0$, $[e_0, e_k] = ke_k$ imply that z acts as the multiplication by c on the whole module $M(h,c)$ (the physicists would have said, the central charge is c) and e_0 acts on $M(h,c)_k$ as the multiplication by $h + k$.

4.3.2. Examples of reducible Verma modules. As before, it is important to find out, for which h, c the Verma module $M(h,c)$ is reducible, that is, contains singular vectors in some $M(h,c)_k$ with $k > 0$. A vector $w \in M(h,c)_k$, $k > 0$ is singular (and then it is of type $(h+k,c)$), if and only if $e_{-1}v = 0$, $e_{-2}v = 0$ (e_{-1} and e_{-2} generate $\mathfrak{n}_+$). For relatively small values of k, it is not hard to determine all h, c for which $M(h,c)_k$ contains a singular vector; for brevity's sake, we do it here assuming that $c = 0$.

For $k = 1$, we observe that $e_{-1}e_1 v = 2hv$, $e_{-2}e_1 v = 0$; hence, $M(h,c)_1$ contains a singular vector if and only if $h = 0$.

For $k = 2$, we observe that

$$\begin{aligned}
e_{-1}e_1^2 v &= 2(2h+1)e_1 v, & e_{-2}e_1^2 v &= 6hv,\\
e_{-1}e_2 v &= 3e_1 v, & e_{-2}e_2 v &= 4hv;
\end{aligned}$$

thus, a singular vector in $M(h,c)_2$ exists if and only if

$$\det \begin{bmatrix} 2(2h+1) & 6h \\ 3 & 4h \end{bmatrix} = 16h^2 - 10h = 0,$$

that is, if and only if $h = 0$ or $\dfrac{5}{8}$.

For $k = 3$, we find that

$$\begin{aligned} e_{-1}e_1^3v &= 6(h+1)e_1^2v, & e_{-2}e_1^3v &= 6(3h+1)e_1v,\\ e_{-1}e_1e_2v &= 3e_1^2v + 2(h+2)e_2v, & e_{-2}e_1e_2v &= (4h+9)e_1v,\\ e_{-1}e_3v &= 4e_2v, & e_{-2}e_3v &= 5e_1v; \end{aligned}$$

thus, $M(h,c)_3$ contains a singular vector if and only if

$$\det \begin{bmatrix} 6(h+1) & 0 & 6(3h+1) \\ 3 & 2(h+2) & 4h+9 \\ 0 & 4 & 5 \end{bmatrix} = -12(3h^2 - 7h + 2) = 0,$$

that is, if and only if $h = 2$ or $\dfrac{1}{3}$.

Further computations (which become more and more involved when k grows) show that $M(h,0)_4$ contains singular vectors (only) for $h = 1, \dfrac{1}{8}, \dfrac{33}{8}$; $M(h,0)_5$ contains singular vectors for $h = 0, 2, 7$; and $M(h,0)_6$ contains singular vector for $h = -\dfrac{1}{24}, 1, \dfrac{10}{3}, \dfrac{85}{8}$.

Remark. An attentive reader could notices that all the values of h, for which the module $M(h,0)$ has a singular vector have the form $\dfrac{m^2-1}{24}$. Actually, this is true and will be proved below; see Section 4.8.1.1.

4.3.3. Criterion of reducibility: statement. Below (in Section 4.5), we will prove the following criterion of reducibility of Verma modules over the Virasoro algebra.

For $p, q \in \mathbb{Z}_{>0}$, let Φ_{pq} be a curve in the plane $\mathbb{C}^2(h,c)$ with parametric equations

$$h = \frac{1-p^2}{4}t + \frac{1-pq}{2} + \frac{1-q^2}{4}t^{-1},$$
$$c = 6t + 13 + 6t^{-1}.$$

Obviously, $\Phi_{pq} = \Phi_{qp}$; if $p = q$ then it is the straight line $24h = (1-p^2)(c-1)$; if $p \neq q$, then Φ_{pq} is a conic, in $\mathbb{R}^2(h,c)$ a hyperbola.

The curves $\Phi_{p,q}$ are presented on a diagram on the next page.

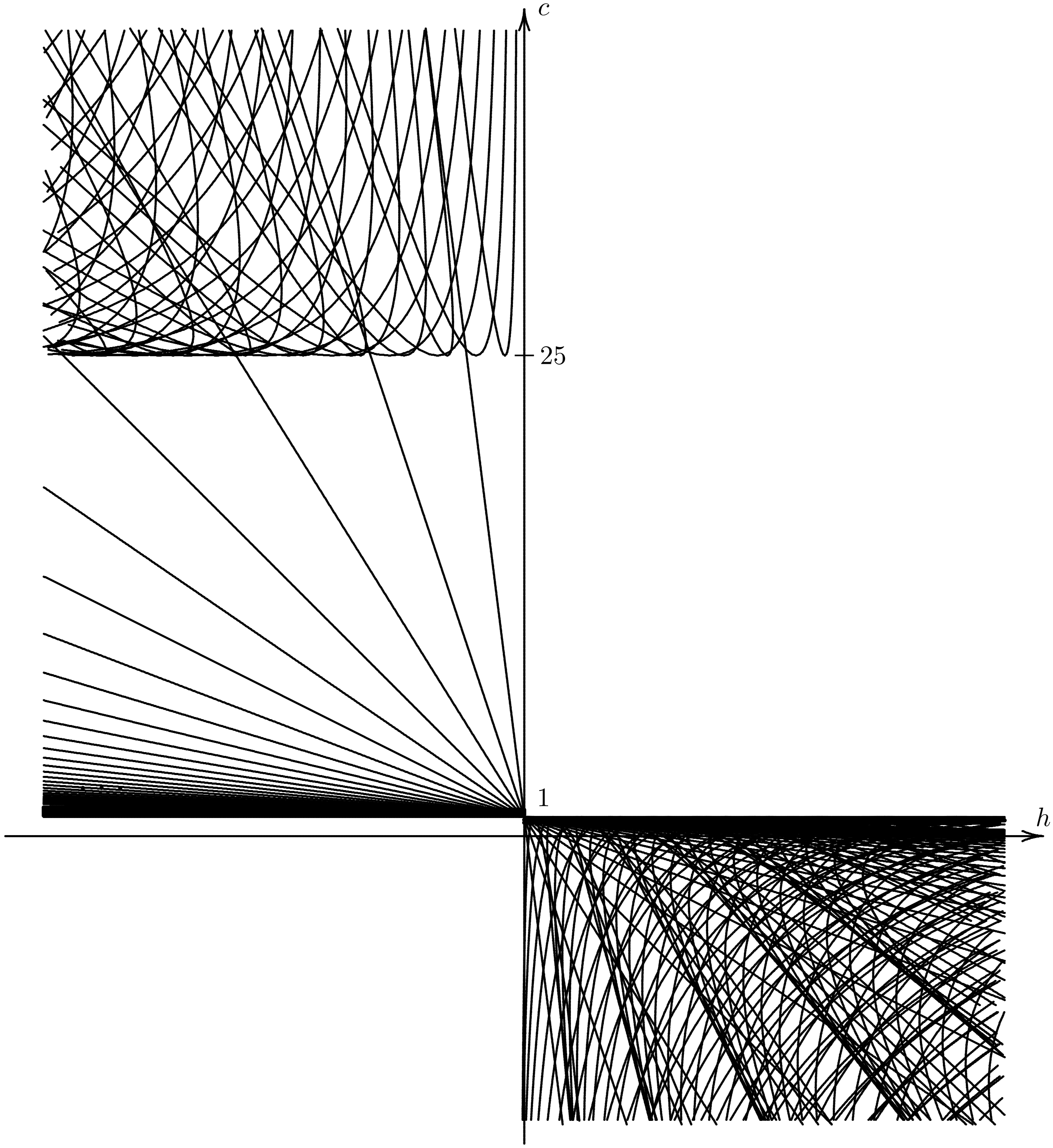
The lower branches of hyperbolas Φ_{pq} fill densely the domain $c \leq 0, h \geq \dfrac{1}{24}c$, the upper branches are contained in the domain $c \geq 25, h \leq 0$ but their union is nowhere dense.

Theorem 4.5. (1) *If (h,c) is not contained in any of Φ_{pq} then the Verma module $M(h,c)$ is irreducible.*

(2) *If $(h,c) \in \Phi_{pq}$ but is not contained in any $\Phi_{p'q'}$ with $p'q' < pq$, then there is a singular vector in $M(h,c)_{pq}$ and no singular vectors in $M(h,c)_k$ with $1 \leq k < pq$.*

We will give a proof and a more detailed statement in Section 4.5 below.

If $(h,c) \in \Phi_{pq}$ and $c = 0$, then $t = -\frac{2}{3}$ or $t = -\frac{3}{2}$ and $h = \frac{(2p-3q)^2-1}{24}$ or $\frac{(3p-2q)^2-1}{24}$. This gives, at least partially, an explanation to the computations above in this section.



4.3.4. Modules contragredient to Verma modules. There are several important two-parameter families of $\mathfrak{Vir}$-modules. One is the family of Verma modules. Another family will be constructed and studied in Section 4.4. One more will appear in Section 5.1.2. But now we will consider an immediate example: $\mathfrak{Vir}$-modules contragredient to

Verma modules.

The module $\overline{M(h,c)}$ has a significant resemblance to the module $M(h,c)$. First, $\dim \overline{M(h,c)}_k = \dim M(h,c)_k$. Second, $\overline{M(h,c)}_0 = \mathbb{C}\overline{v}$, where $\overline{v}(v) = 1$, and $\overline{v}$ is a singular vector of the type (h,c). Third, if $M(h,c)$ is irreducible, then $\overline{M(h,c)} \cong M(h,c)$, and if $M(h,c)$ is reducible, then $\overline{M(h,c)}$ is also reducible. But in this last case, the modules $M(h,c)$ and $\overline{M(h,c)}$ are sharply different.

PROPOSITION 4.6. (1) *The module $\overline{M(h,c)}$ has no singular vectors not proportional to $\overline{v}$.*

(2) *The submodule B of $\overline{M(h,c)}$ generated by $\overline{v}$ is isomorphic to the (irreducible) quotient of the module $M(h,c)$ over its maximal submodule $L(h,c) \not\ni v$.*

(3) $\overline{M(h,c)}/B \cong L(h,c)$.

Proof. (1) Let $f \in \overline{M(h,c)}_k$, $k > 0$ be a singular vector. Then, for any $x \in M(h,c)_{k-i}$, $f(e_i x) = e_{-i} f(x) = 0$. But $M(h,c)_k$ is spanned by the sets $e_i(M(h,c)_{k-i})$, which provides a contradiction.

(2) There exists a $\mathfrak{Vir}$-epimorphism $M(h,c) \to B$. Thus B is a quotient of $M(h,c)$. If B were reducible, it would have contained a singular vector not in $\mathbb{C}\overline{v}$, which is impossible by (1).

(3) follows from (2).

4.4. The $\mathfrak{Vir}$-module of semi-infinite forms.

For us, these modules (as well as the whole notion of semi-infinite forms) will serve as a tool for proving our main result (Theorem 4.5). To establish reducibility or irreducibility of Verma modules, we need the information about singular vectors in this modules. But constructing of these vectors turns out to be a very difficult problem (we will discuss some results in this direction in Section 4.6). Our main idea is find sufficiently many singular vectors in some other modules, and with their help to approach the problem of reducibility of Verma modules. These "other modules" will be the modules of semi-infinite forms.

4.4.1. Definition. Let $\lambda, \mu \in \mathbb{C}$. In Section 4.1.2, we considered $\mathfrak{Witt}$- (and $\mathfrak{Vir}$-) modules $\mathcal{F}_{\lambda\mu}$ with a basis $\{f_j\}$. Now we consider infinite monomials

$$\ldots \wedge f_{j_3} \wedge f_{j_2} \wedge f_{j_1}, \ \ldots < j_3 < j_2 < j_1, \ j_k = -k \text{ for almost all } k.$$

For a monomial F as above, we set $\deg F = \sum_k (j_k + k)$. Then the number of monomials of a given degree k is $\mathbf{p}(k)$, the number of partitions $k = k_1 + \ldots + k_s$, $k_1 \geq \ldots \geq k_s$: for a partition $\tau = \{k_1, \ldots, k_s\}$, we set

$$F_\tau = \ldots \wedge f_{-s-1} \wedge f_{-s+k_s} \wedge \ldots \wedge f_{-2+k_2} \wedge f_{-1+k_1}. \tag{11}$$

Finite linear combinations of monomial (of degree k) are called *semi-infinite forms* (of degree k), and the (graded) space of semi-infinite forms is denoted as $\mathcal{H}(\lambda,\mu)$. (The physicists call it the *fermionic Fock space*). We will define on this space a structure of a graded $\mathfrak{Vir}$-module; this structure (unlike the space $\mathcal{H}(\lambda,\mu)$ itself) will depend on λ and μ.

For $i \neq 0$, we define the action of e_i on $\mathcal{H}(\lambda, \mu)$ by the formula

$$e_i(\ldots \wedge f_{j_3} \wedge f_{j_2} \wedge f_{j_1}) = \sum_{k=1}^{\infty} (\ldots \wedge e_i f_{j_k} \wedge \ldots \wedge f_{j_2} \wedge f_{j_1}) \tag{12}$$

(We may need to rearrange the factors to make the sequence of subscripts monotonic; this may result in a sign change). This works, since $e_i f_{j_k}$ is proportional to f_{j_k+i}, and for almost all k, $j_k + i$ is contained among j_ℓ's. Notice that $e_i \colon \mathcal{H}(\lambda, \mu)_k \to \mathcal{H}(\lambda, \mu)_{k+i}$.

For $i = 0$ this definition does not work. We take for the definition of the action of e_0 the equality $e_0 = \frac{1}{2}[e_{-1}, e_1]$. Finally, the action of z is defined by the equality $[e_{-2}, e_2] = 4e_0 + \frac{1}{2}z$. The following proposition shows that these operators e_i and z satisfy the relations in the Virasoro algebra and provide an additional information on the action of e_0 and z.

Proposition 4.7. (1) *If neither of $i, j, i+j$ is zero, then $[e_i, e_j] = (j-i)e_{i+j}$.*

(2) $[e_0, e_i] = ie_i$.

(3) $[e_i, z] = 0$.

(4) $[e_{-i}, e_i] = 2ie_0 + \frac{1}{12}(i^3 - i)z$.

(5) *e_0 acts in $\mathcal{H}(\lambda, \mu)_k$ as multiplication by $\frac{1}{2}\mu(\mu - 2\lambda - 1) + k$.*

(6) *z acts in $\mathcal{H}(\lambda, \mu)$ as multiplication by $-2(6\lambda^2 + 6\lambda + 1)$.*

Proof. (1) If we apply e_i and then e_j to $\ldots \wedge f_{k_3} \wedge f_{k_2} \wedge f_{k_1}$, then we apply first e_i to one of the factors (since $i \neq 0$) and then e_j to one of the factors (since $j \neq 0$). If these factors are different, then the ordering e_i, e_j is irrelevant. Hence, the application of $e_i e_j - e_j - e_i$ consists in successive application of $e_i e_j - e_j e_i = (j-i)e_{i+j}$ to $f_{k_1}, f_{k_2}, \ldots$. Thus, $(e_i e_j - e_j e_i)(\ldots \wedge f_{k_3} \wedge f_{k_2} \wedge f_{k_1} \ldots \wedge f_{k_3} \wedge f_{k_2} \wedge f_{k_1}) = (j-i)e_{i+j}(\ldots \wedge f_{k_3} \wedge f_{k_2} \wedge f_{k_1} \ldots \wedge f_{k_3} \wedge f_{k_2} \wedge f_{k_1})$ (since $i + j \neq 0$).

(2) If $i \neq 0$, then $[\,[e_{-1}, e_1], e_i] = [e_{-1}, [e_1, e_i]\,] - [e_1, [e_{-1}, e_i]\,] = (i-1)[e_{-1}, e_{i+1}] - (i+1)[e_1, e_{i-1}] = [(i-1)(i+2) - (i+1)(i-2)]e_i = 2ie_i$; thus the equality $[e_0, e_i] = ie_i$ also holds.

(3) $[e_i, [e_{-2}, e_2]] = [[e_i, e_{-2}], e_2] + [e_{-2}, [e_i, e_2]] = (-2-i)[e_{i-2}, e_2] + (2-i)[e_{-2}, e_{i+2}] = [(-2-i)(4-i) - (2-i)(i+4)]e_i = -4ie_e = 4[ei, e_0]$. Hence, $[e_1, z] = \frac{1}{2}[e_i, [e_{-2}, e_2] - 4e_0] = 0$.

(4) For $i = 1$ and 2, it is the definition. For $i > 2$ we use induction:

$$\begin{aligned}
[e_{-i}, e_i] &= \frac{1}{i-2}[e_{-i}, [e_1, e_{i-1}]\,] \\
&= \frac{1}{i-2}\left([\,[e_{-i}, e_1], e_{i-1}] + [e_1, [e_{-i}, e_{i-1}]\,]\right) = \frac{i+1}{i-2}[e_{-i+1}, e_{i-1}] + \frac{2j-1}{i-2}[e_1, e_{-1}] \\
&= \left(\frac{2(i-1)(i+1)}{i-2} - \frac{2(2i-1)}{i-2}\right) e_0 + \frac{i+1}{12(i-2)}((i-1)^3 - (i-1))z \\
&\qquad = 2ie_0 + \frac{1}{12}(i^3 - i)z.
\end{aligned}$$

(5) We need to compute

$$\frac{1}{2}[e_{-1}, e_1](\ldots \wedge f_{-(s+2)} \wedge f_{-(s+1)} \wedge f_{-s+k_s} \wedge f_{-(s-1)+k_{s-1}} \wedge \ldots \wedge f_{-1+k_1})$$

where $1 \leq k_s \leq \ldots \leq k_1$ and $k_s + \ldots + k_i = k$. For this, we need to apply to one factor e_1, then apply to one factor (the same or other) e_1, take the sum of all the forms obtained; then do the same in the other order (first e_{-1} and then e_1) and to subtract the second result from the first. As we noted in Proof of (1), if we apply e_{-1} and e_1 to *different* factors, then the order does not matter, so we cancel these terms. Furthermore, the application of $e_{\pm 1}$ to $f_{-(s+2)}, f_{-(s+3)}, \ldots$ turns the whole product to zero, as well as the application of e_{-1} to $f_{-(s+1)}$. What remains, we need to apply successively $e_{-1}e_1$ to $f_{-(s+1)}$ and $e_{-1}e_1 - e_1e_{-1} = 2e_0$ to $f_{-s+k_s}, f_{-(s-1)+k_{s-1}}, \ldots, f_{-1+k_1}$. Since $e_{-1}e_1 f_{-(s+1)} = (-(s+1)+\mu-2\lambda)(-s+\mu)f_{-(s+1)}$ and $e_0 f_j = (j+\mu-\lambda)e_j$, the whole operation consists in multiplication by one half of the sum of these coefficients:

$$(-(s+1)+\mu-2\lambda)(-s+\mu)/2 + (-s+k_s+\mu-\lambda) + \ldots + (-1+k_1+\mu-\lambda).$$

The first of the summands is $\frac{1}{2}(-(s+1)+\mu-2\lambda)(-s+\mu) = \frac{1}{2}(\mu^2 - 2\lambda\mu - \mu(2s-1) + 2\lambda s + s(s+1))$; the sum of the other summands is $(-s+k_s-(s-1)+k_{s-1}+\ldots-1+k_1+s\mu-s\lambda = -\frac{s(s+1)}{2} + k + s\mu - s\lambda$. The total sum is $\frac{1}{2}(\mu^2 - 2\lambda\mu - \mu) + k$, as was stated.

(6) The proof is similar to that of (5), but shorter. We need to apply to the same product as in (5) $[e_{-2}, e_2] + 4e_0 = [e_{-2}, e_2] + 2[e_{-1}, e_1]$ and then to multiply the result by 2. But since in $\mathcal{F}_{\lambda,\mu}$ $[e_{-2}, e_2] + 4e_0 = 0$, almost everything cancels, and all we need is to calculate $e_{-2}e_2 f_{-(s+2)}, e_{-2}e_2 f_{-(s+1)}, -2e_{-1}e_1 f_{-(s+1)}$, the add up the coefficients arising, and the multiply the result by 2. These three coefficients are

$$(-(s+2)+\mu-3\lambda)(-s+\mu+\lambda), (-(s+1)+\mu-3\lambda)(-(s-1)+\mu+\lambda),$$
$$-2(-(s+1)+\mu-2\lambda)(-s+\mu),$$

and their sum is $-6\lambda^2 - 6\lambda - 1$. This completes the proof.

Proposition 4.6 establish a structure of a $\mathfrak{Vir}$-module on $\mathcal{H}(\lambda,\mu)$. Notice that the modules $\mathcal{H}(\lambda,\mu)$ and $\mathcal{H}(-1-\lambda, \mu-2\lambda-1)$ are contragredient (because the corresponding modules $\mathcal{F}$ are contragredient) and the modules $\mathcal{H}(\lambda,\mu)$ and $\mathcal{H}(\lambda, -\mu+2\lambda+1)$ are isomorphic. The isomorphism is established by the formula

$$\ldots \wedge f_{-3-k_3} \wedge f_{-2-k_2} \wedge f_{-1-k_1} \leftrightarrow \ldots \wedge f_{-3-\overline{k}_3} \wedge f_{-2-\overline{k}_2} \wedge f_{-1-\overline{k}_1}$$

where $(k_1, k_2, k_3, \ldots)$ and $(\overline{k}_1, \overline{k}_2, \overline{k}_3, \ldots)$ are dual partitions, $\overline{k}_i = \#\{j \mid k_j \geq i\}$.

The fact that this is a module isomorphism, is Exercise 6.19.

4.4.2. Modules of semi-infinite forms and Verma modules. Throughout this section, we put

$$h = \frac{1}{2}\mu(\mu - 2\lambda - 1),\ c = -2(6\lambda^2 + 6\lambda + 1).$$

Modules $\mathcal{H}(\lambda,\mu)$ and $M(h,c)$ have a lot in common: $\dim M(h,c)_k = \dim \mathcal{H}(\lambda,\mu)_k$, and the action of e_0 and z on $M(h,c)$ and $\mathcal{H}(\lambda,\mu)$ is the same: multiplication by $h+k$ and c. Let us call a graded module $M = \oplus_{k\geq 0} M_k$ *irreducible in degrees* $\leq m$, if for any submodule L of M, the intersection $L \cap \oplus_{k=0}^m M_k$ is either $\oplus_{k=0}^m M_k$ or 0.

PROPOSITION 4.8. *The module $M(h,c)$ is irreducible if and only if so is $\mathcal{H}(\lambda,\mu)$. Moreover, the module $M(h,c)$ is irreducible in degrees $\leq m$ if and only if so is $\mathcal{H}(\lambda,\mu)$.*

Proof. Since $F = \ldots \wedge f_{-3} \wedge f_{-2} \wedge f_{-1} \in \mathcal{H}(\lambda,\mu)$ is a singular vector of type (h,c), there is a canonical homomorphism $\varphi\colon M(h,c) \to \mathcal{H}(\lambda,\mu)$, $\varphi(v) = F$. If φ is an isomorphism in degrees $\leq m$, then the modules $M(h,c)$ and $\mathcal{H}(\lambda,\mu)$ are isomorphic in degrees $\leq m$ and hence these modules are irreducible in degrees $\leq m$ simultaneously. If φ is not an isomorphism in degrees $\leq m$, then $0 \neq \operatorname{Ker}\varphi \cap \oplus_{k=0}^m M(h,c)_k \neq \oplus_{k=0}^m M(h,c)_k$ and $0 \neq \operatorname{Im}\varphi \cap \oplus_{k=0}^m \mathcal{H}(\lambda,\mu)_k \neq \oplus_{k=0}^m \mathcal{H}(\lambda,\mu)_k$, so neither of the two modules is irreducible in degrees $\leq m$.

With this proposition in mind, to study the reducibility of Verma modules, we will consider modules of semi-infinite forms. In the next section, we will demonstrate an explicit construction of singular vectors in many such modules.

4.4.3. Singular vectors in modules of semi-infinite forms. Let $s \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$. Consider the "expression"

$$\varphi_{s,n} = \sum_{\substack{j_1+\ldots+j_n = s+\binom{n}{2} \\ j_1<\ldots<j_n}} \frac{\prod\limits_{1\leq u<v\leq n} (j_v - j_u)}{\prod\limits_{1\leq u<v\leq n} (v-u)} f_{j_1} \wedge \ldots \wedge f_{j_n}.$$

We could have regarded this expression as an element of $\Lambda^n(\mathcal{F}_\lambda \mu)$ but the sum is infinite. For example, $\varphi_{2t,2} = f_t \wedge f_{t+1} + 3f_{t-1} \wedge f_{t+2} + 5f_{t-2} \wedge f_{t+3} + \ldots$. Still, it makes sense to apply e_i to this expression.

PROPOSITION 4.9.

$$e_i \varphi_{s,n} = n(\mu^* - \lambda^*(i+1))\varphi_{s+i,n}$$

where

$$\lambda^* = \lambda + \frac{(n-1)(n+2)}{2n},\ \mu^* = \mu + n + \frac{s-1}{n}.$$

In particular, if $\lambda = -\dfrac{(n-1)(n+2)}{2n}$ *and* $\mu = -n - \dfrac{s-1}{n}$, *then* $e_i\varphi_{s,n} = 0$.

It is not hard to prove this by a direct computation, but we will postpone the proof, since we will prove a more general statement below (Section 5.2.3).

Let us assume now that $\lambda = -\dfrac{(n-1)(n+2)}{2n}$ and $\mu = -n - \dfrac{s-1}{n}$. For a semi-infinite form Φ, consider the expression $\Psi = \operatorname{Shift}_{-nk}\left(\Phi \wedge (\varphi_{s,n})^k\right)$. In other words, we multiply Φ k times by $\varphi_{s,n}$ and then subtract nk from every subscript at every f in the result. We again obtain a semi-infinite form. (The shift of the indices is necessary to keep the condition $j_k = -k$ for k large.)

From now on we assume that n *is even.*

Proposition 4.10. *For our choice of* λ *and* μ*, the map* $\Phi \mapsto \Psi$ *is a* $\mathfrak{Vir}$*-homomorphism* $\mathcal{H}(\lambda,\mu) \to \mathcal{H}(\lambda,\mu+nk)$ *of degree* $k\left(s - \dfrac{n^2(k-1)}{2}\right)$.

Proof. It follows from Proposition 4.9, that $e_i\left(\Phi \wedge (\varphi_{s,n})^k\right) = \left(e_i\Phi \wedge (\varphi_{s,n})^k\right)$; as to the shift by $-nk$, it commutes with e_1, if we shift μ accordingly. Commuting with z is obvious: z is a multiplication by $c = -2(6\lambda^2 + 6\lambda + 1)$ both before and after the homomorphism. It remains to determine the degree. The contribution of $\mathrm{Shift}_{-nk}\Phi$ into $\deg\Psi$ is $\deg\Phi$ (in each summand $j_\ell + \ell$, j_ℓ loses nk and ℓ gains nk). The sum of the subscripts in $\varphi^k_{s,n}$ is $k\left(s + \binom{n}{2}\right)$; the shift by nk results in a subtracting $(nk)^2$, and adding the numbers (ℓ to j_ℓ) contributes $\dfrac{nk(nk+1)}{2}$. The final result for the degree is

$$\begin{aligned} k\left(s + \binom{n}{2}\right) - (nk)^2 + \frac{nk(nk+1)}{2} &= k\left(s + \frac{n^2 - n - 2n^2k + n^2k + n}{2}\right) \\ &= k\left(s - \frac{n^2(k-1)}{2}\right). \end{aligned}$$

Proposition 4.11. *If* $s > \dfrac{n^2(k-1)}{2}$*, then the* $\mathfrak{Vir}$*-homomorphism* $\mathcal{H}(\lambda,\mu) \to \mathcal{H}(\lambda,\mu+nk)$ *is not trivial.*

The proof of this we also postpone to Section 5.2.5.

Corollary 4.12. *If*

$$\lambda = -\frac{(n-1)(n+2)}{2n} \text{ and } \mu = -n - \frac{s-1}{n},$$

then the module $\mathcal{H}(\lambda,\mu+nk)$ *has a singular vector of degree* $k\left(s - \dfrac{n^2(k-1)}{2}\right)$.

4.5. Proof of reducibility criterion of Verma modules over $\mathfrak{Vir}$.

Using the above results on the module of semi-infinite forms, we can prove the Theorem 4.5. We use the parametric equations

$$h_{pq}(t) = \frac{1-p^2}{4}t + \frac{1-pq}{2} + \frac{1-q^2}{4}t^{-1},$$
$$c_{pq}(t) = c(t) = 6t + 13 + 6t^{-1}.$$

of the curve Φ_{pq} given in Section 4.4.3.

Lemma. *If*

$$\lambda = -\frac{(n-1)(n+2)}{2n},\ \mu = (k-1)n - \frac{s-1}{n},\ p = k,\ q = s - \frac{n^2(k-1)}{2},\ t = -\frac{n^2}{2},$$

then

$$h_{pq}(t) = \frac{1}{2}\mu(\mu - 2\lambda - 1), \ c(t) = -2(6\lambda^2 + 6\lambda + 1).$$

Proof is a direct computation, but we will do it. First,

$$\lambda = -\frac{n}{2} - \frac{1}{2} + \frac{1}{n}, \ \lambda + 1 = -\frac{n}{2} + \frac{1}{2} + \frac{1}{n}, \ \lambda(\lambda+1) = \left(-\frac{n}{2} + \frac{1}{n}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{n^2}{4} - \frac{5}{4} + \frac{1}{n^2},$$

$$-2(6\lambda(\lambda+1) + 1) = -3n^2 + 13 - \frac{12}{n^2} = 6t + 13 + 6t^{-1} = c(t)$$

with $t = -\frac{n^2}{2}$. Next,

$$\mu - 2\lambda - 1 = (k-1)n - \frac{s-1}{n} + n + 1 - \frac{2}{n} - 1 = kn - \frac{s+1}{n},$$

$$\frac{1}{2}\mu(\mu - 2\lambda - 1) = \frac{k(k-1)}{2}n^2 - \frac{2ks - s - 1}{2} + \frac{s^2 - 1}{2n^2}.$$

On the other hand,

$$\frac{1-p^2}{4} \cdot t + \frac{1-pq}{2} + \frac{1-q^2}{4} \cdot t^{-1} = \frac{(k^2-1)n^2}{8} + \frac{1-ks}{2} + \frac{n^2k(k-1)}{4} + \frac{s^2-1}{2n^2} - \frac{s(k-1)}{2} + \frac{n^2(k-1)^2}{8} = \frac{k(k-1)}{2}n^2 - \frac{2ks - s - 1}{2} + \frac{s^2-1}{2n^2}.$$

Thus, $\frac{1}{2}\mu(\mu - 2\lambda - 1) = h_{pq}(t)$.

PROPOSITION 4.13. *For any positive integers p and q, and any positive even n, the module $M(h_{pq}(t), c(t))$, where $t = -\frac{n^2}{2}$, has a singular vector of positive degree $\le pq$.*

Proof. Put $k = p$, $s = q + \frac{n^2(k-1)}{2}$, $\lambda = -\frac{(n-1)(n+2)}{2n}$, $\mu = (k-1)n - \frac{s-1}{n}$, and, as above, $t = -\frac{n^2}{2}$. Then $\mu - kn = -n - \frac{s-1}{n}$, and, by Corollary 4.12, the module $\mathcal{H}(\lambda, \mu)$ has a singular vector of degree $k\left(s - \frac{n^2(k-1)}{2}\right) = pq$. In virtue of Proposition 4.8 and the last lemma, this shows that the module

$$M\left(\frac{1}{2}\mu(\mu - 2\lambda - 1), -2(6\lambda^2 + 6\lambda = 1)\right) = M\left(h_{pq}\left(-\frac{n^2}{2}\right), c\left(-\frac{n^2}{2}\right)\right)$$

has a singular vector of degree $\le pq$.

Now, let us make use of the Shapovalov form. Similar to the Kac-Moody case, we define $F_{h,c;k}(x, y) \in \mathbb{C}$ for $x, y \in U(\mathfrak{n}_-)_k$, by the formula $\sigma(x)yv = F_{h,c;k}(x, y)v$, where v is the vacuum vector in $M(h, c)$. This is a symmetric bilinear form on the space $U(\mathfrak{n}_-)_k =$

$M(h,c)_k$ of dimension $\mathbf{p}(k)$, and the kernel of this form is precisely the intersection of the maximal submodule of $M(h,c)$ and $M(h,c)_k$. The determinant of the form $F_{h_{p,q}(t),c_{p,q}(t);pq}$ in $M(h_{p,q}(t),c_{p,q}(t))_{pq}$ is a polynomial of t; but Proposition 4.13 shows that it is zero for $t=-\dfrac{n^2}{2}$ for all even n. Hence, this determinant is zero for all $t\in\mathbb{C}$. Now we can formulate our main technical result.

For $q\geq p\geq 1$, let $G_{p,q}$ be a polynomial in h and c, of degree 2, if $p<q$, and of degree 1, if $p=q$, such that $G_{p,q}(h,c)=0$ is the equation of the curve Φ_{pq}.

THEOREM 4.14. (conjectured by Kac [8], proved by Feigin and me [9]).

$$\det F_{h,c;k}=C\prod_{\substack{q\geq p\geq 1\\ pq\leq k}}[G_{p,q}(h,c)]^{\mathbf{p}(k-pq)} \tag{13}$$

where C is a non-zero constant.

Obviously, this result implies Theorem 4.5 from Section 4.3.3.

Proof of Theorem consists of two parts: (1) the two sides of (13) have the same degree; (2) $\det F_{h,c;k}$ is divisible by the product in the right hand side.

Proof of (1). The matrix of the Shapovalov form has rows and columns labeled by partitions of k: the entry $a_{\tau\rho}$ corresponding to partitions $\tau=\{k_1,\dots,k_s\},\ \rho=\{\ell_1,\dots,\ell_t\}$ is determined by the formula $e_{-\ell_1}\dots e_{-\ell_t}e_{k_1}\dots e_{k_s}v=a_{\tau\rho}v$. The degree of the polynomial $a_{\tau\rho}$ does not exceed $\min(s,t)$, and, precisely as in Section 3.3.4, we find that in the determinant, the only monomial of the highest degree is the product of the diagonal entries. The degree of $a_{\tau\tau}$ is $s=\ell(\tau)$, the number of parts in the partition τ. Thus, $\deg\det F_{h,c;k}$ (with respect to h and c) is $\sum\limits_{\tau\in\mathcal{P}(k)}\ell(\tau)$ where $\mathcal{P}$ is the set of partitions of k. On the other hand,

$$\deg\prod_{\substack{q\geq p\geq 1\\ pq\leq k}}[G_{p,q}(h,c)]^{\mathbf{p}(k-pq)}=\sum_{\substack{q>p\geq 1\\ pq\leq k}}2\mathbf{p}(k-pq)+\sum_{\substack{q=p\geq 1\\ pq\leq k}}\mathbf{p}(k-pq)=\sum_{\substack{q\geq 1,p\geq 1\\ pq\leq k}}\mathbf{p}(k-pq).$$

Thus, we need to prove a simple combinatorial fact:

$$\sum_{\tau\in\mathcal{P}(k)}\ell(\tau)=\sum_{\substack{q\geq 1,p\geq 1\\ pq\leq k}}\mathbf{p}(k-pq).$$

Probably, it is well known, but I will provide a proof. Let $\mathcal{Q}(k)$ be the set of pairs $(\tau,(p,q))$ where τ is a partition of k which contains p at least $q(>0)$ times. For a given partition τ there are precisely $\ell(\tau)$ of different (p,q) satisfying the condition. (Indeed, let the partition τ contains α_m parts equal to m. Then the pairs (p,q), such that $(\tau,(p,q))\in\mathcal{Q}(k)$ are $(1,1),\dots,(1,\alpha_1);(2,1),\dots,(2,\alpha_2);\dots$, and the number of such pairs (p,q) is $\alpha_1+\alpha_2+\ldots=\ell(\tau)$). On the other hand, for a given (p,q) there are $\mathbf{p}(k-pq)$ partitions containing p at least q times. Thus, the two sides of our projected equality are equal to the number of elements in $\mathcal{Q}(k)$.

Proof of (2) is based on a lemma:

LEMMA. *Let $A(t)$ be an $N \times N$ matrix whose entries a_{ij} are infinitely differentiable functions of t, and let $D(t) = \det A(t)$. In, for some t_0, $\operatorname{rank} A(t_0) \le N - r$, then $D^{(s)}(t_0) = 0$ for $s < r$.*

Proof of Lemma. For an entry a_{ij} of A, denote as A_{ij} the matrix obtained from A by removing i-th row and j-th column. Assume (by induction) that for matrices of order $N-1$ the Lemma has been already proved. Obviously,

$$D'(t) = \sum_{i,j} (-1)^{i+j} a'_{ij}(t) \det A_{ij}(t),$$

and the s-th derivative $D^{(s)}(t)$ involves, at most $(s-1)$-st derivatives of $\det A_{ij}(t)$, and these derivatives are all zeroes at t_0, since $\operatorname{rank} A_{ij}(t_0) \le \operatorname{rank} A(t_0) \le N - r = (N-1) - (r-1)$.

Back to (2). Let $(h,c) \in \Phi_{p,q}$, $pq \le k$. Then $M(h,c)$ contains a Verma submodule generated by a singular vector of degree pq (or less) and hence the Shapovalov form $F_{h,c;k}$ has a kernel of dimension at least $\mathbf{p}(k-pq)$. Consider a curve $\gamma = \{\gamma(t)\}$ in the plane $\mathbb{C}^2(h,c)$ transverse to $\Phi_{p,q}$ and such that $\gamma(0) = (h,c)$; for the parameter t, we can take $G_{p,q}$. By Lemma, $\det F_{\gamma(t);k}$ is divisible by $t^{\mathbf{p}(k-pq)}$, which certainly means that $\det F_{h,c;k}$ is divisible by $G_{p,q}(h,c)^{\mathbf{p}(k-pq)}$, as stated.

4.6. Explicit formula for singular vectors.

4.6.1. Examples and existing results. It follows from the results of previous section that for $h = h_{p,q}(t)$, $c = c_{p,q}(t)$, there exists a unique, up to a non-zero constant factor, singular vector in $M(h,c)_{pq}$. As a function of t, it is a polynomial in t, t^{-1}, and to make it unique, we normalize it as $\sigma_{p,q}(t)v$, where

$$U(\mathfrak{n}_-) \ni \sigma_{p,q}(t) = \sum_{\substack{j_1 \ge \ldots \ge j_s > 1 \\ j_1 + \ldots + j_s = pq}} P_{p,q}^{j_1,\ldots,j_s}(t) e_{j_1} \ldots e_{j_s},\ P_{p,q}^{1,\ldots,1} = 1.$$

For relatively small values of p and q, this $\sigma_{p,q}(t)$ may be explicitly found:

$$\begin{aligned}
&\sigma_{1,1}(t) = e_1;\\
&\sigma_{2,1}(t) = e_1^2 + te_2;\\
&\sigma_{3,1}(t) = e_1^3 + 4te_1e_2 + (4t^2 - 2t)e_3;\\
&e_{2,2}(t) = e_1^4 + 2ue_1^2e_2 + (u^2-4)e_2^2 + (6-2u)e_1e_3 + (3u-6)e_4\\
&\qquad\qquad\qquad\qquad\qquad \text{where } u = t + t^{-1};\\
&e_{4,1}(t) = e_1^4 + 10te_1^2e_2 + (24t^2 - 10t)e_1e_3 + 9t^2e_2^2 + (36t^3 - 24t + 6t)e_4.
\end{aligned}$$

However, no general formula of this kind for $\sigma_{p,q}(t)$ exists. I am aware of several attempts of writing such a formula. First, I will mention a work of a group of physicists [17], who found a recursive procedure for finding a formula; but I doubt that it may give even the results listed above. Second, A. Kent [16] extended to the Virasoro case the formula of Feigin-Fuchs-Malikov discussed above (see Sections 3.4). One should also mention a beautiful work of Benoit and Sent-Aubin [15], a formula for $\sigma_{1,q}(t)$.

4.6.2. The calculation mod $\mathbf{e_3}$. Here I present a partial result obtained in mid-80's by Feigin and me [12]. Let $\pi\colon U(\mathfrak{n}_-) \to \mathbb{C}[e_1, e_2]$ be the projection of $U(\mathfrak{n}_-)$ onto the quotient over the ideal generated by e_3. There is a very explicit formula for $\pi\sigma_{p,q}(t)$; it looks especially attractive when $q = 1$:

$$\pi\sigma_{2r,1}(t) = (e_1^2 + te_2)(e_1^2 + 9te_2)(e_1^2 + 25te_2)\dots(e_1^2 + (2r-1)^2te_2),$$

$$\pi\sigma_{2r+1,1}(t) = e_1(e_1^2 + 4te_2)(e_1^2 + 16te_2)(e_1^2 + 36te_2)\dots(e_1^2 + (2r)^2te_2).$$

(compare with the example above). The general formula is this:

$$[\pi\sigma_{p,q}(t)]^2 = \prod_{\substack{-p<r<p,\ -q<s<q \\ r\not\equiv p \bmod 2,\ s\not\equiv q \bmod 2}} \left(e_1^2 + (r\theta + s\theta^{-1})^2 e_2\right) \tag{14}$$

where $\theta = \sqrt{t}$. Notice that every factor in this product appears twice, with a possible exception of e_1^2 which corresponds to $r = s = 0$, so extracting the square root of this product cause no difficulties. For example,

$$\pi\sigma_{3,2}(t) = (e_1^2 + (4t + 4 + t^{-1})e_2)(e_1^2 + t^{-1}e_2)(e_1^2 + (4t - 4 + t^{-1})e_2),$$
$$\pi\sigma_{3,3}(t) = e_1(e_1^2 + 4te_2)(e_1^2 + 4t^{-1}e_2)(e_1^2 + 4(t + 2 + t^{-1})e_2)(e_1^2 + 4(t - 2 + t^{-1})e_2).$$

The idea of the proof will be explained in Section 4.6.4.

4.6.3. The action in $\mathcal{F}_{\lambda\mu}$. The article [12] contains another partial formula for the singular vectors. It describes the action of $\sigma_{p,q}$ in the module $\mathcal{F}_{\lambda\mu}$. Namely, let $\sigma_{p,q}(t)f = P_{p,q}f_{pq}$. Then

$$\begin{aligned}P_{p+1,q+1}(\lambda,\mu,t)^2 = \prod_{i=0}^{p}\prod_{j=0}^{q} \{&(\mu - 2\lambda)^2 - [(p-2i)\theta^{-1} + (q-2j)\theta]^2\lambda \\ &+ [(2i(p-i) + p)t^{-1} + p(1-2j) + q(1-2i) - 4ij + (2j(q-j) + q)t](\mu - 2\lambda) \\ &+ ((i\theta^{-1} + j\theta)((i+1)\theta^{-1} + (j+1)\theta)\times \\ &\times ((p-i)\theta^{-1} + (q-j)\theta)(p-i+1)\theta^{-1} + (q-j+1)\theta)\}\end{aligned} \tag{15}$$

This formula may look less attractive than (14), but it is of the same nature: the polynomial $P_{p,q}(\lambda,\mu,t)$ is expressed as of a product of polynomial of degree 2. Again the idea of the proof is presented in a section below.

4.6.4. The proof of formulas in Sections 4.6.2 and 4.6.3. For some values of t, the singular vector in $M(h(t), c(t))_{pq}$ is contained in intermediate Verma modules $M(h(t) + p'q', c(t)$; this corresponds to intersections of the curve $\Phi_{p,q}$ with curves $\Phi_{p',q'}$ with $p'q' < pq$. For these t, $\sigma_{p,q}(t) \in U(\mathfrak{n}_-)$ becomes the product of several factors $\sigma_{p',q'}(t)$, and we can assume by induction that corresponding $\pi\sigma_{p',q'}(t)$ and $P_{p'.q'}(t)$ are known. This gives a certain amount of values for $\pi\sigma_{p,q}$ and $P_{p,q}$, and it is obvious that this amount is sufficient to fully determine these polynomials. Thus, all we need to do to prove formulas (14) and (15) is to check that they do not contradict to these product rules which is easy to do (especially for intersections $\Phi_{p,q}\cap\Phi_{p',q'}$ with $p'\le p, q'\le q, p'q'<pq$).

4.6.5. The upper and lower degrees of $\sigma_{\mathbf{p},\mathbf{q}}(\mathbf{t})$. The following statement was proved by A. Astashkevich and me [18]. I omit here some details of computations, they all can be found in the article cited.

PROPOSITION 4.15.

$$\sigma_{p,q}(t) = (q-1)!^{2p}e_q^pt^{-(q-1)p}+\ldots+(p-1)!^{2q}e_p^qt^{(p-1)q},$$

where "…" denotes the terms of intermediate degrees in t. Thus, the upper and lower degrees of $\sigma_{p,q}(t)$ in t are $q(p-1)$ and $-p(q-1)$

Proof. It follows from the formula (15) that in $\mathcal{F}_{\lambda,\mu}$,

$$\sigma_{p,q}(t)f_0=\Bigg[(p-1)!^{2q}t^{(p-1)q}\prod_{v=0}^{q-1}(\mu-(p+1)\lambda+pv)+\ldots \\ +(q-1)!\prod_{u=0}^{p-1}(\mu-(q-1)\lambda+qu)^{2p}t^{-(q-1)p}\Bigg]f_{pq},$$

or, in other words,

$$\sigma_{p,q}(t)f_0=\left[(p-1)!^{2q}t^{(p-1)q}e_p^q+\ldots+(q-1)!^{2p}t^{-(q-1)p}e_q^p\right]f_{pq}.$$

Let us denote upper and lower degrees of a polynomial P in t,t^{-1} as $d_+(P)$ and $d_-(P)$. The last formula shows that

$$\max_{j_1+\ldots+j_s=pq} d_+\left(P_{p,q}^{j_1,\ldots,j_s}\right)\ge(q-1)p,$$
$$\min_{j_1+\ldots+j_s=pq} d_-\left(P_{p,q}^{j_1,\ldots,j_s}\right)\le-(p-1)q.$$

For a positive integer j, put

$$\varphi(j)=j-1-\left\lfloor\frac{j-1}{p}\right\rfloor.$$

MAIN LEMMA. $d_+\left(P_{p,q}^{j_1,\ldots,j_s}\right)\le\sum_{r\le s}\varphi(j_r)$.

This lemma is proved by induction by the lexicographical ordering $(j_1',\ldots,j_{s'}')\prec(j_1,\ldots,j_s)$. We assume that $j_u>j_{u+1}=1$ and deduce the desired inequality from $e_{-\ell}v=0$ for $\ell=\begin{cases}j_u-1, & \text{if } j_u\not\equiv 1 \bmod p,\\ p, & \text{if } j_u\equiv 1 \bmod p,\end{cases}$ using the induction hypothesis and some obvious properties of the function φ.

Main Lemma immediately implies the statement concerning the upper and lower degrees of $\sigma_{p,q}(t)$. The computation of the coefficients $(p-1)!^{2q}$ and $(q-1)!^{2p}$ is straightforward (and, actually, we will never need it).

4.7. Casimir operators in modules over the Virasoro algebra.

This section contains a short presentation of a note [19] of Feigin and me, where we tried to describe"Casimir operators" in the case of Virasoro algebra. Our goal was to pave a way to description of the structure of reducible Verma modules $M(h,c)$. We managed to describe these structure later (I will present our results about them in Section 4.8 below, but we did not use any seriously the Casimir operators. Our Casimir operator work remained unfinished and mostly forgotten. Still it had at least one reader, and this reader was Victor Kac. He used not our results, but our approach to get a more or less full description of all Casimir operators (not just one as in the Kac-Kazhdan article) over Kac-Moody algebras [20]. I want to refresh my memories about this work to demonstrate, how much the Virasoro algebra is different from the Kac-Moody algebras.

4.7.1. Central series. Consider a series

$$\sum_{|I|=|J|} e_{i_q} \dots e_{i_1} \lambda_J^I(e_0, z) e_{-j_1} \dots e_{-j_r} \tag{16}$$

where $I = \{i_1, \dots, i_q\}, J = \{j_1, \dots, j_r\}, i_1 \leq \dots \leq i_q, j_1 \leq \dots \leq j_r |I| = i_1 + \dots + i_q, |J| = j_1 + \dots + j_r$, and λ_J^I are entire function of two complex variables.

This series is not element of the universal enveloping algebra $U(\mathfrak{Vir})$, but it has the ability to act in (e_0, z)-diagonalizable (that is spanned by common eigenvectors of e_0 and z) virtually $\mathfrak{n}_+$-nilpotent $\mathfrak{Vir}$-modules. Such modules are usually (and justly) called *Bernstein-Gelfand-Gelfand modules.*

Remark. If v is a singular vector of the type (h,c), then the operator (16) takes it into $\lambda_\emptyset^\emptyset(h,c)v$.

The series (16) is called *central*, if its action commutes with the action of all e_i and z.

Theorem 4.16. (1) *If the series* (16) *is central, then all the functions* λ_J^I *can be reconstructed from* $\lambda_\emptyset^\emptyset$

(2) *A function* $\lambda(h,c)$ *can be* $\lambda_\emptyset^\emptyset$ *for some central series if and only if it satisfies the following condition:*

$(*)$ *If* $(h,c) \in \Phi_{pq}$ *for some* $p, q \in \mathbb{Z}_{\geq 0}$, *then* $F\lambda(h+pq, c) = \lambda(h,c)$.

Remark. For example, any function $\lambda(h,c)$ periodic in h with period 1 satisfies the condition $(*)$. But this example is not interesting for us: we are studying central series because if the Verma module $M(h,c)$ contains a singular vector of the type (h',c), then $\lambda(h',c) = \lambda(h,c)$; but it is certainly true that h' is $h+$ a positive integer.

Proof of Theorem 4.16. Let us try to construct, for a given function $\lambda(h,c)$ a central series with $\lambda_\emptyset^\emptyset(e_0,z) = \lambda(e_0,z)$. We already know that the action of the series (16) on the whole module $M(h,c)$ is the multiplication by $\lambda(h,c)$.

Suppose that we already know the functions λ_J^I with $|I| = |J| < s$. Present the series (16) as the sum of three sums,

$$\sum_{|I|=|J|<s} + \sum_{|I|=|J|=s} + \sum_{|I|=|J|>s},$$

and consider the action of these three sums on the space $M(h,c)_s$ with the basis $e_I v$, $|I| = s$. The action of the first sum is known to us; let $A = \|A^I_J(h,c)\|$ be its matrix. The action of the third sum is zero. The second sum takes $e_I v \in M(h,c)_s$ into

$$\sum_{|K|=|J|=s} e_J(\lambda^J_K(h,c)e_{-K}e_I)v = \sum_J \Lambda_s(h,c)Sh(h,c;s))^I_J)e_J v),$$

where $\Lambda_s(h,c)$ is the matrix $\|\lambda^I_J(h,c)\|$ and $Sh(h,c;s)$ is the matrix of the Shapovalov form $F_{h,c;s}$. We obtain the equation for $\Lambda_s(h,c)$:

$$(A(h,c) + \Lambda_s(h,c))Sh(h,c;s) = \lambda(h,c)E \quad (E \text{ is the identity matrix}). \tag{16}$$

Thus, if the Shapovalov form is non-degenerate (that is if $(h,c) \notin \Phi_{pq}$ with $pq \leq s$), then

$$\Lambda_s(h,c) = (\lambda(h,c)E - A(h,c))Sh(h,c;s)^{-1}.$$

This completes the proof of part (1): the functions λ^I_J, if they exist, are determined by $\lambda^\emptyset_\emptyset$ in the complement to the curves Φ_{pq}, and hence in the whole plane $\mathbb{C}^2(h,c)$. It remains to prove that Condition $(*)$ is sufficient (its necessity is obvious) for removing singularities on these curves. Suppose that (h,c) belongs to only one curve Φ_{pq} with $pq \leq s$. Then $M(h,c)$ has a unique proper submodule $L \cong M(h+pq,c)$ with a non-zero intersection $L \cap M(h,c)_s$. The operator (16) on this intersection with the multiplication by $\lambda(h+pq,c)$, and if $\lambda(h+pq,c) = \lambda(h,c)$, then the operator $\lambda(h,c)E - A(h,c)$ is zero on $L \cap M(h,c)_s = \mathrm{Ker}(F_{h,g;s})$, and the equation (17) is solvable on this intersection. This shows that the function $\Lambda_s(h,c)$ can have singularities only in the isolated points of intersections of curves $\Phi_{p,q}$, and hence it has no singularities at all. This completes the proof of Theorem 3.15.

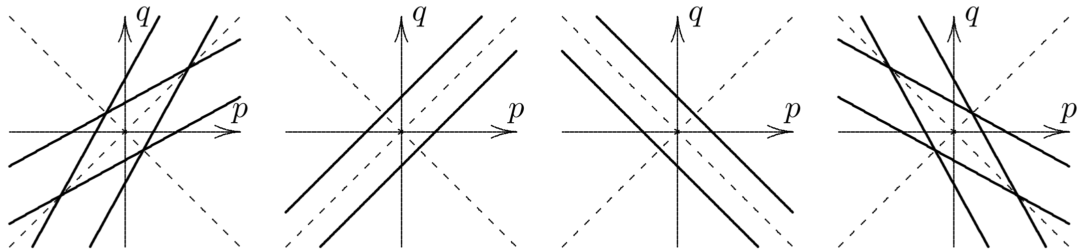
4.7.2. The relations between (h,c) and (p,q). These relations are two-sided, and, accordingly, this section will consists of two (closely related) parts. The first part will be needed in Section 4.8, while the second part will be needed in Sections 4.7.3 and 4.7.4.

First. let us locate, for given h,c, let us first locate all pairs $p,q \in \mathbb{C}$ such that $(h,c) \in \Phi_{pq}$. (It is important to notice that now we consider curves Φ_{pq} with p,q being not positive integers, rather arbitrary complex numbers. The result is: excluding the exceptional cases $c = 1, 25$, the set of these p,q is a union of four lines invariant with respect to the transformations $(p,q) \mapsto (-p,-q), (q,p))$ (no wonder: $\Phi_{pq} = \Phi_{qp} = \Phi_{-p,-q} = \Phi_{-p,-q}$); in other words, the four lines form a rhombus invariant with respect to the "diagonals" $p = \pm q$. Calculations:

- for a given c, the equation (with respect to t) of the equation $c = 6t + 13 + 6t^{-1}$ has two solutions (with the product 1), $t = \dfrac{c - 13 \pm \sqrt{(c-1)(c-25)}}{12}$. Put $\theta = \sqrt{t}$; obviously, θ has 4 values: if θ is one of them, then the other 3 are $-\theta, \theta^{-1}, -\theta^{-1}$;
- the formula for h gives

$$\begin{gathered} h = \frac{1}{4}[(t+t^{-1}) - (p\theta + q\theta^{-1})^2] = \frac{c-1}{24} - \frac{(p\theta + q\theta^{-1})^2}{4}, \\ p + t^{-1}q = \pm\sqrt{\frac{c-1-24h}{6t}}. \end{gathered} \tag{18}$$

The latter is an equation of a line, actually, of four lines (generically, they all are different, but they may become a pair of parallel or crossing lines, or even one line), since there are two possible values for t (that is, t may be replaced by t^{-1}) and two values of the square root. The slopes of these lines are $-t$ and $-t^{-1}$. These lines are real, if c is real and either ≤ 1, or ≥ 25. In the first case the slope is positive, in the second case it is negative. In the boundary cases $c = 1$ and $c = 25$ the four lines become a pair of parallel lines. See the picture below.



$c < 1$ $\qquad$ $c = 1$ $\qquad$ $c = 25$ $\qquad$ $c > 25$

Let us now consider the relation between (h, c) and (p, q) in the "opposite" direction. Let ℓ be a line in the plane $\mathbb{C}^2(p, q)$, *not parallel to the* p- *and* q-*axes; let* $\alpha p + \beta q = \gamma$ *be its equation (so,* $\alpha \neq 0$ *and* $\beta \neq 0$*). Then, as we have just seen, all the curves* Φ_{pq} *with* $(p, q) \in \ell$ *have a common point* (h, c), and we can find these h and c, since we know, from (17), the equation(s) of the line(s), which contain (h, c), and these equations have the form $p + t^{-1}q = \sqrt{\dfrac{c - 1 - 24h}{6t}}$. From this,

$$t = \frac{\alpha}{\beta}, \text{ so } c = 6t + 13 + 6t^{-1} = 6\frac{\alpha}{\beta} + 13 + 6\frac{\beta}{\alpha} = \frac{6\alpha^2 + 13\alpha\beta + 6\beta^2}{\alpha\beta} = \frac{(3\alpha + 2\beta)(2\alpha + 3\beta)}{\alpha\beta}$$

and

$$c - 1 - 24h = \frac{\gamma^2}{\alpha^2} \cdot 6t = \frac{6\gamma^2}{\alpha\beta}, \text{ so } h = \frac{c - 1 - 6\gamma^2}{24\alpha\beta} = \frac{6(\alpha + \beta)^2 - 6\gamma^2}{24\alpha\beta} = \frac{(\alpha + \beta)^2 - \gamma^2}{4\alpha\beta}.$$

With these formulas in mind, let us return to central series.

4.7.3. The case of multivalued functions $\lambda_{\mathbf{J}}^{\mathbf{I}}$. The construction and results of the previous section can be applied to the case, when functions λ_J^I are multivalued, more precisely, are defined on the same finite branched cover of the plane $\mathbb{C}^2(h, c)$.

For this branched cover, we take the manifold L of lines in the plane $\mathbb{C}^2(p, q)$ not parallel to the axes. The "projection" $\pi\colon L \to \mathbb{C}^2(h, c)$ takes the line $\ell = \{\alpha p + \beta q = \gamma\}$ into the point $(h, c) = \left(\dfrac{(\alpha^2 + \beta^2) - \gamma^2}{4\alpha\beta}, \dfrac{(3\alpha + 2\beta)(2\alpha + 3\beta)}{\alpha\beta}\right)$, which is a 4-fold covering branched over the lines $c = 1$ and $c = 25$. (Indeed, a generic (h, c) has 4 inverse images, $\alpha p \pm \beta q = \pm\gamma$; if $c = 1$ or 25, then $\alpha = \beta$ or, accordingly, $\alpha = -\beta$, and the 4 lines become 2 lines.) Let us assume now that the functions λ_I^J are functions on L, which we can consider as multivalued functions of h and c. For a line $\ell \in L$ and a point $A \in \mathbb{C}^2(p, q)$, we denote as $A\ell$ the line, which passes through A and is parallel to ℓ. The following statement is proved by a direct computation.

Lemma. *A function λ (considered as a function os h and c) satisfies the condition* $(*)$ *from Theorem 3.15 if and only if it satisfies the following condition*

$(**)$ *if the line ℓ passes through a point (p, q) with integral coordinates, then*

$$\lambda(\ell) = \lambda((p, -q)\ell).$$

Assymetry of this statement with respect to p and q is illusory.

It is not hard to construct a function λ, which satisfies this condition: we can take any function invariant with respect to the transformation of $\tau\colon L \to L$ induced by the transformation $(p, q) \mapsto (p, q + 2)$ of $\mathbb{C}^2(p, q)$. For example, we can take the function $\lambda(\{p + \beta q = \gamma\}) = \exp(\pi i \gamma)$. As a function of h and c, this function is determined by the formula

$$\lambda(h, c) = \exp \pi i \sqrt{\frac{13 - c + \sqrt{(c-1)(c-25)}}{12} \cdot \frac{24h + 1 - c}{6}}. \tag{19}$$

The transformation τ plays for the Virasoro algebra the role of a Weyl group transformation. Probably, a reasonable analogy of the Weyl group for the Virasoro algebra is the group of transformations of L in “horizontal” lines $q = k$ with integral k. The previous construction involves only the transformations, which preserve c; the other transformations acs on c as $c \mapsto 26 - c$.

4.7.4. Application: the composition factors of indecomposable Bernstein-Gelfand-Gelfand modules. Irreducible sub-quotient-modules of a Bernstein-Gelfand-Gelfand module M are called *composition factors* of this module. They all have the form $M(h, c)/L(h, c)$, where $L(h, c)$ is the maximal submodule of the Verma module $M(h, c)$; (h, c) is the weight of this composition factor. The invariance property of Casimir operators shows that the function λ of the formula (19) takes equal values on the weights of all composition factors of an indecomposable Bernstein-Gelfand-Gelfand module.

Consider, for example, the case $c = 0$. The previous argument show that *if one of the weights of an indecomposable Bernstein-Gelfand-Gelfand module has the form $(h, 0)$, then all of these weights are $(h_i, 0)$ and the functions*

$$\exp 2\pi i \frac{\sqrt{24 + 1}}{2} \text{ and } \exp 2\pi i \frac{\sqrt{24 + 1}}{3} \tag{20}$$

take equal values for all $h = h_i$. This statement shows that if one of the composition factors has the weight $(0, 0)$, then all the other have weights $\left(\frac{3m^2 \pm m}{2}, 0\right)$, and if one of the weights is $\left(\frac{1}{8}, 0\right)$, then all the rest have the form $\left(\frac{(2m-2)(6m-1)}{2}, 0\right)$. And it is true that the singular vectors in the Verma modules $M(0, 0)$ and $M\left(\frac{1}{8}, 0\right)$ have precisely these weights (see the next section). However it does not seem likely that the Casimir operators provide a complete answer to the question whether two particular weights may be weights of the composition factors of an indecomposable Bernstein-Gelfand-Gelfand module (precisely as in the Kac-Moody case, see Section 3.3.3). For example, the values of functions (19) are the same for $h = \frac{175}{32}$ and $\frac{207}{32}$, but it does not seem plausible that $\left(\frac{175}{32}, 0\right)$ and $\left(\frac{207}{32}, 0\right)$ are weights of composition factors of the same irreducible Bernstein-Gelfand-Gelfand module.

4.8. Structure of reducible Verma modules and modules of semi-infinite forms.

In this Section, I have to restrict myself to a sketchy presentation. Most of the results were proven by Feigin and myself in [13], and then Astashkevich [14] found a shorter proof (which covers also the case of Neveau-Schwarz superalgebra.).

4.8.1. Preliminary computations and examples. Consider a pair (h, c). According to Section 7.3, pairs (p, q), for which $\Phi_{pq} \ni (h, c)$ form 4 lines in the plane $\mathbb{C}^2(p, q)$. Take any of these four lines, and denote it as $\ell_{h,c}$. Roughly, there are three possibilities. First: $\ell_{h,c}$ contains no integral points, or just one integral point, but the product of its coordinates is not positive. In this case the module $M(h, c)$ is irreducible. Second: $\ell_{h,c}$ contains only one integral point, (p, q), with $pq > 0$. Then $M(h, c)$ contains only one proper submodule, this submodule is generated by a singular vector of degree pq, is isomorphic to $M(h + pq, c)$, and is irreducible. (By the way, the fact that the module $M(h + pq, c)$ is irreducible requires a separate proof. The proof is based on the following obvious fact which will be also useful in the future:

$$\textit{if } (h, c) \in \Phi_{p,q}, \textit{ then } (h + pq, c) \in \Phi_{-p,q}; \tag{21}$$

this implies that the line $\ell_{h+pq,c}$ contains an integral point with a negative product of coordinates, and it cannot contain any other integral points, since it is parallel to $\ell_{h,c}$.) Third case: $\ell_{h,c}$ contains infinitely many integral points; this case deserves a separate consideration.

To contain infinitely many integral points, a line must have rational slope. Thus, let slope be $t = -\frac{r}{s}$, an irreducible fraction. Then the line has the equation $rp + sq = m$. According ton the computations in the end of Section 4.7.2,

$$h = \frac{(r+s)^2 - m^2}{4rs}, \; c = \frac{(3r + 2s)(2r + 3s)}{rs} = \left(3 + \frac{2s}{r}\right)\left(3 + \frac{2r}{s}\right).$$

Now, let us consider examples.

4.8.1.1. First example: $\mathbf{h = 0, c = 0}$. Seemingly the simplest, this one, actually, is one of the most complicated. For $c = 0$, we can take in the formulas above $r = 3$, $s = -2$. Then $h = \frac{m^2 - 1}{24}$ *, and the line $\ell_{h,c}$ has the equation $3p - 2q = m$; for $h = 0$, we should take $m = 1$. The line $3p - 2q = 1$ has integral points $\ldots, (-5, -8), (-3, -5), (-1, -2), (1, 1), (3, 4), (5, 7), \ldots$ with the products of coordinates

* By the way, this explains the observation made in Section 4.3.2 (see Remark there)

$1, 2;\ 12, 15;\ 35, 40; \ldots$. Thus the module $M(0,0)$ has singular vectors of each of these degrees; does it have more? The singular vector of degree 1 generates a submodule of $M(0,0)$ isomorphic to $M(1,0)$. The line $\ell_{1,0}$ has the equation $3\alpha - 2\beta = 5$; the integral points are $\ldots(-3,-7),(-1,-4),(1,-1),(3,2),(5,5),\ldots$, the products of coordinates are $\ldots,21,4,-1,6,25,\ldots$. The negative product -1 is not a degree of a singular vector; its meaning is that $M(1,0)$ is generated by a singular vector of degree 1 in another Verma module (which we already know: it is $M(0,0)$)*. But singular vectors of degrees $4,6;\ 21,25;\ldots$ have degrees $5,7;\ 22,26;\ldots$ in $M(0,0)$ which are *not* among the degrees listed above. Go further: $\ell_{2,0}$ has the equation $3\alpha - 2\beta = 7$, the products of coordinates of the integral points on this line are $-2;\ 3,5;\ 20,24;\ldots$, which show that $M(2,0)$ is generated by a singular vector of degree 2 in another Verma module (it really is!) and has singular vectors which in $M(0,0)$ have degrees $5,7;\ 22,26;\ldots$ – the same as before. Next: the line $\ell_{5,0}$ has the equation $3\alpha - 2\beta = 11$, the integral points are $\ldots,(-3,-10),(-1,-7),(1,-4),(3,-1),(5,2),(7,5),\ldots$ with the product of coordinates $-4,-3;\ 7,10;\ 30,35;\ldots$, that is, $M(5,0)$ brings to $M(0,0)$ singular vectors of degrees $12,15;\ 35,40;\ldots$, but we already had these degree. Thus $M(0,0)$ has singular vectors of degrees **1**, **2**, 5, 7, **12**, **15**, 22, 26, **35**, **40**, …; "half" of them (boldface in the sequence above) come to $M(0,0)$ along the hyperbolas $\Phi_{p,q}$. But some of them do not appear as a result of a deformation. I mean that, for instance, $M(0,0)$ has a singular vector of degree 5, but no other module $M(h,c)$ with (h,c) in some neighborhood of $(0,0)$ has a singular vector of this degree.

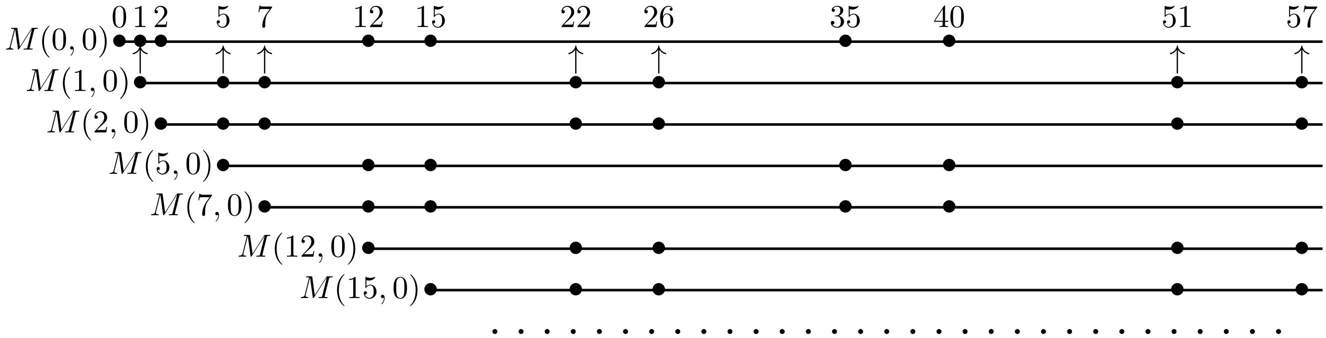
The reader may recognize in numbers $1,2;\ 5,7;\ 12,15;;\ 22,26;\ 35,40$ the Euler's "pentagonal numbers", $\dfrac{3n^2 \pm n}{2}$. Let us provide the explanation of this in general form. Let $(h,c) = \left(\dfrac{3n^2+n}{2}, 0\right)$. The corresponding line in the plane $\mathbb{C}(p,q)$ is $3p - 2q = 6n+1$. This line contains an integral point $(2n+1, 1)$ and all the point, which are obtained from it by adding a multiple of the vector $(3,2)$. The corresponding integral points with positive product of coordinates are $(2n+1+2k, 1+3k)$ and $(-1-2k, -2-3n-3k)$, where k is a non-negative integer. The degrees of singular vectors in the module $M\left(\dfrac{3n^2+n}{2}, 0\right)$ are product of their coordinates; the types of these singular vectors are

$$\left(\frac{3n^2+n}{2} + (2n+1+2k)(1+3k), 0\right) = \left(\frac{3(n+2k+1)^2 - (n+2k+1)}{2}, 0\right),$$
$$\left(\frac{3n^2+n}{2} + (1+2k)(2+3n+3k), 0\right) = \left(\frac{3(n+2k+1)^2 + (n+2k+1)}{2}, 0\right).$$

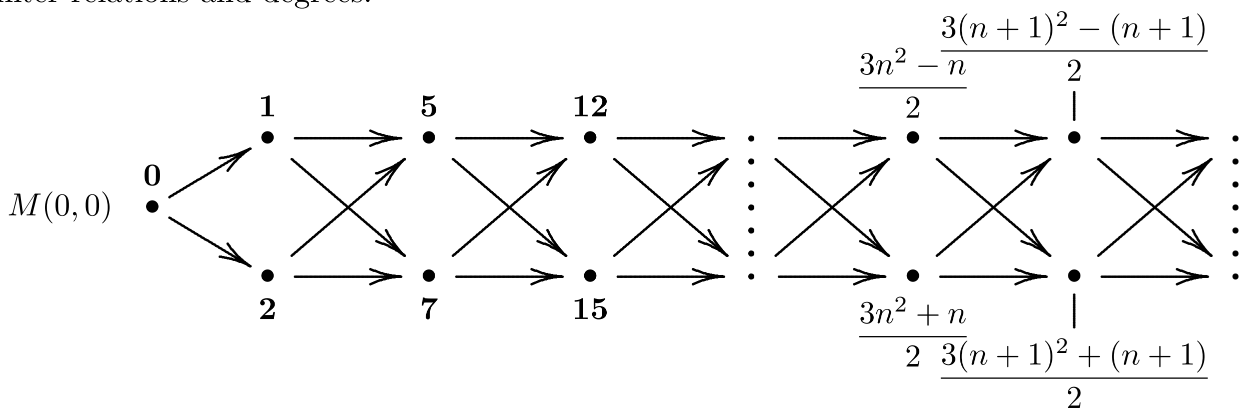
A similar computation shows that the module $M\left(\dfrac{3n^2 n}{2}, 0\right)$ contain singular vectors *of*

* I have in mind the following fact. If $(p,-q) \in \ell_{h,c}$ for some positive integers p, q, then, according to (21), $(h-pq, c) \in \Phi_{pq}$. So $M(h-pq,c)$ contains a singular vector of the type $(h-pq+pq, c) = (h,c)$, and hence $M(h,c) \subset M(h-pq,c)$.

the same type. The scheme of relations between the modules $M\left(\frac{3n^2 \pm n}{2}, 0\right)$ and degrees of their singular vectors is shown below.

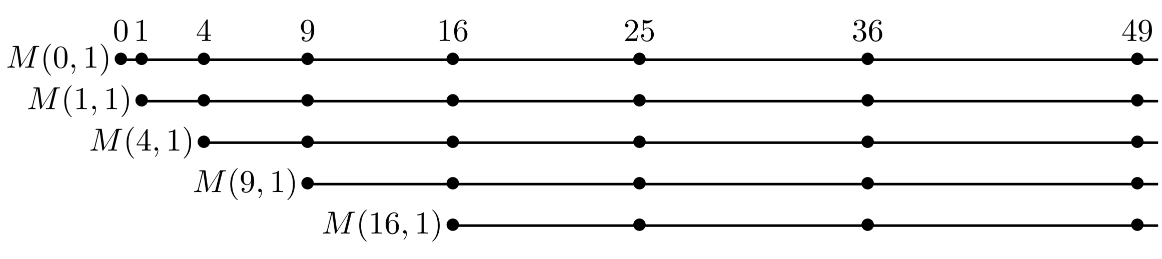


Below, we show a scheme of the singular vectors of the module $M(0,0)$ with their inter-relations and degrees.



4.8.1.2. Second example: h = 0, c = 1. The line $\ell_{0,1}$ has an equation $\alpha - \beta = 0$, the integral points are $\ldots, (-2,-2), (-1,-1), (0,0), (1,1), (2,2), \ldots$, the products of coordinates are the squares: $1, 4, 9, 16, 25, \ldots$. These are the degrees of singular vectors.

The submodules $M(k^2, 1)$ of $M(0,0)$ do not add any new singular vectors. indeed, an easy computation shows that the line $\ell_{k^2,1}$ has the equation $p - q = 2k$. Thus, the module $M(k^2, 1)$ has singular vectors of degrees $p(p+2k)$. The degrees of these vectors in the module $M(0,1)$ are $k^2 + p(p+2k) = (p+k)^2$, and singular vectors of these degrees have been already constructed before. Thus, the structure of the module $M(0,1)$ looks as shown below:



4.8.2. Structure of reducible Verma modules: theorems and comments. The main results of this Section (Theorems 4.18 – 4.21) provide a full classification and a description of interdependence of singular vectors in all reducible Verma modules over the Virasoro algebra. All these results can be proved by computations similar to those in the examples in Section 4.8.1, and I do not return to them here. The reader can find the details in the article [13] of Feigin and me. A clarification of our proofs and a partial "superization" of them are contained in the article [14] of A. Astashkevich.

THEOREM 4.17. *All submodules of Verma modules over the Virasoro algebra are generated by singualr vectors.*

(To show that this theorem is non-trivial we mention the fact that Verma modules over affine algebras (of sufficiently big range) are not always generated by singular vectors. For semi-infinite forms this will be explained in Section 4.8.4. A more radical example: modules $\overline{M(h,c)}$ contragredient to the Verma modules over the Virasoro algebra *never* contain singular vectors of positive degrees – see Section 4.3.4.)

A sketch of proof of Theorem 4.17 will be given in Section 4.8.3. (A more detailed versioin is contained in [13].)

4.8.2.1. Introduction. Let $h, c \in \mathbb{C}$, and let $\ell_{h,c} \subset \mathbf{C}^2(p,q)$ be the line defined in Section 4.8.1. The following result has been already discussed above, but we repeat it here for the completeness sake.

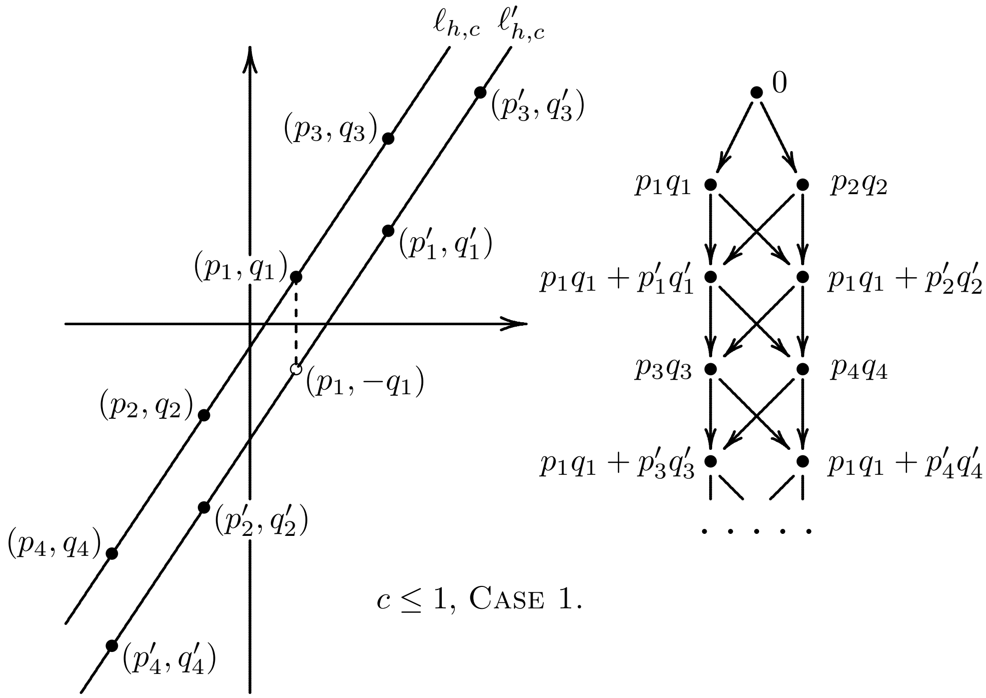
THEOREM 4.18. (1) *If the line $\ell_{h,c}$ contains no integral points with positive product of coordinates, then the module $M(h,c)$ is trreducible.* (2) *If the line $\ell_{h,c}$ contains precisely one integral point with the positive product of coordinates, and this product is m, then $M(h,c)$ has precisely one proper submodule, this submodule is isomorphic to $M(h+m,c)$ and is irreducible.*

The remaining case is when $\ell_{h,c}$ contains infinitely many integral points. As we already know, in this case

$$h = \frac{m^2 - (r+s)^2}{4rs},\ c = \frac{(3r+2s)(2r+3s)}{rs}\ (r, s \in \mathbb{Z}_{\neq 0},\ m \in \mathbb{Z});$$

the line $\ell_{h,c}$ has the equation $rp + sq = m$. We need to distinguish 6 cases. First, either $c \leq 1$ (equivalently, the line $\ell_{h,c}$ has a positive slope), or $c \geq 25$ ($\ell_{h,c}$ has a negative slope. Second, the line $\ell_{h,c}$ may intersect in integral points no coordinate axes (we refer to this case as to CASE 1), one axis (CASE 2), or both of them (CASE 3). We describe the structure of the module $M(h,c)$ below in each of these cases.

4.8.2.2. c ≤ 1, Case 1. THEOREM 4.19. *Let $(p_1,q_1),(p_2,q_2),(p_3,q_3),\dots$ are integral points of the line $\ell_{h,c}$ with positive products of coordinates ordered in such a way that $p_1q_1 < p_2q_2 < p_3q_3 < \dots$.* (If $p_iq_i = p_jq_j$ for some $i \neq j$, then $\ell_{h,c}$ intersects both axes in integral points). *Let then $\ell'_{h,c}$ be the line parallel to $\ell_{h,c}$ and passing through the point $(p_1,-q_1)$, and $(p'_1,q'_1),(p'_2,q'_2),(p'_3,q'_3),\dots$ are integral points of the line $\ell'_{h,c}$ with positive products of coordinates ordered in a similar way. Then the module $M(h,c)$ has one (up to a non-zero factor) singular vector of each of the degrees p_kq_k, $p_1q_1 + p'_kq'_k$ and no other singular vectors; the interdependence of the singular vectors is shown on the diagram below on the previous page.*



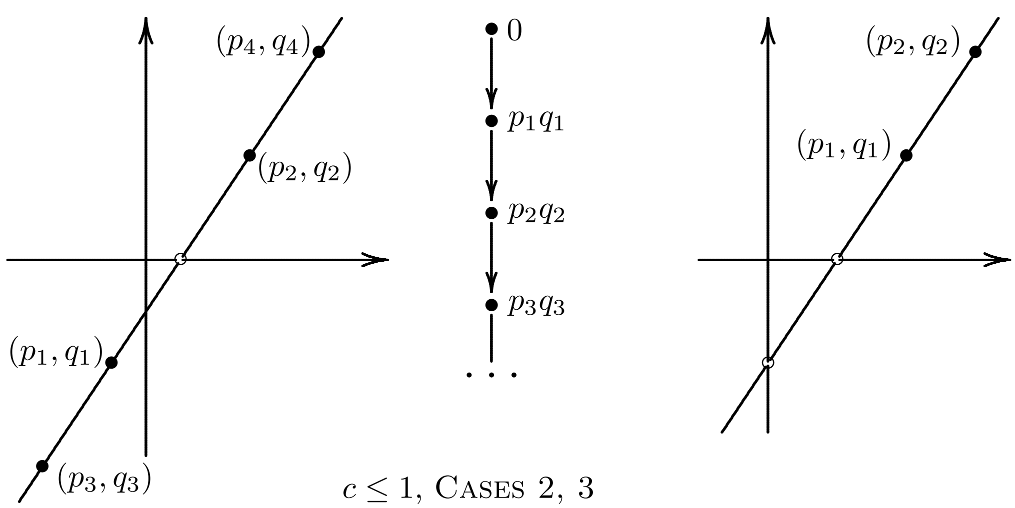
$c \leq 1$, Case 1.

Thus, the module $M(h,c)$ contains singular vectors of degrees p_kq_k, $k = 1, 2, \dots$ and of degrees $p_1q_1 + p'_kq'_k$, $k = 1, 2, \dots$. Similarly to the situation described in Section 4.8.1.1, singular vectors of degrees P_kq_k “arrive” at $M(h,c)$ along the curves Φ_{p_k,q_k}, while Verma modules $M(h',c')$ with (h',c') close, but not equal to (h,c) do not contain singular vectors of degrees $p_1q_1 + p'_kq'_k$.

4.8.2.3. c ≤ 1, Case 2 and 3. Theorem 4.20. *Let $\ell_{h,c}$ intersect one axis in an integral point, and let $(p_1,q_1), (p_2,q_2), (p_3,q_3), \dots$ be integral points of the line $\ell_{h,c}$ with positive products of coordinates ordered as above. Then the module $M(h,c)$ has one singular vector of each of the degrees p_kq_k, and no other singular vectors.*

If $\ell_{h,c}$ intersect both axes in integral points, then all the same, but we consider only integral points with both coordinates positive.

The interdependence of singular vectors is shown on the diagrams below.

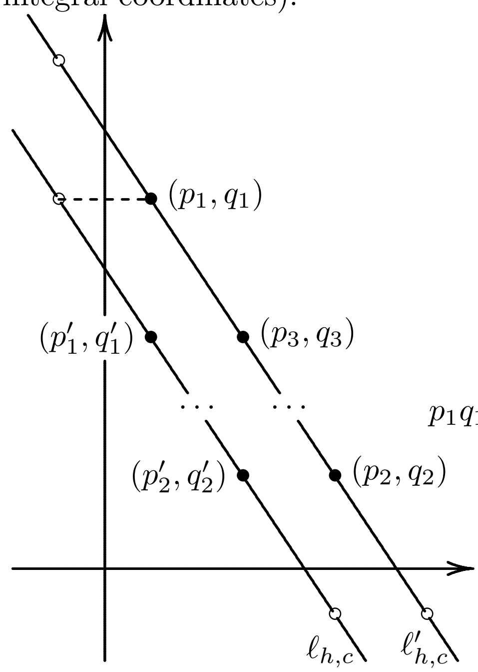


$c \leq 1$, Cases 2, 3

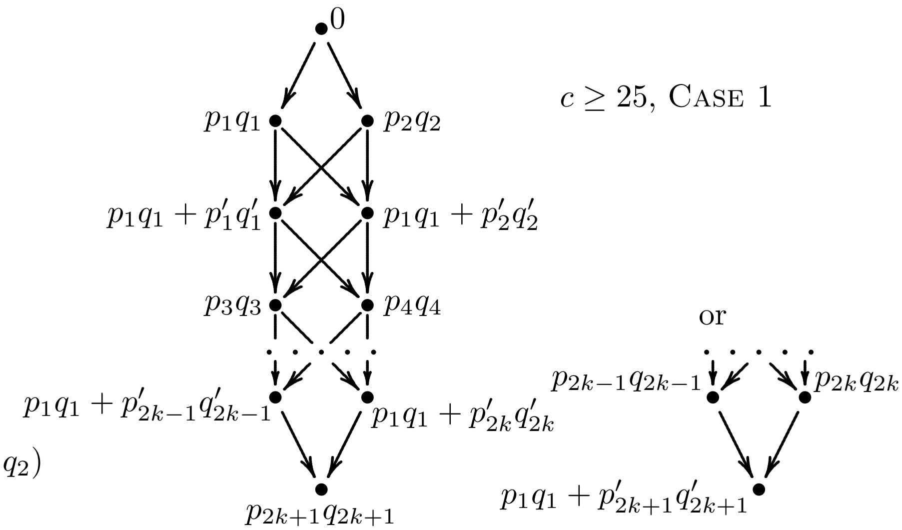
4.8.2.4. c ≥ 25, Cases 1-3. THEOREM 4.21. *See the diagrams below.*

In Case 1, the total amount of singular vectors is odd, but it may be 1 or 4 modulo 4, whence two options for the lower end.

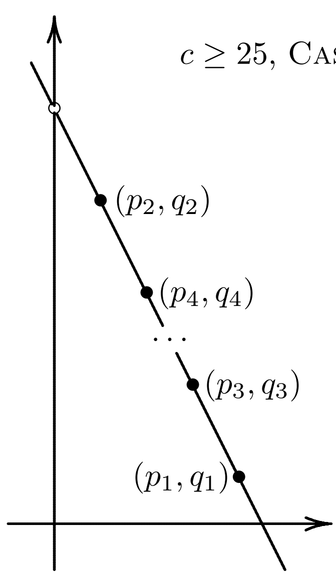
In Case 3, we apply the notation (p_i, q_i) only to the point below the midpoint of the interval between the two intersections with the axes (including the midpoint itself, if it has integral coordinates).

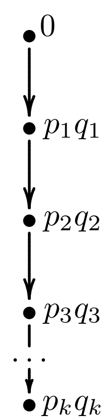


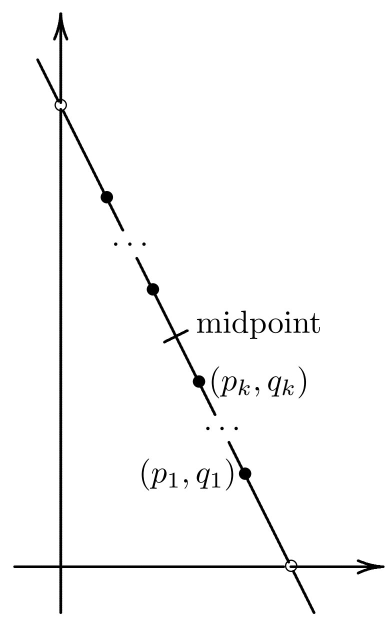
$c \geq 25$, CASE 1



$c \geq 25$, CASES 2,3







4.8.2.5. Final remarks. Notice that the cases $c \leq 1$ and $c \geq 25$ are symmetric to each other: there is a 1–1 correspondence between embeddings

$$M(h, c) \to M(h', c) \quad \text{and} \quad M(1 - h', 26 - c) \to M(1 - h, 26 - c)$$

In all cases, the module $M(h,c)$ is contained in other Verma module(s) if and only if the line $\ell_{h,c}$ contains integral point(s) with negative product(s) of coordinates. In Cases 1 and 2, these "bigger" Verma modules correspond to these points, in Case 3 they correspond to pairs of these points with equal products of coordinates. Therefore, in the case $c \leq 1$ the module $M(h,c)$ can be contained in finitely many different Verma modules, while in the case $c \geq 25$ this set of "bigger" Verma modules is either empty, or infinite. These embracing Verma modules are related by diagrams similar to those in Sections 4.8.2.2 – 4.8.2.4.

4.8.3. Submodules of Verma modules. This Section contains a proof of Theorem 4.17: all submodules of Verma modules over the Virasoro algebra are generated by singular vectors. We will assume in this proof that the Statements of Theorems 4.18 – 4.20 are true (see comments in Section 4.8.2).

4.8.3.1. The Shapovalov form of a quotient of a Verma module. Consider the curve $\Phi_{p,q} = \{h = h(t) = h_{p,q}(t), c = c(t)\} \subset \mathbb{C}^2(h,c)$. For every t, the module $M(h(t),c(t))$ contains a singular vector $\eta_{p,q}(t) = \sigma_{p,q}(t)v \in M(h(t),c(t))_{pq}$ (we use the notations of Section 4.6). This $\sigma_{p,q}(t)$ is a polynomial in t, t^{-1} with values in $U(\mathfrak{n}_-)$, which has no zeroes and its upper and lower degree are, correspondingly, $(p-1)q$ and $q(p-1)$ (see Proposition 4.15 in Section 4.6.5).

Let $M(t) \cong M(h(t)+pq, c(t))$ be the submodule of $M(h(t),c(t))$ generated by $\eta_{p,q}(t)$, and let $L(t) = M(h(t),c(t))/M(t)$. The Shapovalov form

$$F(h(t),c(t))\colon M(h(t),c(t)) \to M(h(t),c(t))^*$$

is homogeneous of degree zero and is zero on $M(t)$ and takes values in $L(t)^* \subset M(h(t),c(t))^*$. Thus, it induces a non-degenerate symmetric bilinear form

$$S_k^{p,q}(t) = S_k(t)\colon L(t)_k \to (L(t)_k)^*.$$

The determinant of this form is defined up to a non-zero factor, but we can speak of its zeroes and their multiplicities. Let d_k be the sum of these multiplicities for all $t \in \mathbb{C}$. In Sections 4.8.3.2 snd 4.8.3.3 below, we provide two computations of d_k. The first computation is based on the degrees of polynomials $\sigma_{p,q}(t)$ (see above). The second computation is done under the assumption that all submodules of the Verma modules $M(h,c)$ are generated by singular vectors (Theorem 4.17). The results of these two computations will agree, and we will use it to complete the proofs of Theorems 4.18–4.21.

4.8.3.2. The first computation of $\mathbf{d_k}$. We augment the curve $\Phi_{p,q}$ by $t=0$ and ∞. We obtain a rational curve $\widehat{\Phi}_{p,q} \subset \mathbb{C}P^2$. The spaces $M(h(t),c(t))_k$ form a trivial $\mathbf{p}(k)$-dimensional bundle $\mathcal{M}(h(t),c(t))_k$ over $\widehat{\Phi}_{p,q}$ (we denote as $\mathbf{p}$ the "partition function": $\mathbf{p}(k)$ is the number of partitions of k). The subspaces $M(h(t)+pq, c(t))_{k-pq} = M(t)_k$ of $M(h(t),c(t))_k$ form a $\mathbf{p}(k-pq)$-dimensional subbundle $\mathcal{M}(t)_k$ of $\mathcal{M}(h(t),c(t))_k$. The fibers of the quotient $\mathcal{M}(h(t),c(t))_k/\mathcal{M}(t)_k$ are spaces $L(t)_k$, and we denote this quotient as $\mathcal{L}(t)_k$.

Further, we denote the line bundle $\mathcal{M}(t)_{pq}$ as $\eta_{p,q}$. This line bundle has a section $\eta_{p,q}(t)$ without zeroes and with two poles of degrees $(p-1)q$ and $(q-1)p$ (at $t=0$ and ∞). Hence,

$$\operatorname{Eu}\eta_{p,q} = -(p-1)q - (q-1)p = p+q-pq.$$

It is also obvious that $\mathcal{M}(t)_k \cong \mathbf{p}(k-pq)\eta_{p,q}$.

The determinant $\det S_k^{p,q}$ form a section of the line bundle

$$\omega_{p,q}^k = (S^2\Lambda^{\dim \mathcal{L}_k}\mathcal{L}_k)^*,$$

isomorphic to $\bigotimes^{2\mathbf{p}(k-pq)} \eta_{p,q}$ (since $\mathcal{L}(t)_k$ is isomorphic to [trivial bundle $-\ \mathbf{p}(k-pq)\eta_{p,q}$]). Hence, $\operatorname{Eu}(\omega_{p,q}^k) = 2\mathbf{p}(k-pq)\operatorname{Eu}\eta_{p,q}$. Let $P_k(0)$ and $P_k(\infty)$ be the multiplicities of poles of the section $\det S_k^{p,q}$ at $t=0$ and ∞. Then for the total multiplicity d_k of zeroes of $\det S_k^{p,q}$ we have the formula

$$d_k = P_k(0) + P_\infty(0) + 2\mathbf{p}(k-pq)\operatorname{Eu}\eta_{p,q}.$$

It remains to calculate $P_k(0)$ and $P_k(\infty)$. It is not hard.

Near ∞, the determinant of the form $S^{p,q}$ may be calculated as the determinant of the principal minor of the corresponding Shapovalov matrix of the whole $M(h(t), c(t)$, which correspond to the part of the base $\{e_{j_1}\dots e_{j_s}v\}$, in which e_q is present less than p times (recall that the fiber of $\eta_{p,q}$ over ∞ is spanned by $e_q^p v$). For $t\to\infty$, $h \sim \dfrac{(1-p^2)t}{4}$, $c\sim 6t$. The degree of the minor considered is the sum of degrees of diagonal entries, and this sum is easy to compute, since

$$e_{-i}e_iv = \left[2ih + \frac{1}{12}(i^3-i)c\right]v \sim \frac{(i-p)(1-p^2)}{2}tv.$$

which has degree 1, if $i\neq p$, and degree 0, if $i=p$. Thus $P_k(\infty)$ is equal to the total amount of elements not equal to p in all partitions of k, which contain q less than p times. $P_k(0)$ is described in the same way with $p\leftrightarrow q$. The results of the computations may be described by the formulas

$$\sum P_k(\infty)t^k = \mathbf{p}(t)(1-t^{pq})\left(s(t) - \frac{t^p}{1-t^p}\right), \sum P_k(0)t^k = \mathbf{p}(t)(1-t^{pq})\left(s(t) - \frac{t^q}{1-t^q}\right),$$

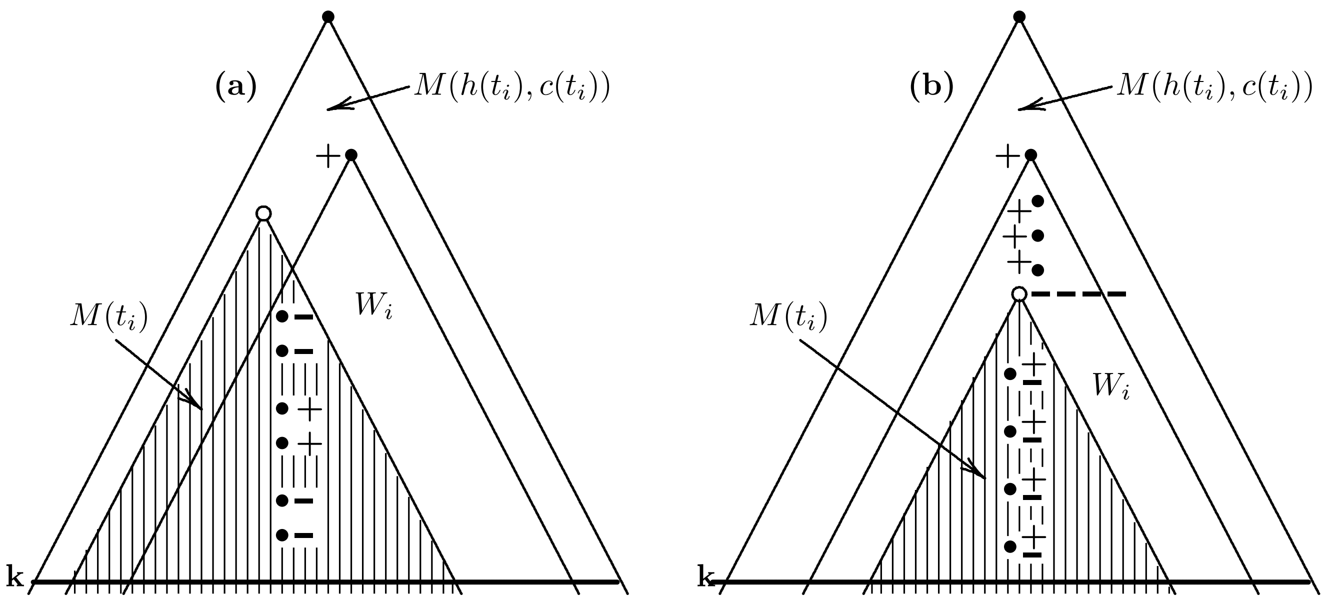
where $\mathbf{p}(t) = \sum_{k=0}^{\infty}\mathbf{p}(k)t^k,\ s(t) = \sum_{u=1}^{\infty}\sum_{v=1}^{\infty}t^{uv}$.

4.8.3.3. The second computation of $\mathbf{d_k}$. Let $w_i \in M(h_{p,q}(t_i), c(t_i)),\ i=1,\dots,N$ be all singular vectors of positive degrees $\le k$. For each i we denote by b_i the dimension of the intersection of $L(t_i)_k$ with the submodule of $L(t_i)$ generated by the projection of w_i. Then we put $r_i=2$, if the pair $h_{p,q}(t_i), c(t_i)$ belongs to the Case 3, and $r_i=1$ otherwise and define

$$c_k = \sum_{i=1}^{N} r_ib_i.$$

Our aim is to calculate c_k on the base of Theorems 4.18 – 4.20 and then to check that c_k coincides with d_k.

Denote by W_i the submodule of $M(h(t_i), c(t_i))$ generated by w_i. According to Theorems 4.18 – 4.20, there are two possibilities of mutual disposition of the modules W_i and $M(t_i)$ (see Figure below).



In both cases, we must take the dimensions of the submodules of $M(h(t_i), c(t_i))$ generated by singular vectors with the signs indicated (in the case (**b**) the singular vectors inside $M(t_i)$ make no contribution, but it is convenient for us successively with the signs + and −). We observe that all the singular vectors, which come to the module $M(h_{p,q}(t), c(t))$ along the curves $\Phi_{p',q'}$ with $p'q' \le pq$ are taken with the sign +, and all the singular vectors which come in the similar way in modules $M(t) = M(h_{p,q}(t) + pq, c(t)$ are taken with the sign −. Furthermore, the coefficient r_i is equal to 2 (which may happen only in the case (**b**)) if and only if either the singular vector comes at once along two different curves Φ, ot it comes along a curve tangent to the curve $(h_{p,q}(t), c(t))$ or $(h_{p,q}(t)+pq, c(t))$. In addition to that, we must take with the sign − and some coefficient $N(p,q)$ (not depending on k) the dimension of $M(t_i)_k$. The final result is

$$c_k = \sum_{\substack{p' \ge q',\, p'q' \le k \\ \{p',q'\} \ne \{p,q\}}} \alpha_{p',q'} \mathbf{p}(k - p'q') - \sum_{\substack{p'' \ge q'' \\ k''l'' \le k - pq}} \beta_{p'',q''} \mathbf{p}(k - pq - p''q'') - N(p,q)\mathbf{p}(k - pq), \quad (22)$$

where $\alpha_{p'q'}$ is the number of intersection points of the curves $\Phi_{p',q'}$ and $\Phi_{p,q}$, and $\beta_{p'',q''}$ is the number of intersection points of the curves $\Phi_{p'',q''}$ and $(h_{p,q}(t)+pq, c(t))$ (the tangency points are regarded as double intersection points). These numbers generically are equal to 4, but they can be 3, 2, or even 1, if one or both "curves" are straight lines, or if the sets $\{p, q\}, \{p', q'\}$ (or $\{p'', q''\}\}$) have non-empty intersection. The result of these computations is:

$$\sum_k \sum_{p',q'} \alpha_{p',q'} \mathbf{p}(k - p'q') t^k = \mathbf{p}(t) \left[2s(t) - \frac{t^p}{1 - t^p} - \frac{t^q}{1 - t^q} - 2t^{pq} \right],$$

$$\sum_k \sum_{p'',q''} \beta_{p'',q''} \mathbf{p}(k - pq - p''q'') t^k = \mathbf{p}(t) \left[2s(t) - \frac{t^p}{1 - t^p} - \frac{t^q}{1 - t^q} \right] t^{pq}.$$

Hence,

$$\sum_k c_k t^k = \mathbf{p}(t)(1-t^{pq})\left[2s(t) - \frac{t^p}{1-t^p} - \frac{t^q}{1-t^q} - 2t^{pq}\right] - (N(p,q)+2)\mathbf{p}(t)t^{pq}.$$

It remains to prove:

LEMMA.

$$N(p,q) = 2(2pq - p - q - 1).$$

Proof. $N(p,q)$ is composed of $\sum r_i$, where the summation is taken over those i, for which $W_i \supset M(t_i)$ (case **(b)**), and the doubled number of pairs $(p', q') \neq (p,q), (q,p)$ with $p'q' - pq$ (these pairs were not excluded from the first sum in formula (22), but the singular vectors, which come along the corresponding curves $\Phi_{p',q'}$ are contained in $M(t_i)_{pq}$ and thus do not contribute in c_k). That is, we must draw all the lines through the point (p,q) not parallel to the axes, and then mark on them all integral points $(p', q') \neq (\pm p, \pm q)$ with $0 < p'q' \leq pq$ if the intersection point with the set $\{xy = 0, x \geq 0, y \geq 0$ is integral, and all such points but one otherwise. Then $N(p,q)$ will be the full number of marked points. We observe that (i) the lines symmetric in the line $x = p$ bear the equal numbers of marked points; (ii) the number of marked points on a line ℓ of positive slope is equal to $2a + b - c$, where a, b, c are the number of integral points in the intersections of ℓ with the sets $\{0 < x < p, 0 < y < 1\}, \{xy = 0, x \geq 0, y \geq 0\}, \{(-p, -q), (-q, -p)\}$ (so b and c are equal to either 0 or 1). Hence

$$N(p,q) = 2[2(p-1)(q-1) + (p+q-1) - 2] = 2(2pq - p - q - 1).$$

We see that $d_k = c_k$. It follows from this that all submodules of $M(h(t), c(t))$ are generated by singular vectors, since a submodule not generated by singular vectors, would have increased some c_k, and the above equation would have been impossible.

4.8.4. Final remark: unitary representations. In conclusion, we mention a description of all values of h and c, for which the irreducible representation $L(h,c)$ of the Virasoro algebra possesses the structure of a unitary representation. This result belongs to D. Friedan, Z. Qiu, S. Shenker, P. Goddard, A. Kent, and D. Olive; the relevant reference is [21].

PROPOSITION 4.22. *The irreducible representation of* $\mathfrak{Vir}$ *with the highest weight* h, c *possesses a structure of a unitary representation if and only if*

$$c = 1 - \frac{6}{m(m+1)}, \ h = \frac{((m+1)^2 k - m\ell^2 - 1}{4m(m+1)}$$

for some k, ℓ, m *satisfying the conditions* $m \geq 2, 1 \leq \ell \leq k < m$.

For the proof see the article cited above.

Examples: $c = 0, h = -\frac{1}{24}$; $c = \frac{1}{2}, h = 0, \frac{1}{16}, \frac{1}{2}$.

Notice that (h,c) given by the formula in Proposition 4.22 belongs to the curve $\Phi_{m,m+1}$ and the Verma nodule $M(h,c)$ has infinitely many singular vectors.

4.8.5. Structure of modules of semi-infinite forms. The main goal of this section (which contains no proofs, proofs can be found in [13]) is to demonstrate that the highly reducible modules of semi-infinite forms are considerably different from the corresponding Verma modules. We will use the following notation: $L(h,c)$ is the irreducible quotient of the Verma module $M(h,c)$.

Let Ψ_{pq} a curve in the plane $\mathbb{C}^2(\lambda,\mu)$ given by the parametric equations

$$\lambda = \lambda(\theta) = \frac{\theta - \theta^{-1}}{\sqrt{2}} - \frac{1}{2},\ \mu = \mu_{pq}(\theta) = \frac{(p+1)\theta - (q+1)\theta^{-1}}{\sqrt{2}}.$$

A direct computation shows that

$$-1 - \lambda(\theta) = \lambda(\theta^{-1}),\ \mu_{pq}(\theta) = -\mu_{qp}(\theta^{-1}),\ \mu_{pq}(\theta) - 2\lambda(\theta) - 1 = \mu_{-p,-q}(\theta^{-1}),$$

that is, the transformations $(\lambda,\mu) \mapsto -(1-\lambda, \mu - 2\lambda - 1),\ (\lambda,\mu) \mapsto (-1-\lambda, -\mu)$ which take the modules of semi-infinite forms into contragredient or isomorphic modules, map the curve Ψ_{pq} onto Ψ_{qp} or $\Psi_{-p,-q}$. Also

$$-2(6\lambda(\theta)^2 + 6\lambda(\theta) + 1) = c(t),\ \frac{1}{2}\mu_{pq}(\theta)(\mu_{pq}(\theta) - 2\lambda(\theta) - 1) = h_{pq}(t)$$

where $t = -\theta^2$, so the image of the curve Ψ_{pq} with respect to the map $\mathbb{C}^2(\lambda,\mu) \to \mathbb{C}^2(h,c),\ (\lambda,\mu) \mapsto \left(\frac{1}{2}\mu(\mu - 2\lambda - 1), -2(6\lambda^2 + 6\lambda + 1)\right)$ is Φ_{pq}. Differently: the inverse image of $\Phi_{pq} = \Phi_{qp}$ consists of the curves $\Psi_{pq}, \Psi_{qp}, \Psi_{-p,-q}, \Psi_{-q,-p}$.

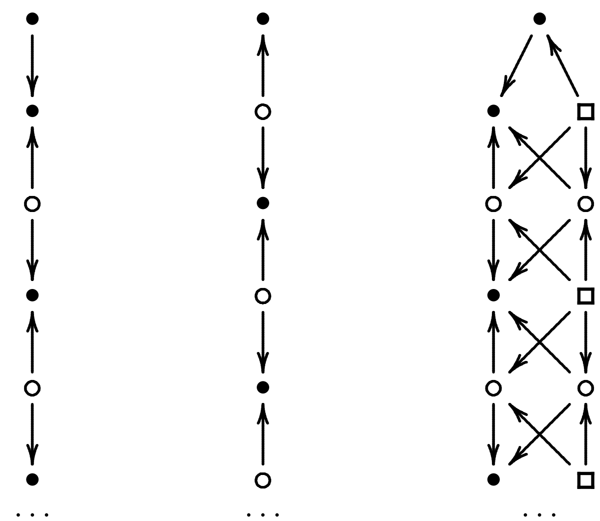
Let us fix $\lambda, \mu \in \mathbb{C}$ and corresponding $h = \frac{1}{2}\mu(\mu - 2\lambda - 1),\ c = -2(6\lambda^2 + 6\lambda + 1)$. The set $\{(\alpha,\beta) \mid (\lambda,\mu) \in \Psi_{\alpha\beta}\}$ consists of two lines symmetric to each other with respect to the diagonal $\alpha = \beta$ (two of the four similar lines for (h,c), see Section 4.8.2). Choose one of these lines and denote it as $\ell_{\lambda\mu}$. The structure of the module $\mathcal{H}(\lambda,\mu)$ depends on the set of integral points on this line.

If the line $\ell_{\lambda\mu}$ contains no integral points, or only one integral point with a non-positive product of coordinates, then the module $\mathcal{H}(\lambda,\mu)$ is irreducible and is isomorphic to $M(h,c)$. If the line $\ell_{\lambda\mu}$ contains only one integral point, (p,q), and $p > 0, q > 0$, then $\mathcal{H}(\lambda,\mu)$ is again isomorphic to $M(h,c)$, the latter contains a proper irreducible submodule isomorphic to $M(h+pq,c)$ with the quotient $L(h,c)$. In the similar case with $p < 0, q < 0$, the module $\mathcal{H}(\lambda,\mu)$ is isomorphic to the module $\overline{M}(h,c)$ *contragredient* to $M(h,c)$. In this case, the highest weight vector of $\mathcal{H}(\lambda,\mu)$ generates a proper irreducible submodule isomorphic to $L(h,c)$, and the quotient, also irreducible, is isomorphic to $M(h+pq,c)$.

If the line $\ell_{\lambda\mu}$ contains infinitely many integral points and crosses the both axes in integral points, then

$$\mathcal{H}(\lambda,\mu) \cong L(h,c) \oplus L(h + p_1q_1, c) \oplus L(h + p_2q_2, c) \oplus \dots$$

where the notations come from the diagrams in Theorem 4.20 of Section 4.8.2 corresponding to Case 3. The sum above is infinite, if the slope of $\ell_{\lambda,\mu}$ is positive, and finite, if the slope is negative. If the line $\ell_{\lambda\mu}$ contains infinitely many integral points, has a positive slope and crosses at most one of the axes in an integral point, then the possible structure of the module $\mathcal{H}(\lambda, \mu)$ is presented on the diagrams below.



The black dots, white dots, and squares on these diagrams mean generators of modules; two symbols are connected with an oriented sequence of arrows if and only if the second one is contained in a submodule generated by the first one; the arrows directed upward (downward) correspond to polynomials in e_i with negative (positive) i. Black dots denote singular vectors; white dots become singular vectors after factorizing over the submodule generated by singular vectors, squares become singular vectors after second such operation. The two left diagrams (straight) correspond to the case when the line $\ell_{\lambda\mu}$ intersect one of the axes in an integral point; the first of them corresponds to the case $p_1 < 0, p_2 < 0$ (in the notation of the diagram in Theorem 4.19 of Section 4.8.2), the second one corresponds to the case $p_1 > 0, p_2 > 0$. The right diagram reflects the case when the line $\ell_{\lambda\mu}$ (of positive slope) does not cross any axis in integral points. Notice that in all the three cases the submodule generated by singular vectors is an infinite direct sum of modules which are irreducible, with one possible exception of an extension of one irreducible module with another one (corresponds to the picture $\overset{\bullet}{\underset{\bullet}{\downarrow}}$). After the factorization over this submodule, the singular vectors again generate a direct sum of irreducible modules, and the same after the second factorization.

If the slope of $\ell_{\lambda\mu}$ is negative, the structure of the module is similar; the main difference is that all the diagrams are finite.

5. The overview from the point of view of Heisenberg algebra.

The main goal of this part is to show that the whole theory of representations of the Virasoro algebra (and, actually, the affine algebras) may be understood as a projection of (much simpler) theory of representations of the (infinite-dimensional) Heisenberg algebra. As a by-product of this understanding, we will obtain a more "industrial" way of constructing singular vectors in the modules of semi-infinite forms, developed in Section 4.4.3. While the construction in Section 4.4.3 provides some sporadic examples (which was sufficient for the purposes there), our new way will give explicit formulas for *all* singular vectors in these modules.

5.1. The Heisenberg algebra and its canonical representation.

5.1.1. Definitions. Let $\mathfrak{H}$ be a Lie algebra with the basis consisting of F_j, $j \in \mathbb{Z}_{\neq 0}$ and $\mathbb{1}$ and the commutator relations

$$[F_i, F_j] = j\delta_{-i,j}\mathbb{1},\ [F_j, \mathbb{1}] = 0.$$

There exists a "canonical" $\mathfrak{H}$-module $\mathcal{D}$ which is the space $\mathbb{C}[x_1, x_2, \ldots]$ of polynomials of infinitely many variables with the action of $\mathcal{H}$ described by the formulas

$$F_j = x_j\cdot \text{ for } j > 0,\ F_{-j} = j\frac{\partial}{\partial x_j} \text{ for } j > 0,\ \mathbb{1} = \mathrm{id}\,.$$

With respect to the decomposition $\mathfrak{H} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, $\mathfrak{n}_- = \mathrm{span}(F_j, j > 0), \mathfrak{n}_+ = \mathrm{span}(F_j, j < 0), \mathfrak{h} = \mathbb{C}\mathbb{1}$, $\mathcal{D}$ may be described either as the Verma module $M(\lambda)$ with $\lambda(\mathbb{1}) = 1$ (and $v_\lambda = 1$) or as a unique irreducible $\mathfrak{H}$-module with $\mathbb{1}$ acting as id and a virtually nilpotent action of $\mathfrak{n}_-$.

5.1.2. The $\mathfrak{H}$-module $\mathcal{D}$ as a $\mathfrak{Vir}$-module. Put

$$e_i = \sum_{\substack{r>s\\ r+s=i\\ r\neq 0,\, s\neq 0}} F_rF_s \left(+\frac{1}{2}F_{\frac{i}{2}}^2, \text{ if } i \text{ is even and} \neq 0\right),\ z = \mathbb{1}.$$

Although the sum in the definition of e_i is infinite, it may be regarded as a valid operation in $\mathcal{D}$: for any $p \in \mathcal{D}$, there exists a $C \in \mathbb{Z}$ such that $F_jp = 0$ for any $j < C$.

PROPOSITION 5.1. *The operators $e_i, z\colon \mathcal{D} \to \mathcal{D}$ equip $\mathcal{D}$ with a $\mathfrak{Vir}$-module structure.*

Proof. Let $i + j \neq 0$.

$$[e_i, e_j] = \left[\sum_{r+s=i} F_rF_s \left(+\frac{1}{2}F_{\frac{i}{2}}^2\right), \sum_{u+v=j} F_uF_v \left(+\frac{1}{2}F_{\frac{j}{2}}^2\right)\right]$$

(we abbreviate the notations). The term F_wF_{i+j-w} appears in the last commutator twice:

$$[F_wF_{i-w}, F_{i+j-w}F_{w-i}] = (w-i)F_wF_{i+j-w},\ [F_{i+j-w}F_{w-j}, F_wF_{j-w}] = (j-w)F_wF_{i+j-w};$$

the total is $(j-i)F_wF_{i+j-w}$ (the formulas will look slightly differently, if w is $\frac{i}{2}, \frac{j}{2},$ or $\frac{i+j}{2}$; we leave the details to the reader). The case of $i + j = 0$ is substantially different. Notice that

$$[F_aF_{-b}, F_bF_{-a}] = bF_aF_{-a} - aF_bF_{-b}.$$

Hence (we write the formula for odd $j = 2k+1$, for even j a small modification is needed)

$$[e_{-j}, e_j] = \left[\sum_{r=1}^{k} F_{-r}F_{r-2k-1} + \sum_{s=1}^{\infty} F_s F_{-2k-s-1}, \sum_{r=1}^{k} F_{2k+1-r}F_r + \sum_{s=1}^{\infty} F_{2k+s+1}F_{-s},\right]$$

$$= \sum_{r=1}^{k}[F_{-r}F_{r-2k-1}, F_{2k+1-r}F_r] + \sum_{s=1}^{\infty}[F_s F_{-2k-s-1}, F_{2k+s+1}F_{-s}]$$

$$= \sum_{r=1}^{k}((2k+1-r)F_{-r}F_r + rF_{2k+1-r}F_{r-2k-1})$$

$$+ \sum_{s=1}^{\infty}((2k+s+1)F_sF_{-s} - sF_{2k+s+1}F_{-2k-s-1}).$$

In the first of the two sums in the last expression, we replace $F_{-r}F_r$ by $F_rF_{-r} + r\mathbb{1}$. We get

$$[e_{-j}, e_j] = \sum_{u=1}^{k}(2k+1-u)F_uF_{-u} + \sum_{u=k+1}^{2k+1}(2k+1-u)F_uF_{-u} + \sum_{u=1}^{\infty}(2k+1+u)F_uF_{-u}$$

$$+ \sum_{u=2k+2}^{\infty}(2k+1-u)F_uF_{-u} + \sum_{u=1}^{k}(2k+1-u)u\mathbb{1}$$

$$= \sum_{u=1}^{\infty} 2(2k+1)F_uF_{-u} + \frac{(2k+1)^3-(2k+1)}{12}\mathbb{1} = 2je_0 + \frac{j^3-j}{12}\mathbb{1},$$

as required. The case of $j = 2k$ is similar.

As a $\mathfrak{Vir}$-module, we will denote $\mathcal{D}$ as $\mathcal{K}$. If we put $\deg x_j = j$ (so $\deg F_j = j$), then $\mathcal{K}$ becomes a graded $\mathfrak{Vir}$-module: $\mathcal{K}_0 = \mathrm{span}(1)$, $\mathcal{K}_1 = \mathrm{span}(x_1)$, $\mathcal{K}_2 = \mathrm{span}(x_1^2, x_2)$, $\mathcal{K}_3 = \mathrm{span}(x_1^3, x_1x_2, x_3)$, and so on; in particular, $\dim \mathcal{K}_n = \mathbf{P}(n)$. This makes $\mathcal{K}$ analogous to $M(h,c)$ and $\mathcal{H}(\lambda,\mu)$; but unlike $\mathcal{K}$, each of these two depends on two complex parameters. We want now to extend the definition of $\mathcal{K}$ to a two-parameter family.

PROPOSITION 5.2. *For any $\alpha, \beta \in \mathbb{C}$, the operators*

$$\widetilde{e}_i = \begin{cases} e_i + (\alpha i + \beta)F_i & \text{for } i \neq 0, \\ e_0 + \dfrac{\beta^2 - \alpha^2}{2}\mathbb{1} & \text{for } i = 0, \end{cases} \qquad z = (1 - 12\alpha^2)\mathbb{1}$$

satisfy the Virasoro commutator relations.

LEMMA: $[e_i, F_j] = \begin{cases} jF_{i+j}, & \text{if } i+j \neq 0, \\ 0, & \text{if } i+j = 0. \end{cases}$

Proof of Lemma:

$$[e_i, F_j] = \sum_{r+s=i}([F_r, F_j]F_s + F_r[F_s, F_j])$$

$$\left(+\frac{1}{2}([F_{\frac{i}{2}}, F_j]F_{\frac{i}{2}} + F_{\frac{i}{2}}[F_{\frac{i}{2}}, F_j]), \text{ if } i \text{ is even and } \neq 0\right).$$

If $i+j=0$, then everything in this sum is 0; if $i+j\neq 0$, then one term survives, and it is jF_{i+j}.

Proof of Proposition 5.2. If $i\neq 0, j\neq 0, i+j\neq 0$, then $[\widetilde{e}_i,\widetilde{e}_j]=$

$$\begin{aligned}[e_i+(\alpha i+\beta)F_i, e_j+(\alpha j+\beta)F_j] &= [e_i,e_j]+(\alpha j+\beta)[e_i,F_j]-(\alpha i+\beta)[e_j,F_i]\\ &=(j-i)e_{i+j}+[j(\alpha j+\beta)-i(\alpha i+\beta)]F_{i+j}\\ &=(j-i)e_{i+j}+(j-i)(\alpha(i+j)+\beta)F_{i+j}=(j-i)\widetilde{e}_{i+j}.\end{aligned}$$

If $i=j\neq 0$, then $[\widetilde{e}_0,\widetilde{e}_j]=$

$$\left[e_0+\frac{\beta^2-\alpha^2}{2}\mathbb{1}, e_j+(\alpha j+\beta)F_j\right]=[e_0,e_j]+[e_0,(\alpha j+\beta)F_j]=je_j+j(\alpha j+\beta)F_j=j\widetilde{e}_j.$$

The case of $[\widetilde{e}_i,\widetilde{e}_0]$ is similar. And the last case: if $j\neq 0$, then $[\widetilde{e}_{-j},\widetilde{e}_j]=$

$$\begin{aligned}&[e_{-j}+(-\alpha j+\beta)F_{-j}, e_j+(\alpha j+\beta)F_j]=[e_{-j},e_j]+(-\alpha j+\beta)(\alpha j+\beta)[F_{-j},F_j]\\ &\quad=2je_0+\frac{j^3-j}{12}\mathbb{1}+j(\beta^2-j^2\alpha^2)\mathbb{1}=2j\left(e_0+\frac{\beta^2-\alpha^2}{2}\mathbb{1}\right)+\frac{j^3-j}{12}(1-12\alpha^2)\mathbb{1}\\ &\quad=2j\widetilde{e}_0+\frac{j^3-j}{12}z.\end{aligned}$$

This completes the proof.

Proposition 5.2 establishes a two parameter family $\mathcal{K}(\alpha,\beta)$ of $\mathfrak{Vir}$-modules. From now on, we will use the notation e_i instead of $\widetilde{e}_i$; what used to be e_i, corresponds to the case of $\alpha=\beta=0$.

Our next goal is to compare it with the family $\mathcal{H}(\lambda,\mu)$ of semi-infinite forms modules.

5.1.3. Bosonic-fermionic correspondence. We will use this term, borrowed from physics, for the following result.

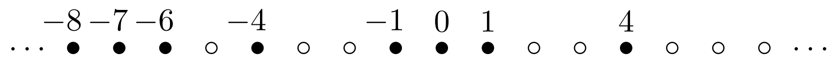
THEOREM 5.3. *There exists a (unique up to a non-zero constant factor)* $\mathfrak{Vir}$*-isomorphism* $\mathcal{K}(\alpha,\beta)\cong\mathcal{H}(\lambda,\mu)$ *where* $\lambda=-\alpha-\frac{1}{2},\ \mu=\beta-\alpha.$

Proof: the whole Section.

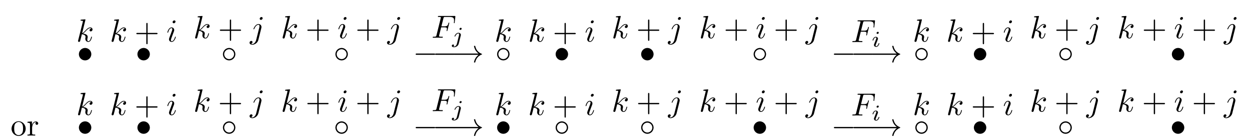
To begin, we will construct an action of the Heisenberg algebra in $\mathcal{H}=\mathcal{H}(\lambda,\mu)$ (it will not depend on λ and μ). The construction is simple:

$$F_i(\ldots\wedge f_{j_3}\wedge f_{j_2}\wedge f_{j_1})=\sum_{r=1}^{\infty}\left(\ldots\wedge f_{j_{r+1}}\wedge f_{j_r+i}\wedge f_{j_{r-1}}\wedge\ldots\wedge f_{j_1}\right).$$

We are going to prove that $[F_i,F_j]=0$, if $i+j\neq 0$ and $[F_{-j},F_j]=j\cdot\mathrm{id}$. Here and below, we will use the following graphic presentation of semi-infinite monomials: for $\ldots\wedge f_{j_3}\wedge f_{j_2}\wedge f_{j_1}$, we will place, on a number line, black dots at $\ldots,j_3,j_2,j_1$ and white dots at all other integers. For example, $\ldots\wedge f_{-8}\wedge f_{-7}\wedge f_{-6}\wedge f_{-4}\wedge f_{-1}\wedge f_0\wedge f_1\wedge f_4$ will be presented as

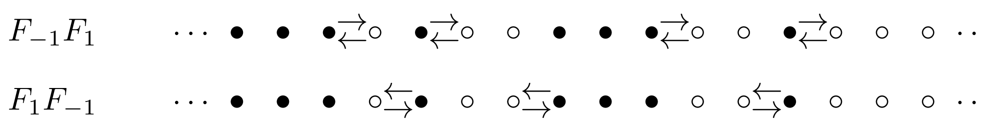


The action of F_i is presented by the diagram $\overset{k}{\bullet} \longrightarrow \overset{k+i}{\circ}$. Each of the compositions F_iF_j and F_jF_i ($i+j \neq 0$) may transform $\overset{k}{\bullet}\ \overset{k+i}{\circ}\ \overset{\ell}{\bullet}\ \overset{\ell+j}{\circ}$ into $\overset{k}{\circ}\ \overset{k+i}{\bullet}\ \overset{\ell}{\circ}\ \overset{\ell+j}{\bullet}$ (the order may be different) or $\overset{k}{\bullet}\quad \overset{k+i+j}{\circ}$ into $\overset{k}{\circ}\quad \overset{k+i+j}{\bullet}$. The equality $F_iF_j = F_jF_i$ is obvious in the first case and also in the second case, if the dots $k+i$ and $k+j$ are both black or both white. If, say $k+i$ is black and $k+j$ is white, then F_jF_i cannot transform $\overset{k}{\bullet}\quad \overset{k+i+j}{\circ}$ into $\overset{k}{\bullet}\quad \overset{k+i+j}{\circ}$ at all, and F_iF_j can do it in two different ways:

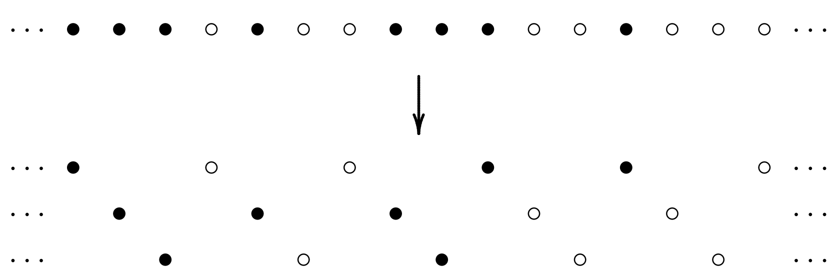


and the results cancel, since the first composition reverses the order of k and $k+i$, and the second one preserves this order.

Now consider the commutator $[F_{-j}, F_j]$. Let, for the beginning, $j = 1$. The compositions $F_{-1}F_1$ and F_1F_{-1} are presented on the diagram below.



(top arrows show the transformation performed first, bottom arrows indicate the second transformation). We see that $F_{-1}F_1$ multiplies the monomial by the number of adjacent pairs ($\bullet\ \circ$) and F_1F_{-1} multiplies the monomial by the number of adjacent pairs ($\circ\ \bullet$), or, differently, the first number is the number of transitions (black $\to$ white) we make moving from far left to far right, and the second one is the number of transitions (white $\to$ black). Since we begin with $\bullet$ and end with $\circ$, the first number is one more than the second one, and the difference is 1. Thus, $[F_{-1}, F_1] = 1 \cdot \mathrm{id}$ (we do not count the possibilities of applying $\to$ and $\leftarrow$ to different dots; they are the same for $F_{-1}F_1$ and F_1F_{-1}, and cancel in the difference $F_{-1}F_1 - F_1F_{-1}$). At last, for $j > 1$ we split our sequence of black and white dots into j sequences according to the residue modulo j, like this (for $j = 3$):



Each of the j sparse sequences contributes 1 to $[F_{-j}, F_j]$ as above, so $[F_{-j}, F_j] = j \cdot \mathrm{id}$.

Thus, $\mathcal{H}$ becomes an $\mathfrak{H}$-module, and

$$x_{i_1}^{k_1} \dots x_{i_s}^{k_s} = F_{i_1}^{k_1} \dots F_{i_s}^{k_s} 1 \mapsto F_{i_1}^{k_1} \dots F_{i_s}^{k_s} (\dots \wedge f_{-3} \wedge f_{-2} \wedge f_{-1})$$

is an $\mathfrak{H}$-isomorphism $\mathcal{K} \to \mathcal{H}$.

Let us return to the proof. If we fix α and β, then $\mathcal{K}$ acquires a structure of a $\mathfrak{Vir}$-module, and our isomorphism carries this structure into $\mathcal{H}$. It remains to check that the last structure is the same as that of $\mathcal{H}(\lambda, \mu)$ with $\lambda = -\alpha - \dfrac{1}{2}$, $\mu = \beta - \alpha$.

First, the action of z with respect to the two structures is the same:

$$\begin{aligned} -12\lambda^2 - 12\lambda - 2 = -12\lambda(\lambda+1) - 2 &= -12\left(-\alpha - \frac{1}{2}\right)\left(-\alpha + \frac{1}{2}\right) - 2 \\ &= -12\alpha^2 + 3 - 2 = 1 - 12\alpha^2. \end{aligned}$$

Next, we compare the actions of e_i for the two structures. It would be sufficient to do it, say, for e_3 and e_{-1}, since these two generate $\mathfrak{Vir}$, but we will do it for an arbitrary odd i (the case of an even i is not much different, but we prefer to avoid the coefficients $\dfrac{1}{2}$ and the squares of F's). For the beginning, assume that $\alpha = \beta = 0$. Then, with respect to the structure coming from $\mathcal{K}$, $e_i = \sum F_r F_s$ ($r + s = i$, $r > s$, $r \neq 0$, $s \neq 0$). This e_i can transform $\overset{k}{\bullet}\,\overset{\ell}{\bullet}\,\overset{\ell+s}{\circ}\,\overset{k+r}{\circ}$ into $\overset{k}{\circ}\,\overset{\ell}{\circ}\,\overset{\ell+s}{\bullet}\,\overset{k+r}{\bullet}$ in two different ways:

$$\overset{k}{\bullet}\,\overset{\ell}{\bullet}\,\overset{\ell+s}{\circ}\,\overset{k+r}{\circ} \xrightarrow{F_s} \overset{k}{\bullet}\,\overset{\ell}{\circ}\,\overset{\ell+s}{\bullet}\,\overset{k+r}{\circ} \xrightarrow{F_r} \overset{k}{\circ}\,\overset{\ell}{\circ}\,\overset{\ell+s}{\bullet}\,\overset{k+r}{\bullet}$$

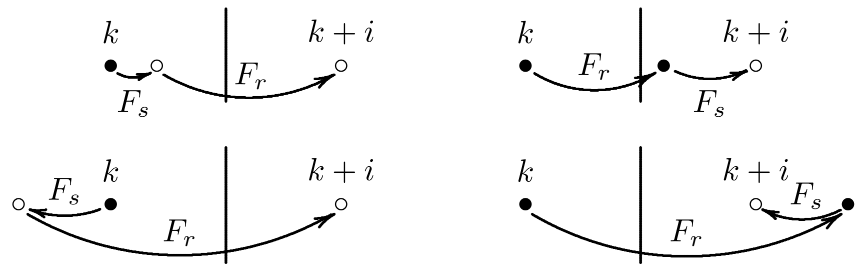
$$\text{and}\quad \overset{k}{\bullet}\,\overset{\ell}{\bullet}\,\overset{\ell+s}{\circ}\,\overset{k+r}{\circ} \xrightarrow{F_{\ell+s-k}} \overset{k}{\circ}\,\overset{\ell}{\bullet}\,\overset{\ell+s}{\bullet}\,\overset{k+r}{\circ} \xrightarrow{F_{k+r-\ell}} \overset{k}{\circ}\,\overset{\ell}{\circ}\,\overset{\ell+s}{\bullet}\,\overset{k+r}{\bullet}$$

(the order of operators in the second line may be different), and the results cancel because of different signs.

It remains to evaluate the amount of transformations $\overset{k}{\bullet} \longrightarrow \overset{k+i}{\circ}$ arising in the action of e_i. We will need the following simple combinatorial observation. Consider the diagram of a monomial as above,

$$\begin{array}{ccccccccccc|ccccccccc} & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \cdots & \bullet & \bullet & \bullet & \circ & \bullet & \circ & \circ & \bullet & \bullet & & \bullet & \circ & \circ & \bullet & \circ & \circ & \circ & \cdots \end{array}$$

(we mark black and white dots as well) and draw a vertical line between the dots marked as $j-1$ and j. Let w be the number of white dots to the left of the vertical line and b be the number of black dots to the right of vertical line. Then $j = w - b$ (on the picture, $1 = 3 - 2$). The fact is obvious (the proof is left to the reader). Now take the diagram of our monomial and draw a vertical line through the midpoint of the interval between $\overset{k}{\bullet}$ and $\overset{k+i}{\circ}$. The combination $F_r F_s$ with $s < r$, $r + s = i$, $r \neq 0$, $s \neq 0$ which take $\overset{k}{\bullet}$ into $\overset{k+i}{\circ}$ correspond to white dots to the left of the vertical line (then this white point has the label $k + s$) and to black dots to the right of the vertical line (then the black dot has the label $k + i - s$); see the diagram below.



Let w be the number of white points to the left of vertical line (in the initial diagram), b be the number of black points to the right of vertical line, and q be the number of black points between $\overset{k}{\bullet}$ and $\overset{k+i}{\circ}$ (all in the initial diagram). The transformation on the left of the diagram above produce the sign $(-1)^q$, the transformation on the right produce the sign $(-1)^{q-1}$. Thus, the total coefficient at $\overset{k}{\bullet} \to \overset{k+i}{\circ}$ in e_i is $(-1)^q(w-b) = (-1)^q\left(k + \dfrac{i+1}{2}\right)$ $= (-1)^q(\mu + k - \lambda(i+1))$ with $\mu = 0$, $\lambda = -\dfrac{1}{2}$, as it should be for $\alpha = \beta = 0$.

For arbitrary α and β, e_i acquires the summand $(\alpha i+\beta)F_i$ which leads to the addition of $\alpha i + \beta$ to $k + \dfrac{i+1}{2}$. But

$$k + \frac{i+1}{2} + \alpha i + \beta = \beta - \alpha + k + \left(\alpha + \frac{1}{2}\right)(i+1) = \mu + k - \lambda(i+1)$$

where $\lambda = -\alpha - \dfrac{1}{2}$, $\mu = \beta - \alpha$, as it should be.

This completes the proof of Theorem.

5.2. Vertex operators.

5.2.1. The definition. Vertex operators $B_k(\alpha)\colon \mathcal{H} \to \mathcal{H}$, $\alpha \in \mathbb{C}, k \in \mathbb{Z}$ are defined by the formula

$$\sum_{k=-\infty}^{\infty} B_k(\alpha)t^k = \exp\left(\sum_{i>0} \frac{\alpha F_i t^i}{i}\right) \cdot \exp\left(\sum_{i<0} \frac{\alpha F_i t^i}{i}\right).$$

In other words,

$$\sum_{k=-\infty}^{\infty} B_k(\alpha)t^k = \left(1 + \alpha F_1 t + \frac{1}{2}\alpha(F_2 + \alpha F_1^2)t^2 + \frac{1}{6}\alpha(2F_3 + 3\alpha F_1F_2 + \alpha^2F_1^3)t^3 + \dots\right) \cdot$$
$$\left(1 - \alpha F_{-1}t^{-1} + \frac{1}{2}\alpha(-F_{-2} + \alpha F_{-1}^2)t^{-2} + \frac{1}{6}\alpha(-2F_{-3} + 3\alpha F_{-1}F_{-2} - \alpha^2F_{-1}^3)t^{-3} + \dots\right)$$

So, for example,

$$B_0(\alpha) = 1 + \alpha^2F_1F_{-1} + \frac{1}{4}\alpha^2(F_2 + \alpha F_1^2)(-F_{-2} + \alpha F_{-1}^2)$$
$$+ \frac{1}{36}\alpha^2(2F_3 + 3\alpha F_1F_2 + \alpha^2F_1^3)(-2F_{-3} + 3\alpha F_{-1}F_{-2} - \alpha^2F_{-1}^3) + \dots,$$
$$B_1(\alpha) = \alpha F_1 - \frac{1}{2}\alpha^2(F_2 + \alpha F_1^2)F_{-1} + \frac{1}{12}\alpha^2(2F_3 + 3\alpha F_1F_2 + \alpha^2F_1^3)(-F_{-2} + \alpha F_{-1}^2) + \dots$$

The following properties of vertex operators are obvious:

PROPOSITION 5.4. (1) $B_k(0) = \begin{cases} \text{id}, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$

(2) *For any $\alpha \in \mathbb{C}$, $B_0(\alpha)v = v$ where $v \in \mathcal{H}$ denotes the vacuum vector.*

(3) *The operators $B_k(\alpha)$, $B_{-k}(\alpha)$ are taken into each other by the anti-isomorphism $F_i \leftrightarrow -F_{-i}$.*

Now we prove the key property of the vertex operators.

PROPOSITION 5.5: $[F_i, B_k(\alpha)] = \alpha B_{k+i}(\alpha)$.

Proof. Obviously, $\left[F_i, \exp\frac{\alpha F_j t^j}{j}\right] = 0$, if $j \neq -i$. Also, $[F_i, F_{-i}^k] = -ikF_{-i}^{k-1}$ ($-i\mathbb{1}$, if $k = 1$). Hence,

$$\left[F_i, \exp\frac{\alpha F_{-i}t^{-i}}{-i}\right] = \sum_{k=1}^{\infty} \frac{\alpha^k [F_i, F_{-i}^k] t^{ki}}{k!(-i)^k} = \sum_{k=1}^{\infty} \frac{\alpha^k (-i) k F_{-i}^{k-1} t^{-ki}}{k!(-i)^k}$$
$$= \alpha t^{-i} \sum_{k=1}^{\infty} \frac{\alpha^{k-1} F_{-i}^{k-1} t^{-(k-1)i}}{(k-1)!(-i)^{k-1}} = \alpha t^{-i} \exp\frac{\alpha F_{-i} t^{-i}}{-i}.$$

Therefore,

$$\sum_{k=-\infty}^{\infty} [F_i, B_k(\alpha)] t^k = \left[F_i, \sum_{k=-\infty}^{\infty} B_k(\alpha) t^k\right] = \left[F_i, \exp\left(\sum_{j>0} \frac{\alpha F_j t^i}{j}\right) \cdot \exp\left(\sum_{j<0} \frac{\alpha F_j t^j}{j}\right)\right]$$
$$= \alpha t^{-i} \exp\left(\sum_{j>0} \frac{\alpha F_j t^i}{j}\right) \cdot \exp\left(\sum_{j<0} \frac{\alpha F_j t^j}{j}\right)$$
$$= \alpha \sum_{k=-\infty}^{\infty} B_k(\alpha) t^{k-i} = \sum_{k=-\infty}^{\infty} \alpha B_{k+i}(\alpha) t^k.$$

Using this proposition, we can give an axiomatic description of vertex operators.

PROPOSITION 5.6. *Let $\alpha \in \mathbb{C}$. Assume that for each $k \in \mathbb{Z}$ a $\mathbb{C}$-linear operator $C_k : \mathcal{H} \to \mathcal{H}$ of degree k is given such that $C_0(v) = v$ and $[F_i, C_k] = \alpha C_{k+i}$. Then $C_k = B_k(\alpha)$.*

Proof. Let $D_k = C_k - B_k(\alpha)$. We need to prove that $D_k = 0$. It follows from our assumptions and Proposition 5.11 that $D_0(v) = 0$ and $[F_i, D_k] = \alpha D_{k+i}$.

If $\alpha = 0$, then the last equality takes the form $[F_i, D_k] = 0$, and $F_i D_k v = D_k F_i v = 0$ for $i < 0$ (since $F_i v = 0$ for $i < 0$). Hence, $D_k(v) = 0$ for all $k \neq 0$; for $k = 0$ this is also true by assumption. Therefore, $D_k F_{i_1} \ldots F_{i_s} v = F_{i_1} \ldots F_{i_s} D_k v = 0$ for all $k, i_1, \ldots, i_s$, that is, $D_k = 0$.

If $\alpha \neq 0$, then $D_k = \dfrac{1}{\alpha}[F_k, D_0]$, so it is sufficient to prove that $D_0 = 0$. Let $D_0(x) = 0$ for $\deg x < j$ and $D_0(F_{j_1} F_{j_2} \ldots F_{j_r} v) \neq 0$ for some positive $j_1, \ldots, j_r$ with $j_1 + \ldots + j_r = j$ ($j > 0$ since $F_0(v) \neq 0$). Then $F_{-i} D_0(F_{j_1} \ldots F_{j_r} v) \neq 0$ for some $i > 0$.

We will use the following computation:

$$\begin{aligned}F_uD_0F_v &= F_uF_vD_0 - \alpha F_uD_v\\ &= F_uF_vD_0 - \alpha D_vF_u - \alpha^2 D_{u+v}\\ &= F_uF_vD_0 - F_sD_0F_u + D_0F_v\\ &\qquad - \alpha F_{u+v}D_0 + \alpha D_0F_{u+v}.\end{aligned}$$

If we put $u=-i$, $v=j_1$ and apply the both sides of the last equality to $F_{j_2}\dots F_{j_r}v$, then on the left hand side we will get $F_{-i}D_0(F_{j_1}\dots F_{j_r}v)\neq 0$, and on the right hand side we will apply D_0 to vectors of degrees, respectively, $j-j_1, j-i, j-j_1-i, j-j_1, j-i$, all less than j; so, on the right hand side we will get 0. This completes the proof.

5.2.2. The vertex operators and the forms $\varphi_{k,n}$. In Section 4.4.3, we considered the forms

$$\varphi_{k,n} = \sum_{\substack{j_1+\dots+j_n=k+\binom{n}{2}\\ j_1<\dots<j_n}} \frac{\mathcal{V}(j_1,\dots,j_n)}{\mathcal{V}(1,\dots,n)} f_{j_1}\wedge\dots\wedge f_{j_n}$$

and also a linear map

$$\mathrm{Shift}_{-n}\circ(\wedge\varphi_{k,n})\colon\mathcal{H}\to\mathcal{H}. \tag{23}$$

Now we are going to prove

PROPOSITION 5.7. *For every $k\in\mathbb{Z}$, $n\in\mathbb{Z}_{>0}$, the mapping* (23) *coincides with $B_k(n)$.*

Proof. In view of Proposition 5.6, we need to check two things: (a) the mapping (23) with $k=0$ takes v into v, and (b) $F_i\varphi_{k,n}=n\varphi_{k+i,n}$. The assertion (a) is obvious: $\varphi_{0,n}=f_0\wedge f_1\wedge\dots\wedge f_{n-1}+$ terms containing f_j with negative j, and

$$\mathrm{Shift}_{-n}((\dots\wedge f_{-2}\wedge f_{-1})\wedge(f_0\wedge f_1\wedge\dots\wedge f_{n-1})) = \dots\wedge f_{-2}\wedge f_{-1}.$$

As to (2),

$$\begin{aligned}F_i\varphi_{k,n} &= \sum_{\substack{j_1+\dots+j_n=k+\binom{n}{2}\\ j_1<\dots<j_n}} \frac{\mathcal{V}(j_1,\dots,j_n)}{\mathcal{V}(1,\dots n)}\sum_{s=1}^n f_{j_1}\wedge\dots\wedge f_{j_s+i}\wedge\dots\wedge f_{j_n}\\ &= \sum_{\substack{j_1+\dots+j_n=k+i+\binom{n}{2}\\ j_1<\dots<j_n}} \frac{\sum_{s=1}^n\mathcal{V}(j_1,\dots,j_s-i,\dots,j_n)}{\mathcal{V}(1,\dots n)} f_{j_1}\wedge\dots\wedge f_{j_n},\end{aligned}$$

and all we need to check is $\sum_{s=1}^n \mathcal{V}(j_1,\dots,j_s-i,\dots,j_n) = n\mathcal{V}(j_1,\dots,j_n)$. Let $\sum_{s=1}^n\mathcal{V}(j_1,\dots,j_s-i,\dots,j_n)=P(i)$. Then $P(i)=P(0)+iP'(0)+\frac{i^2}{2}P''(0)+\dots$; but $P(0)=n\mathcal{V}(j_1,\dots,j_n)$

and all the derivatives $P'(0)$, $P''(0), \dots$ are skew-symmetric polynomials of $j_1, \dots, j_n$ of degree $< \deg \mathcal{V} = \binom{n}{2}$, hence they all are zero.

5.2.3. Explicit formula for vertex operators. For a partition $\sigma = \{s_1, \dots, s_u\}, s_1 \geq \dots \geq s_u > 0$, we put

$$f_\sigma = \dots f_{-u-1} \wedge f_{-u+s_u} \wedge \dots \wedge f_{-1+s_1}, \ J_\sigma = \{\dots, -u-1, -u+s_u, \dots, -1+s_1\}.$$

and

$$h(\sigma) = \prod_{j' \in J_\sigma, j'' \notin J_\sigma, j' > j''} (j' - j'')$$

(the last quantity is known as the dimension of the irreducible representation of $S(u)$ corresponding to the partition σ).

Proposition 5.8.

$$B_k(\alpha) f_\sigma = \sum_{|\tau| = |\sigma| + k} b_\alpha(\sigma, \tau) f_\tau$$

where

$$b_\alpha(\sigma, \tau) = (-1)^{|\sigma|} \frac{\prod\limits_{j' \in J_\sigma, j'' \notin J_\tau, j' > j''} (\alpha - (j' - j''))}{h(\sigma)} \cdot \frac{\prod\limits_{j' \in J_\tau, j'' \notin J_\sigma, j' > j''} (\alpha + (j' - j''))}{h(\tau)} \tag{24}$$

Proof. It is seen from the original definition of $B_k(\alpha)$ that for fixed σ and τ, the function $b_\alpha(\sigma, \tau)$ is a polynomial in α. Therefore, it is sufficient to prove the formula for sufficiently large integral α. By Proposition 5.7, the coefficient $b_\alpha(\sigma, \tau)$ for large integral α can be found in the following way. We shift the set J_τ by α to the right (that is, add α to each element of J_τ), then remove J_σ from the resulting set (if α is large enough, the shifted set contains J_σ) and get a final set $J = \{j_1, \dots, j_r\}$. The coefficient in question is equal to

$$(-1)^N \frac{\mathcal{V}(j_1, \dots, j_r)}{\mathcal{V}(1, \dots, r)},$$

where N is the number of pairs j', j'' such that $j' \in J, j'' \in J_\sigma, j' < j''$. It is easy to see that $N = |\sigma|$, $r = |\alpha|$. Hence

$$b_\alpha(\sigma, \tau) = (-1)^\sigma \frac{\prod\limits_{j', j'' \notin J_\sigma; \, j' - \alpha, j'' - \alpha \in J_\tau; j' > j''} (j' - j'')}{\mathcal{V}(1, \dots, \alpha)}.$$

After cancellations, this formula becomes formula (24).

5.2.4. Commutators with $\mathbf{e_i}$. Let $\lambda, \mu \in \mathbb{C}$.

Proposition 5.9. *The operators*

$$B_k(\alpha) \colon \mathcal{H}(\lambda, \mu) \to \mathcal{H}(\lambda, \mu + \alpha)$$

commute with e_i according to the following rule:

$$[e_i, B_k(\alpha)] = \left[\alpha\mu + k + \frac{\alpha(\alpha-1)}{2} - \frac{(\alpha-1)(\alpha+2)}{2} i - \alpha(i+1)\lambda\right] B_{k+i}(\alpha)$$

.

Proof. If α is a positive integer n, then this proposition is the same as Proposition 4.8 in Section 4.4.3 (the proof of which was postponed there). On the other hand, since the difference between the left hand side and the right hand side of the equality is on operator with coefficients polynomial in α, it is sufficient to prove it for positive integral α. Thus, what we need to prove is $e_i \varphi_{k,n} = B\varphi_{k+i,n}$ where

$$B = n\mu + k + \frac{n(n-1)}{2} - \frac{(n-1)(n+2)}{2} i - n(i+1)\lambda,$$

and we can restrict ourselves to the case $i \neq 0$. In turn, the equality we are proving means precisely that

$$\sum_{s=1}^{n} \mathcal{V}(j_1, \ldots, j_s - i, \ldots, j_n)(\mu + j_s - i - \lambda(i+1)) = B \cdot \mathcal{V}(j_1, \ldots, j_n)$$

provided that $j_1 + \ldots + j_n = k + i + \binom{n}{2}$.

We already know that $\sum\limits_{s=1}^{n} \mathcal{V}(j_1, \ldots, j_s - i, \ldots, j_n) = n\mathcal{V}(j_1, \ldots, j_n)$ (see Section 5.3.2). Now we will need one more relation of this kind.

Lemma.

$$\sum_{s=1}^{n} j_s \mathcal{V}(j_1, \ldots, j_s - i, \ldots, j_n) = \left[\left(\sum_{s=1}^{n} j_s\right) - i\binom{n}{2}\right] \mathcal{V}(j_1, \ldots, j_n).$$

Proof of Lemma. Let $Q(i)$ be the left hand side of the equality in Lemma. Then

$$Q(i) = Q(0) + iQ'(0) + \frac{i^2}{2} Q''(0) + \ldots$$

But all the derivatives of Q starting from $Q''(0)$ are skew-symmetric polynomials in $j_1, \ldots, j_n$ of degree less than $\binom{n}{2}$, so they equal 0. Thus,

$$\begin{aligned} Q(i) = Q(0) + jQ'(0) &= \sum_{s=1}^{n} j_s \mathcal{V}(j_1, \ldots, j_n) - i \sum_{s=1}^{n} j_s \frac{\partial}{\partial j_s} \mathcal{V}(j_1, \ldots, j_n) \\ &= \left(\sum_{s=1}^{n} j_s\right) \mathcal{V}(j_1, \ldots, j_n) - i \deg \mathcal{V} \cdot \mathcal{V}(j_1, \ldots, j_n) \\ &= \left[\left(\sum_{s=1}^{n} j_s\right) - i\binom{n}{2}\right] \mathcal{V}(j_1, \ldots, j_n). \end{aligned}$$

Back to Proposition.

$$\sum_{s=1}^{n} \mathcal{V}(j_1, \ldots, j_s - i, \ldots, j_n)(\mu + j_s - i - \lambda(i+1))$$

$$= \sum_{s=1}^{n} (\mu - i - \lambda(i+1))\mathcal{V}(j_1, \ldots, j_s - i, \ldots, j_n) + \sum_{s=1}^{n} j_s \mathcal{V}(j_1, \ldots, j_s - i, \ldots, j_n)$$

$$= \left[n(\mu - i - \lambda(i+1)) + \left(\sum_{s=1}^{n} j_s \right) - i \binom{n}{2} \right] \mathcal{V}(j_1, \ldots, j_n)$$

$$= \left[n(\mu - i - \lambda(i+1)) + k + i + \binom{n}{2} - i \binom{n}{2} \right] \mathcal{V}(j_1, \ldots, j_n) = B \cdot \mathcal{V}(j_1, \ldots, j_n).$$

Corollary 5.10. *$B_k(\alpha)\colon \mathcal{H}(\lambda, \mu) \to \mathcal{H}(\lambda, \mu + \alpha)$ is a $\mathfrak{Vir}$-homomorphism if and only if*

$$\lambda = -\frac{(\alpha - 1)(\alpha + 2)}{2\alpha},\ \mu = -\frac{\alpha^2 + k - 1}{\alpha}.$$

If we believe that this homomorphism is not 0 at v (see Proposition 4.10 of Section 4.4.3), then we have an (explicit) construction of a singular vector if degree k for a module of semi-infinite forms with h and c equal to $h_{k,1}(t)$ and $c_{k,1}(t)$ with $t = -\frac{\alpha^2}{2}$, that is, for an *arbitrary* point of the curve $\Phi_{k,1}$ (in the notation of Section 4.5).

But what about Φ_{pq} with arbitrary p, q? In Section 4.4.2 we considered a homomorphism $\mathcal{H}(\lambda, \mu) \to \mathcal{H}(\lambda, \mu + pn)$ (we have to change slightly the notations of Section 4.4.2) of degree pq, $q = k - \frac{n^2(p-1)}{2}$ defined as $f \mapsto \mathrm{Shift}_{-np}\left(f \wedge (\phi^p_{k,n})\right)$ and proved that it is a $\mathfrak{Vir}$-homomorphism, if $\lambda = -\frac{(n-1)(n+2)}{2n}$, $\mu = -n - \frac{k-1}{n}$. This gives a singular vector of degree pq in the module $\mathcal{H}(\lambda, \mu + pn)$ for which $h = h_{pq}(t)$, $c = c_{pq}(t)$, $t = -\frac{n^2}{2}$. Is it possible to generalize this construction to the case when n is replaced by a complex number? At least, not directly: the homomorphism considered is the composition

$$B_{k-(p-1)n^2}(n) \ldots B_{k-n^2}(n) B_k(n)$$

and this does not make sense for a non-integral n (well, for a non-integral n^2). What we need, is a "composition" $B_{k_1}(\alpha_1) \ldots B_{k_n}(\alpha_n)$ with complex $k_1, \ldots, k_n$, well, it is desirable that the sum $k_1 + \ldots + k_n$ be integral: this is the degree of our composition. We will discuss these "composition vertex operators" and their applications to singular vectors in subsequent sections. But for the beginning, we will prove some things (not too much) concerning the compositions and the commutators of vertex operators.

5.2.4. Compositions of vertex operators. Proposition 5.11. *For $\alpha, \beta \in \mathbb{C}$,*

$$\sum_{k=0}^{\infty} B_{-k}(\beta) B_k(\alpha) t^k v = (1-t)^{\alpha\beta} v;$$

in other words,

$$B_{-k}(\beta)B_k(\alpha)v = \frac{\alpha\beta(\alpha\beta-1)\dots(\alpha\beta-k+1)}{k!}v.$$

Proof. We must prove that the degree 0 component of the series

$$\left(\sum_{k=-\infty}^{\infty} B_k(\beta)t_2^k\right)\left(\sum_{k=-\infty}^{\infty} B_k(\alpha)t_1^k\right)v$$

is equal to $(1-(t_1/t_2))^{\alpha\beta}v$. Write out this series in full length:

$$\left(\exp\sum_{i>0}\frac{\beta F_i t_2^i}{i}\exp\sum_{i<0}\frac{\beta F_i t_2^i}{i}\exp\sum_{i>0}\frac{\alpha F_i t_1^i}{i}\exp\sum_{i<0}\frac{\alpha F_i t_1^i}{i}\right)v.$$

Of these four factors, the last one can be dropped, since $F_i v = 0$ for $i<0$. Also, the first factor can be dropped, since F_i applied to anything cannot have a component of degree 0. Thus, we are interested in the degree 0 component of the series

$$\left(\exp\sum_{i<0}\frac{\beta F_i t_2^i}{i}\exp\sum_{i>0}\frac{\alpha F_i t_1^i}{i}\right)v = \prod_{i=1}^{\infty}\left(\sum_{s=0}^{\infty}\frac{\beta^s F_{-i}^s t_2^{-is}}{(-i)^s s!}\right)\left(\sum_{s=0}^{\infty}\frac{\alpha^s F_i^s t_1^{is}}{i^s s!}\right)v$$

Since, obviously, $[F_{-i}^s, F_i^s]v = i^s s! v$, this is

$$\prod_{i=1}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha\beta)^s(t_1/t_2)^{si}}{(-i^2)^s(s!)^2}[F_{-i}^s,F_i^s]v = \prod_{i=1}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha\beta)^s(t_1/t_2)^{is}}{(-i)^s s!}v = \prod_{i=1}^{\infty}\left(\exp\frac{\alpha\beta(t_1/t_2)^i}{-i}\right)v$$
$$= \left(\exp\alpha\beta\sum_{i-1}^{\infty}\frac{(t_1/t_2)^i}{-i}\right)v = \left(\exp\alpha\beta\log\left(1-\frac{t_1}{t_2}\right)\right)v = \left(1-\frac{t_1}{t_2}\right)^{\alpha\beta}v.$$

Corollary 5.12. *If $\alpha\beta$ is a positive integer, then $B_{-k}(\beta)B_k(\alpha)v = 0$ for $k>\alpha\beta$.*

Proposition 5.11 has the following generalization.

Proposition 5.13. *For $\alpha_1,\dots,\alpha_n \in \mathbb{C}$,*

$$\sum_{k_1+\dots+k_n=0} t_1^{k_1}\dots t_n^{k_n}B_{k_1}(\alpha_1)\dots B_{k_n}(\alpha_n)v = \prod_{i<j}\left(1-\frac{t_j}{t_i}\right)^{\alpha_i\alpha_j}v.$$

It is proved in precisely the same way as Proposition 5.11.

Next result is similar to Proposition 5.6 of Section 5.2.1.

Proposition 5.14. *Let $\alpha,\beta\in\mathbb{C}$. Assume that for every $k,\ell\in\mathbb{Z}$ an operator $C_{\ell,k}\colon\mathcal{H}\to\mathcal{H}$ is given such that*

$$\sum_{k=-\infty}^{\infty} C_{-k,k}t^k v = (1-t)^{\alpha\beta}v \quad \text{and} \quad [F_i, C_{\ell,k}] = \beta C_{\ell+i,k} + \alpha C_{\ell,k+i}$$

is given. Then $C_{\ell,k} = B_\ell(\beta)B_k(\alpha)$.

The proof is similar to that of proof of Proposition 5.6 (the reference to Proposition 5.5 should be replaced by a reference to Proposition 5.11 above).

Proposition 5.14 may be used to obtain some formulas connecting different vertex operators. Here is a statement from our the work of Feigin; maybe, more can be obtained in a similar way.

PROPOSITION 5.15. *If* $\alpha\beta \in \mathbb{Z}_{\geq 0}$, *then*

$$B_{\ell-\alpha\beta}(\beta)B_k(\alpha) = (-1)^{\alpha\beta}B_{k-\alpha\beta}(\alpha)B_\ell(\beta).$$

Besides this, if $\alpha\beta = -1$, *then*

$$B_{\ell+1}(\beta)B_k(\alpha) + B_{k+1}(\alpha)B_\ell(\beta) = B_{k+\ell+1}(\alpha+\beta),$$

and if $\alpha\beta = -2$, *then*

$$B_{\ell+2}(\beta)B_k(\alpha) - B_{k+2}(\alpha)B_\ell(\beta) = \frac{(k+1)\beta - (\ell+1)\alpha}{\alpha+\beta}B_{k+\ell+2}(\alpha+\beta).$$

Proof. It is sufficient to verify that (for α and β fixed) the operators

$$C_{\ell,k} = \begin{cases} (-1)^{\alpha\beta}B_{k-\alpha\beta}(\alpha)B_{\ell+\alpha\beta}(\beta), & \text{if } \alpha\beta \in \mathbb{Z}_{\geq 0}, \\ -B_{k+1}(\alpha)B_{\ell-1}(\beta) + B_{k+\ell}(\alpha+\beta), & \text{if } \alpha\beta = -1, \\ B_{k+2}(\alpha)B_{\ell-2}(\alpha) + \dfrac{(k+1)\beta - (\ell+1)\alpha}{\alpha+\beta}B_{k+\ell}(\alpha+\beta), & \text{if } \alpha\beta = -2 \end{cases}$$

satisfy the assumption of Proposition 5.14. The verification is straightforward; for example, if $\alpha\beta \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\sum_{k=-\infty}^{\infty} C_{-k,k}t^k v &= (-1)^{\alpha\beta}\sum_{k=-\infty}^{\infty} B_{k-\alpha\beta}(\alpha)B_{-k+\alpha\beta}(\beta)t^k v \\ &= (-1)^{\alpha\beta}\sum_{k=-\infty}^{\alpha\beta} B_{k-\alpha\beta}(\alpha)B_{-k+\alpha\beta}(\beta)t^k v \\ &= (-1)^{\alpha\beta}\sum_{m=0}^{\infty} B_{-m}(\alpha)B_m(\beta)t^{\alpha\beta-m} = (-t)^{\alpha\beta}\left(1-\frac{1}{t}\right)^{\alpha\beta} v = (1-t)^{\alpha\beta}v.\end{aligned}$$

5.2.5. Application: proof of Proposition 4.11. We postponed the proof of this Proposition 4.11 in Section 4.4.3. We can prove it now.

We must prove that if n is an even integer and $s \geq \dfrac{n^2(k-1)}{2}$, then $\mathcal{B}v \neq 0$, where

$$\mathcal{B} = B_{s-(k-1)n^2}(n)\dots B_{s-n^2}(n)B_s(n).$$

Proposition 5.13 provides a convenient formula for compositions of vertex operators $B_{k_i}(\alpha_i)$, but it works only in the case $k_1 + \ldots + k_n = 0$, while this sum for our composition is equal to

$$(s-(k-1)n^2) + (s-(k-1)n^2)(s-(k-2)n^2) + \ldots + (s-n^2) + s = k\ell,$$

where $\ell = s - \dfrac{n^2(k-1)}{2}$. We will obtain a composition of vertex operators, which satisfies our condition, if we subtract ℓ from every subscript:

$$\mathcal{B}_0 = B_{-\frac{n^2(k-1)}{2}}(n) B_{-\frac{n^2(k-3)}{2}}(n) \ldots B_{\frac{n^2(k-3)}{2}}(n) B_{\frac{n^2(k-1)}{2}}(n)$$

According to Proposition 5.13, $\mathcal{B}_0 v = Cv$, where C is the coefficient of the term with $t_1^{-\frac{n^2(k-1)}{2}} t_2^{-\frac{n^2(k-3)}{2}} \ldots t_k^{\frac{n^2(k-1)}{2}}$ in the product $\prod_{i<j} \left(1 - \dfrac{t_i}{t_j}\right)^{n^2}$, which is the same as the coefficient of the term with $(t_1 \ldots t_k)^{n^2(k-1)}$ in the product $\prod_{i<j}(t_j - t_i)^{n^2}$. This coefficient is known to be equal to $\dfrac{(kn^2/2)!}{(n^2/2)!^k}$ (this is the famous *Dyson's identity*, see [22]), and hence is not zero.

Now, let us show that $\mathcal{B}_0 v$ can be obtained from $\mathcal{B} v$ by means of the operators F_{-i}. First notice that the expression

$$B_{s-(k-1)n^2-m_1}(n) \ldots B_{s-n^2-m_{k-1}}(n) B_{s-m_k}(n)$$

with non-negative integers $m_1, \ldots, m_k$ is symmetric with respect to $m_1, \ldots, m_k$: this follows from Proposition 5.15. Next, let us apply $(F_{-\ell})^k$ to $\mathcal{B} v$. If an $F_{-\ell}$ reaches v, we get 0 (since $F_i v = 0$ for $i < 0$); thus, every factor $F_{-\ell} v$ has to form a commutator with some B's. If all these B's are different, the $\mathcal{B}$ becomes $\mathcal{B}_0$. Thus, $(F_{-\ell})^k \mathcal{B} v = nk! \cdot \mathcal{B}_0 v + \ldots$, where "..." consists of terms obtained by applying $F_{-\ell}$ to repeating B. Each of these terms correspond to partitions $\sigma = (\sigma_1, \ldots, \sigma_r)$ of k, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ and $\sigma_1 + \sigma_2 + \ldots + \sigma_r = k$: we subtract $\sigma_1 \ell, \ldots, \sigma_r \ell$ from subscripts at r *different* B's and multiply the composition arising by n^r. By the symmetry described above, the result does not depend on the choice of these B's; since there are $\dfrac{k!}{(n-r)!}$ choices, the final result will be $n^r \dfrac{k!}{(n-r)!} \mathcal{B}^\sigma v$, where

$$\mathcal{B}^\sigma = B_{s-(k-1)n^2-\ell\sigma_1}(n) \ldots B_{s-(k-r)n^2-\ell\sigma_r}(n) B_{s-(k-r-1)n^2}(n) \ldots B_s(n)$$

Next, we apply to $\mathcal{B} v$ the composition $F_{-2\ell}(F_{-ell})^{k-2}$ and subtract the result, with an appropriate coefficient, from what we already have. We will eliminate B^σ with $\sigma = (2, 1, \ldots, 1)$. Then we do the same for the other partitions of k in any order which is compatible with the natural partial order: $\tau > \sigma$, if σ can be obtained from τ by subdividing some τ_i into smaller parts. As a result, we will obtain a presentation of $\mathcal{B}_0 v$ as a linear combination of terms of the form $F_{-i_1} \ldots F_{-i_t} \mathcal{B} v$. Since $\mathcal{B}_0 v \neq 0$, $\mathcal{B} v$ cannot be zero.

5.3. Composition vertex operators.

5.3.1. Introduction. According to Corollary 5.10, if

$$\lambda = -\frac{(\alpha-1)(\alpha+2)}{2\alpha},\ \mu = \frac{1-k}{\alpha}, \tag{25}$$

then $B_k(\alpha)v$, where v is the vacuum vector in $\mathcal{H}(\lambda, \mu-\alpha)$, is a singular vector in $\mathcal{H}(\lambda,\mu)_k$. A direct comparison shows that the equations (25) are the same as the parametric equations of the curve $\Psi_{-k,-1}$ (with the parameter $\theta = \sqrt{2}\alpha^{-1}$; see the beginning of Section 4.8.4). Thus the operators $B_k(\alpha)$ give rise to (more or less) explicit expressions for singular vectors of degree k in all modules $\mathcal{H}(\lambda,\mu)$ with $(\lambda,\mu)\in\Psi_{-k,-1}$, and, through that, to (well, less explicit) formulas for singular vectors of degree k in Verma modules $M(h,c)$ with $(h,c)\in\Phi_{k,1}$. But for the case $k>1, \ell>1$ we had a construction of singular vectors in$\mathcal{H}(\lambda,\mu)$ only for some isolated points $(\lambda,\mu)\in\Psi_{k,\ell}$. This partial success was attained by applying to v some composition of vertex operators (more precisely, by multiplying v by powers of the forms $\varphi_{s,n}$). Our aim now to find a generalization of a composition of vertex operators, which would give singular vectors in all reducible $\mathcal{H}(\lambda,\mu)$. Our way of doing that is attaching some sense to a "composition"

$$B_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n) = B_{k_1}(\alpha_1)\dots B_{k_n}(\alpha_n) \tag{26}$$

in the case when $k_1,\dots,k_n$ are complex numbers with only the sum $k_1+\dots+k_n$ being integral.

5.3.2. The definition. At the moment, we still assume that the $k_1,\dots,k_n$ are integers, but we will use the notation (26). According to Propositions 5.5 and 5.9, we have:

$$[F_j, B_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n)] = \sum_{s=1}^{n}\alpha_s B_{k_1,\dots,k_{s-1}k_s+j,k_{s+1},\dots,k_n}(\alpha_1,\dots,\alpha_n); \tag{27}$$

$$[e_i, B_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n)] = \sum_{s=1}^{n}(\mu_s+k_s-(i+1)\lambda_s)B_{k_1,\dots,k_{s-1}k_s+i,k_{s+1},\dots,k_n}(\alpha_1,\dots,\alpha_n), \tag{28}$$

where

$$\lambda_s = \alpha_s\lambda + \frac{(\alpha_s-1)(\alpha_s+2)}{2}, \tag{29}$$

$$\mu_s = \alpha_s(\mu+\alpha_s+\alpha_{s+1}+\dots+\alpha_n)-1. \tag{30}$$

Now, let us fix complex numbers $\beta_1,\dots,\beta_n$ with $\beta_1+\dots+\beta_n=0$ and try to attach some sense to a "composition" $B_{k_1+\beta_1}(\alpha_1)\dots B_{k_n+\beta_n}(\alpha_n)$. For this "composition," we will use the notation $B_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)$, and we list the expected properties of these operators in the following

Definition. Let $\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n\in\mathbb{C}$, and let $\beta_1+\dots+\beta_n=0$. A family of operators

$$B_{k_1,\dots,k_n}(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n)\colon \mathcal{H}(\lambda,\mu)\to\mathcal{H}(\lambda,\mu+\alpha_1+\dots+\alpha_n)$$

$(k_1, \ldots, k_n \in \mathbb{Z})$ is called a *family of composition vertex operators* (of type $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$), if formulas (27) – (30) hold with the following changes: $(\alpha_1, \ldots, \alpha_n)$ in formulas (27) and (28) should be replaced by $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$; formula (30) should be replaced by

$$\mu_s = \alpha_s(\mu + \alpha_s + \alpha_{s+1} + \ldots + \alpha_n) - 1 + \beta_s. \tag{31}$$

EXAMPLE.

$$B_{k_1}(\alpha_1) \ldots B_{k_n}(\alpha_n) \colon \mathcal{H}(\lambda, \mu) \to \mathcal{H}(\lambda, \mu + \alpha_1 + \ldots + \alpha_n)$$

is a family of composition vertex operators of type $(\alpha_1, \ldots, \alpha_n; 0, \ldots, 0)$.

5.3.3. The main statement. THEOREM 5.16. (i) *For a family of operators $B_{k_1,\ldots,k_n} \colon \mathcal{H}(\lambda, \mu) \to \mathcal{H}(\lambda, \mu + \alpha_1 + \ldots + \alpha_n)$ the property of being a family of composition vertex operators of type $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ does not depend on λ and μ.*

(This allows us to consider a family of composition vertex operators as a family of operators $\mathcal{H} \to \mathcal{H}$.)

(ii) *For $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n$ with $\beta_1 + \ldots + \beta_n = 0$ fixed, the vector space of families of composition vertex operators of type $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ has dimension $\geq (n-1)!$. Moreover, for almost all sets (α, β) this dimension is equal to $(n-1)!$.*

(The precise meaning of words "almost all" will be clarified in the proof.)

CONJECTURE 5.17 (Still open, as far as I know) *The dimension of the above space is $\leq n!$.*

5.3.4. Sufficiency of degree 0 operators. LEMMA. *Let $\{\gamma_{k_1,\ldots,\gamma_n} \mid k_i \in \mathbb{Z}, k_1 + \ldots + k_n = 0$ be a family of complex numbers. Then for any $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ there exists a unique family of operators $B_{k_1,\ldots,k_n}(\alpha_1, \ldots, \alpha_n) \colon \mathcal{H} \to \mathcal{H}$ $(k_1, \ldots, k_n \in \mathbb{Z})$, which satisfies the conditions* (27) *and*

$$\text{for } k_i \in \mathbb{Z}, k_1 + \ldots + k_n = 0, B_{k_1,\ldots,k_n}(\alpha_1, \ldots, \alpha_n)v = \gamma_{k_1,\ldots,k_n} v. \tag{32}$$

Proof is straightforward.

Using this lemma, we can convert the restrictions imposed on $B_{k_1,\ldots,k_n}$, into restrictions for $\gamma_{k_1,\ldots,k_n}$ $(k_1 + \ldots + k_n = 0)$. Namely:

PROPOSITION 5.18. *Let $\{\gamma_{k_1,\ldots,k_n} \mid k_i \in \mathbb{Z}, k_1 + \ldots + k_n = 0\}$ be a family of complex numbers, and let $\{B_{k_1,\ldots,k_n} \colon \mathcal{H} \to \mathcal{H} \mid k_i \in \mathbb{Z}\}$ be a family of operators satisfying the conditions* (27) *(with some $\alpha_1, \ldots, \alpha_n$) and* (32)*. This family is a family of composition vertex operators $\mathcal{H}(\lambda, \mu) \to \mathcal{H}(\lambda, \mu + \alpha_1, + \ldots + \alpha_n)$ of type $(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ if and only if the following equations hold:*

$$\begin{aligned}\sum_j (k_i - j + \beta_i - \alpha_i(\alpha_1 + \ldots + \alpha_{i-1}))\gamma_{\ldots,k_i-j,\ldots} \\ = \sum_{u=1}^{j-1} \sum_{1 \leq i_1 < i_2 \leq n} \alpha_{i_1}\alpha_{i_2}\gamma_{\ldots,k_{i_1}-u,\ldots,k_{i_2}-(j-u),\ldots} \; (j > 0, k_i \in \mathbb{Z}, \sum_i k_i = j);\end{aligned} \tag{33}$$

$$\sum_j (k_i + j + \beta_i - \alpha_i(\alpha_{i+1} + \ldots + \alpha_n))\gamma_{\ldots,k_i+j,\ldots}$$
$$= \sum_{u=1}^{j-1} \sum_{1\le i_1<i_2\le n} \alpha_{i_1}\alpha_{i_2}\gamma_{\ldots,k_{i_1}+u,\ldots,k_{i_2}+(j-u),\ldots} \ (j>0, k_i\in\mathbb{Z}, \sum_i k_i = -j); \tag{34}$$

Here λ and μ are assumed fixed, but as the equations do not involve them, we may conclude that if $B_{k_1,\ldots,k_n}$ is a family of composition vector operators for some $]\lambda,\mu$, then so is for any λ,μ. Thus, the Part (i) of Theorem 5.16 follows from Proposition 5.18.

Proof of Proposition 5.18. Last lemma shows that the operators $B_{k_1,\ldots,k_n}\colon \mathcal{H}\to\mathcal{H}$, which satisfy conditions (27) and (32), exist and unique. Using the description of e_i via F_j, we can deduce from (27) an expression for $[e_j, B_{k_1,\ldots,k_n}]$. All we need is that $[e_j, B_{k_1,\ldots,k_n}]$ satisfies also the condition (28). In other words, we can calculate the commutator $[e_j, B_{k_1,\ldots,k_n}]$ from (27) and (28), and our requirement is that the results be the same. An immediate calculation shows that this is precisely the equalities (33) and (34). The remarkable fact is that the contributions of λ and μ into the results of the two computations cancel, when we equate them (and this is why Part (i) of Theorem 5.16 holds). Let us show how it happens.

Since the commutators $[F_rF_s, B_{k_1,\ldots,k_n}]$ do not involve λ and μ, the dependence of the result of the first computation on λ and μ comes from the commutator $[(\alpha j + \beta)F_j, B_{k_1,\ldots,k_n}]$. But $\alpha j+\beta = \mu-(j+1)\lambda - \frac{j+1}{2}$. Hence, from (27),

$$[e_j, B_{k_1,\ldots,k_n}] = (\mu-(i+1)\lambda)\sum_s \alpha_s B_{\ldots,k_s+j,\ldots} + \text{expression without } \lambda,\mu.$$

Also, since $\mu_s = \alpha_s\mu + \ldots,\ \lambda_s = \alpha_s\lambda+\ldots$, where $\ldots$ do not involve either λ or μ, the same result follows from (28). This explains the absence of λ and μ in equations (33) and (34).

Notice also that our equations are dependent; for example, we can remove all the equations (33), (34) with $j\ge 3$.

5.3.5. Example: n = 2. A simple computation shows that in this case all the equations (33), (34) are corollaries of one of them, for example, the equation (33) with $j=1$. If we put $\gamma_{-k,k}=\gamma_k$ and $\beta_1=-\beta_2=\beta$, then this equation take the form

$$(-k+\beta)\gamma_k + (k+1-\beta+\alpha_1\alpha_2)\gamma_{k+1} = 0.$$

We see that if neither of β and $\beta-\alpha_1\alpha_2$ is an integer, then all γ_k can be expressed via, say, γ_0. The same is true, if only one of $\beta,\beta-\alpha_1\alpha_2$ is an integer, or idf both are integers, but $\alpha_1\alpha_2\ge 0$. But in the "exceptional case," when $\beta\in\mathbb{Z},\alpha_1\alpha_2\in\mathbb{Z}_{<0}$, there are two independent solutions:

$$\gamma_k = \begin{cases} (-1)^k\binom{-\alpha_1\alpha_2+\beta-k-1}{\beta-k} & \text{if } k\le\beta,\\ 0 & \text{if } k>\beta; \end{cases} \quad \gamma_k = \begin{cases} (-1)^k\binom{k-\beta-1}{k+\alpha_1\alpha_2-\beta} & \text{if } k\ge\beta-\alpha_1\alpha_2\\ 0 & \text{if } k<\beta-\alpha_1\alpha_2 \end{cases}$$

5.3.6. Beginning of the proof of Main Theorem. Probably, there are two independent ways of proving Theorem 5.16 (and possibly Conjecture 5.17) The first one consists in a direct investigation of the system (33), (34), while the second one describes the functions $\gamma_{k_1,\dots,k_n} t_1^{k_1} \dots t_n^{k_n}$ as solutions of system of partial differential equations. We will combine these two ways: they will give us upper and lower bounds for the number of independent solutions for generic α, β.

Proposition 5.20. *The system* (33), (34) *is equivalent to the system*

$$\sum_{\substack{S\subset\{1,\dots,,n\}\\ \text{card } S=j}} \left[\sum_{s\in S}(\beta_s+k_s+1)+\sum_{s\in S, t\notin S, s<t}\alpha_s\alpha_t\right]\gamma_{k_1+w_1,\dots,k_n+w_n}=0 \tag{35}$$

$$(k_1+\ldots+k_n=-j, j+1,\ldots,n)$$

where $w_i = 1$, *if* $i \in S$, *and* $w_i = 0$ *if* $i \notin S$.

Proof. Indeed, if we denote the equation of system (34) by $\text{eq}(k_1, \dots, k_n)$ then the equation (35) is equivalent to $\displaystyle\sum_{S\subset\{1,\dots,,n\},\text{card } S<j} \text{eq}(k_1+s_1,\dots,k_n+s_n)$.

Proposition 5.21. *If all the coefficients of the system* (35) *are non-zero (equivalently: if all the sums*

$$\sum_{s\in S}\beta_s+\sum_{s\in S,t\notin S,s<t}\alpha_s\alpha_t$$

are non-integral), then the system (35) *has no more than* $(n-1)!$ *independent solutions.*

Proof. Fix a large positive integer k and consider the part of the system (35) consisting of equations with $k_1 = \ldots = k_{n-1} = k$. We put

$$\Gamma_{\ell_1,\dots,\ell_{n-1}}=\gamma_{\ell_1,\dots,\ell_{n-1},-(n-1)k-\ell_1-\ldots-\ell_{n-1}},\ \Gamma(\ell_1,\dots,\ell_{n-1})=\sum\Gamma_{\ell_1,\dots,\ell_{n-1}}t_1^{\ell_1}\dots t_{n-1}^{\ell_{n-1}}$$

(Thus, $\Gamma(\ell_1, \dots, \ell_{n-1})$ is a power series. Then asymptotically (for $k \to \infty$) our part of system (35) (after multiplication by k^{-1}) takes the form

$$(j\sigma_j-(n-j)\sigma(j-1=0\ (j=1,\dots,n-1),$$

where σ_j is the j-th elementary symmetric polynomial. To solve the last system is the same as to determine the cokernel of the operator

$$\mathbb{C}[[t_1,\dots,t_{n-1}]]\to\mathbb{C}[[t_1,\dots,t_{n-1}]]^{n-1},\ F\mapsto\{(j\sigma_j-(n-j)\sigma_{j-1})F\mid j=1,\dots,n-1\},$$

which, by duality, has the same dimension as the kernel of the operator

$$\mathbb{C}[t_1,\dots,t_{n-1}]^{n-1}\to\mathbb{C}[t_1,\dots,t_{n-1}],\ (p_1,\dots,p_{n-1})\mapsto\sum(j\sigma_j-(n-j)\sigma_{j-1})i_j,$$

that is, as the quotient

$$\mathbb{C}[t_1,\dots,t_{n-1}]/(j\sigma_j-(n-j)\sigma_{j-1}\mid j=1,\dots,n-1),$$

which is isomorphic to $\mathbb{C}[t_1, \ldots, t_{n-1}]/(\sigma_1, \ldots, \sigma_{n-1})$, and it is well known that the dimension of the latter is $(n-1)!$.Thus, our "asymptotic system" has precisely $(n-1)!$ independent solutions. It remains to notice that if all the coefficients of system (35) are non-zero, then every solution $\{\gamma_{k_1,\ldots,k_n}$ with $k_i \geq k$ for $i = 1, \ldots, n-1$, then this solution is zero. Hence, our system has a part with precisely $(n-1)!$ independent solutions, and therefore the whole system has no more that $(n-1)!$ independent solutions.

5.3.7. End of the proof of Main Theorem: cohomology of a domain in $\mathbb{C}^n$ with local coefficients. PROPOSITION 5.22. *If a formal series*

$$\Gamma(t_1, \ldots, t_n) = \sum_{k_1+\ldots+k_n=0} \gamma_{k_1,\ldots,k_n} t_1^{k_1} \ldots t_n^{k_n}$$

satisfies the system of PDE

$$D_i\Gamma = 0 \ (i = 1, \ldots, n), \ D_i = \frac{\partial}{\partial t_i} - \sum_{j \neq i} \left[\frac{\alpha_i \alpha_j}{t_i - t_j} - \frac{\beta_i - \alpha_i(\alpha_1 + \ldots + \alpha_{i-1})}{t_i} \right], \qquad (36)$$

then $\gamma_{k_1,\ldots,k_n}$ satisfy the system (33), (34).

Proof. Indeed, the equations (33), (34) are nothing but

$$\sum t_i^j D_i\Gamma = 0 \ (j = 2, 3, \ldots), \ \sum t_i^j D_i\Gamma = 0 \ (j = 0, -1, \ldots).$$

To complete the proof of Theorem 5.16, it is sufficient to prove the following

PROPOSITION 5.23. *If no product $\alpha_i\alpha_j$ is integral, then the system $D_i\Gamma = 0$ has from $(n-1)!$ to $n!$ independent solutions.*

We shall prove Proposition 5.23 in an equivalent cohomological form. Let

$$\Delta = \Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_i \neq 0, t_j \neq t_i \text{ for } j \neq i\}$$

An easy check shows that $[D_i, D_j] = 0$; hence, the operators D_i generate a flat connection ∇ on the standard trivial line bundle over Δ. There arises the De Rham complex

$$\Omega_\nabla = \{\Omega^0(\Delta) \xrightarrow{d_\nabla} \Omega^1(\Delta) \xrightarrow{d_\nabla} \ldots \xrightarrow{d_\nabla} \Omega^n(\Delta)\},$$

where $\Omega^i(\Delta)$ is the space of holomorphic differential forms of degree i on Δ.

The homology of this complex is closely related to system (36). Namely, for a linear operator $\varphi\colon H^n(\Omega_\nabla) \to \mathbb{C}$, set

$$\gamma_{k_1,\ldots,k_n} = \varphi[t_1^{k_1} \ldots t_n^{k_n} dt_1 \wedge \ldots \wedge dt_n].$$

Obviosly, $\{\gamma_{k_1,\ldots,k_n}\}$ satisfies the system (36). There arises a linear map

$$(H^n(\Omega_\nabla))^* \to \text{solutions of (36)}.$$

Lemma. *If no product $\alpha_i\alpha_j$ is integral, then this map is one-to-one.*

Proof. It follows from the obvious fact that for a holomorphic form $\omega \in \Omega^{n-1}(\mathbb{C}^n - 0)$ the residue of the form $d_\nabla \omega$ with respect to the plane $t_i = t_j$ is

$$(\exp 2\pi\sqrt{-1}\alpha_i\alpha_j - 1)\omega|_{\{t_i=t_j\}}.$$

In view of this lemma, Proposition 5.23 follows from the following

Proposition 5.24. $(n-1)! \le \dim H^n(\Omega_\nabla) \le n!$.

Proof. The cohomology of the complex Ω_∇ is the cohomology of the domain Δ_n with coefficients in a local system with the stalk $\mathbb{C}$ and the transformation induced by a closed path $\sigma\colon [0,1] \to \Delta_n$ being $z \mapsto e_\sigma z$, where e_σ is a function of the linking numbers a_i, b_{ij} of σ with the planes $t_i = 0, t_i = t_j$:

$$e_\sigma = \prod_i \exp 2\pi\sqrt{-1}\, a_i(\beta_i - \alpha_i(\alpha_1 + \ldots + \alpha_{i-1})) \cdot \prod_{i<j} \exp 2\pi\sqrt{-1}\, b_{ij}\alpha_i\alpha_j.$$

Remarks. (1) We will not need this long (although obvious) formula for e_σ; all we need is if σ is a trajectory of the standard action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ in Δ, then $e_\sigma = 1$ (since $\beta_1 + \ldots + \beta_n = 0$).

(2) The fact that the cohomology of Δ can be calculated from the complex of holomorphic forms depends on the fact that Δ is a Stein manifold, namely the holomorphy domain of the function $(t_1, \ldots, t_n) \mapsto \left(\prod_i t_i \cdot \prod_{i<j}(t_i - t_j)\right)^{-1}$.

Consider the sequence of (holomorphic) fibrations

$$\Delta_n \xrightarrow{p_n} \Delta_{n-1} \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_3} \Delta_2 \xrightarrow{p_2} \Delta_1 = \mathbb{C}^*,$$

where $p_j(t_1, \ldots, t_j) = (t_1, \ldots, t_{j-1})$. The fiber F_j of p_j is $\mathbb{C}$ with j punctures, which is homotopy equivalent to the wedge of j circles.

Denote our local system on $\Delta = \Delta_n$ by $\mathcal{C}$ or $\mathcal{C}^n$. Then there arises a sequence of local systems $\mathcal{C}^j = R^1(p_{j+1})_*\mathcal{C}^{j+1}$ on Δ_j. The Leray spectral sequences of the fibrations p_j provide the chain of isomorphisms

$$H^n(\Delta_n; \mathcal{C}^n) \cong H^{n-1}(\Delta_{n-1}; \mathcal{C}^{n-1}) \cong \ldots \cong H^2(\Delta_2; \mathcal{C}^2) \cong H^1(\Delta_1; \mathcal{C}^1) \tag{37}$$

The stalk of the system $\mathcal{C}^j$ is $H^1(F_{j+1}; \mathcal{C}^{j+1}|_{F_{j+1}}))$. Let d_j be the dimension of this stalk. Then the description of F_j given above shows that $jd_{j+1} \le d_j \le (j+1)d_{j+1}$. Since $d_n = 1$, this shows that $(n+1)! \le d_1 \le n!$. From (37), $H^n(\Delta_n; \mathcal{C}^n) \cong H^1(\Delta_1; \mathcal{C}^1)$. Since the system $\mathcal{C}^1$ is trivial (see Remark (1) above) and Δ_1 is homotopically equivalent to the circle, $\dim H^1(\Delta_1; \mathcal{C}^1) = d_1$.

This completes the proof of Proposition 5.24, and, hence, of Theorem 5.16.

5.3.8. A final remark. Informally speaking, Proposition 5.22 states that

$$\sum_{k_1+\ldots+k_n=0} \gamma_{k_1,\ldots,k_n} t_1^{k_1} \ldots t_n^{k_n} = \prod_{1\le i<j\le n} \left(1 - \frac{t_i}{t_j}\right)^{\alpha_i\alpha_j} \prod_{i=1}^n t_i^{\beta_i}.$$

(Compare Proposition 5.13.) It is true, however, that neither the left hand side of this equality, nor its right-hand side, makes much sense: the first one is a divergent series, the second one is a multivalued function. The equality means only that the both sides satisfy the same system of equation. (See further comments in Section 4.8 of [13].)

5.4. Applications to singular vectors in the modules of semi-infinite forms.

The results of Section 5.3 show that

$$B_{k_1,\ldots,k_n}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)v\in\mathcal{H}(\lambda,\mu)_{k_1+\ldots+k_n}$$

(where v is the vacuum vector of $\mathcal{H}(\lambda,\mu-\alpha_1-\ldots-\alpha_n)$) is a singular vector if and only if it is not zero and

$$\begin{gathered}\alpha_s\lambda+\frac{(\alpha_s-1)(\alpha_s+2)}{2}=0\Longleftrightarrow\alpha_s^2+\alpha_s(2\lambda+1)-2=0,\\ \alpha_s(\mu-\alpha_{s+1}-\ldots-\alpha_n)-1+\beta_s+k_s=0\end{gathered}\tag{38}$$

($s=1,\ldots,n$). Let us begin with the case $\alpha_1=\ldots=\alpha_n=\alpha$. The the sum (over s) of the second equality (38) becomes (since $\beta_1+\ldots+\beta_n=0$)

$$n\left(\alpha\mu-\frac{n-1}{2}\alpha^2\right)=n-k,\ \text{where } k=k_1+\ldots+k_n$$

The system (38) becomes

$$\begin{gathered}\alpha^2+\alpha(2\lambda+1)-2=0,\\ \frac{n(n-1)}{2}\alpha^2-n\mu\alpha+n-k=0.\end{gathered}$$

From this,

$$\lambda=-\frac{\alpha^2-\alpha-2}{2\alpha},\ \mu=\frac{(n-1)\alpha}{2}-\frac{k-n}{n\alpha}.$$

This is the parametric equation of the curve $\Psi_{-n,-k/n}$ (see Sections 4.8.4 and 5.3.1).

If we plug the last expression for μ into the second of equations (38), we get:

$$\beta_s=\left(\frac{n+1}{2}-s\right)\alpha^2+\frac{k}{n}-k.$$

With these β_s, we can state that

$$B_{k_1,\ldots,k_n}(\alpha,\ldots,\alpha;\beta_1,\ldots,\beta_n)v\in\mathcal{H}(\lambda,\mu)_k\tag{39}$$

(in view of an indeterminacy of composition vectors operators, it is better to cay "any vector of this form") is either zero, or a singular vector. (It is easy to see that the vector (39) does not depend on $k_1,\ldots,k_n$: only k matters.) If k is not divisible by n, then $\mathcal{H}(\lambda,\mu)$

(generically) does not have singular vectors of degree k, and hence the vector (39) is zero (which is not obvious *a priori*). If k is divisible by n, then $\mathcal{H}(\lambda,\mu)$ contains singular vectors of degree k, and the vector

$$B_{k/n,\dots,k/n}\left(\alpha,\dots,\alpha;\frac{n-1}{2}\alpha^2,\frac{n-3}{2}\alpha^2,\dots,\frac{1-n}{2}\alpha^2\right)v\in\mathcal{H}(\lambda,\mu)_k$$

is either zero, or a singular vector. One can expect that this vector is not zero for some α's, but we were not able to prove it. If we believe in this, then we obtain explicit formula for singular vectors in all reducible modules $\mathcal{H}(\lambda,\mu)$.

Remark in conclusion that if we drop the assumption $\alpha_1=\ldots=\alpha_n$, we get more vanishing theorems for composition vector operators. For example,

$$B_{k_1,k_2}\left(\alpha,-\frac{2}{\alpha};-\beta,\beta\right)v=0$$

in $\mathcal{H}(\lambda,\mu)$ for arbitrary α and

$$\beta=1+\frac{k_2\alpha^2+2k_1}{2-\alpha^2},\lambda=-\frac{(\alpha-1)(\alpha+2)}{2\alpha},\mu=\frac{(k_1+k_2)}{2-\alpha^2}.$$

5.5. The Japanese Lie algebra $\mathfrak{GL}$.

Our next (and last) goal is to include the Lie algebras $\mathfrak{H}$ and $\mathfrak{Vir}$ into a bigger Lie algebra which, in many senses is the right infinite-dimensional version of $\mathfrak{gl}(n)$. We call it Japanese, because it is one of basic objects in "semi-infinite geometry," created and developed by the so called "Sato school" which was founded in Kyoto, Japan, by Mikio Sato. The most relevant publication on this subject is [23]. I can say that semi-infinite forms also belong to this geometry, as well as the *Japanese Grassmannian*. This is the (infinite-dimensional) manifold of subspaces L of the space V with the basis $\{v_i \mid i\in\mathbb{Z}\}$ such that the quotients $L/(L\cap V_-)$ and $V-/(L\cap V_-)$ (where $V_-=\operatorname{span}\{v_i\mid i\le 0\}$) are finite-dimensional.

The actions of $\mathfrak{H}$ and $\mathfrak{Vir}$ in $\mathcal{H}$ will be the restriction of a certain action of this Lie algebra. We begin with its construction.

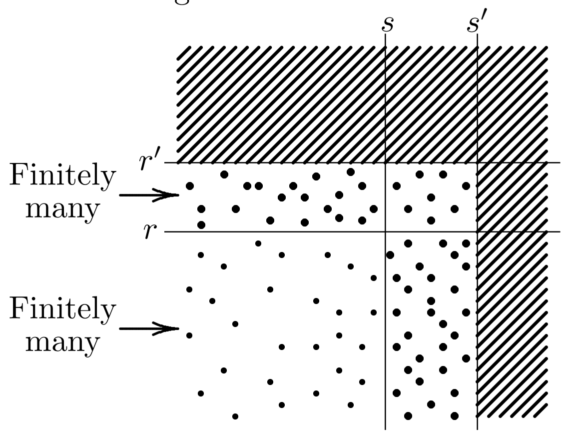
5.5.1. The main construction. What is the right Lie algebra of matrices of infinite order? The most obvious candidate is the limit $\varinjlim_{n\to\infty}\mathfrak{gl}(n)$, which is the Lie algebra of $\infty\times\infty$ matrices with finitely many non-zero entries. We will denote this Lie algebra as $\mathfrak{gl}(\infty)$. Also we can consider the algebra of automorphisms of an ∞-dimensional space. There is two other versions of this. Let V be the (complex) vector space with the basis $\{v_i\mid i\in\mathbb{Z}\}$,and let $\widehat{V}$ be the space of "linear combinations" $\sum_{k\in\mathbb{Z}}a_kv_k$ with the set of non-zero coefficients a_k with positive k being finite. Then $\operatorname{End}V$ is the Lie algebra of matrices $\|a_{ij}\|_{i,j\in\mathbb{Z}}$ with finitely many non-zero entries in every column, and $\operatorname{End}\widehat{V}$ is the Lie algebra of matrices $\|a_{ij}\|_{i,j\in\mathbb{Z}}$ such that, for some N, the set of non-zero entries a_{ij} with $i>N$ or $j<-N$ is finite. However, neither of these three Lie algebras possesses a non-trivial central extension, and by this reason we have to look somewhere else.

We describe below modifications of these two Lie algebras, which are equally satisfactory for our purposes. We will attribute the notation $\mathfrak{GL}_0$ to the last of them, but the reader can replace it by any of the other two.

The first is the Lie algebra of *generalized Jacobi matrices*, that is, matrices $\|a_{ij}\|$ such that for some n, $a_{ij} = 0$, if $|j-i| > N$ (in other words, matrices with finitely many non-zero diagonals).

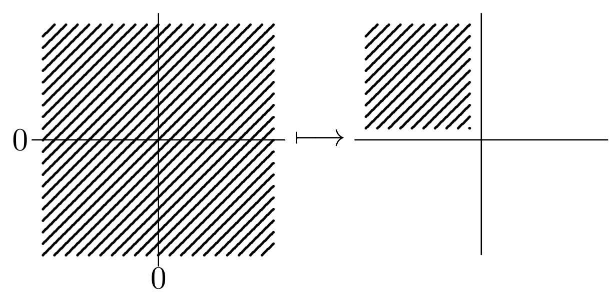
The second is the Lie algebra of endomorphisms of V continuous with respect to topology generated by sets $V_k = \operatorname{span}\{v_i \mid i \leq k\}$. In other words, this is the Lie algebra of matrices with finitely many non-zero entries in every column satisfying the additional condition: for every $r, s \in \mathbb{Z}$, the number of non-zero entries $\|a_{ij}$ with $i > r$ and $j < s$ is finite.

The last one is the same with V replaced by $\widehat{V}$. Its matrix description is the same as in the previous case, with the condition of "finitely many non-zero entries in columns" dropped. This is what we denote as $\mathfrak{GL}_0$.The picture below shows (schematically) the structure of a matrix from this Lie algebra.



The common property of the last three Lie algebras is that they possess a (common to them all) non-trivial central extension. Below, we describe a cocycle $c(A, B), A, B \in \mathfrak{GL}_0$. First, define a transformation $\phi\colon \mathfrak{GL}_0 \to \mathfrak{GL}_0$ which assigns to a matrix A its "upper left corner":

$$A = \|a_{ij}\| \mapsto \phi(A) = \|a^{\phi}_{ij},\ a^{\phi}_{ij} = \begin{cases} a_{ij}, & \text{if } i < 0, j < 0, \\ 0 & \text{otherwise} \end{cases}$$



This transformation "almost commutes" the commutators, in the sense that *the matrix* $\psi(A, B) = [\phi(A), \phi(B)] - \phi[A, B]$ *has finitely many non-zero entries.* Indeed, if $A = \|a_{ij}\|, B = \|b_{ij}\|$, and $\psi(A, B) = \|c_{ij}\|$, then, for $i < 0, j < 0$,

$$c_{ij} = \sum_{k<0}(a_{ik}b_{kj} - b_{ik}a_{kj}) - \sum_{k}(a_{ik}b_{kj} - b_{ik}a_{kj}) = -\sum_{k\geq 0}(a_{ik}b_{kj} - b_{ik}a_{kj}),$$

which has finitely many non-zero values, since both a_{kj} and b_{kj} can be different from zero for finitely many $k \geq 0, j < 0$. We put

$$c(A,B) = \operatorname{tr} \psi(A,B).$$

Let us prove that this c satisfies the cocycle equation. For $A, B, C \in \mathfrak{GL}_0$,

$$\begin{aligned}
&c([A,B],C) + c([B,C],A) + c([C,A],B) = \\
&= \operatorname{tr}\left(\psi([A,B],C) + \psi([B,C],A) + \psi([C,A],B)\right) \\
&= \operatorname{tr}\left([\phi[A,B],\phi(C)] + [\phi[B,C],\phi(A)] + [\phi[C,A],\phi(B)]\right. \\
&\qquad -\phi[[A,B],C] \quad - \phi[[B,C],A] \quad \left.- \phi[[C,A],B]\right) \qquad \text{zero (Jacobi)} \\
&= \operatorname{tr}\left([\phi[A,B],\phi(C)] + [\phi[B,C],\phi(A)] + [\phi[C,A],\phi(B)]\right. \\
&\quad \left.-[\phi(A),\phi(B)],\phi(C)] - [\phi(B),\phi(C)],\phi(A)] - [\phi(C),\phi(A)],\phi(B)]\right) \ \text{zero (Jacobi)} \\
&= \operatorname{tr}\left([\psi(A,B),\phi(C)] + [\psi(B,C),\phi(A)] + [\psi(C,A),\phi(B)]\right)
\end{aligned}$$

The latter is equal to zero, since $\psi(-,-) \in \mathfrak{gl}(\infty)$ (has finitely many non-zero entries) and the following holds:

LEMMA *If $X \in \mathfrak{gl}(\infty)$ and Y is an arbitrary $\mathbb{Z} \times \mathbb{Z}$ matrix, then the trace* $\operatorname{tr}[X,Y]$ *is defined and is equal to zero.*

Proof. It is sufficient to consider when X is a one-entry matrix with the entry of x_{rs}. Let $Y = \|y_{ij}\|$. Then the only (non-zero) diagonal entry of XY is $(XY)_{rr} = x_{rs}y_{sr}$ and the only diagonal entry of YX is $(YX)_{ss} = y_{sr}x_{rs}$. Hence, $\operatorname{tr}[X,Y] = y_{sr} - y_{sr} = 0$.

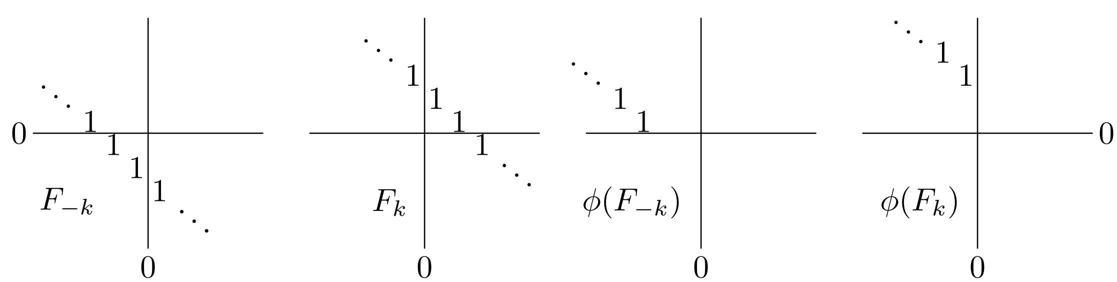
Thus, c is a cocycle, and it gives rise to a one-dimensional central extension of the Lie algebra $\mathfrak{GL}_0$. We denote the extended Lie algebra as $\mathfrak{GL}$ and denote the generator of the center by z.

We will see in the next section that this central extension is not trivial. Actually, $\mathfrak{GL}$ is the unique (up to isomorphism) non-trivial central extension of $\mathfrak{GL}_0$

5.5.2. Some important subalgebras of $\mathfrak{GL}$.

5.5.2.1. The Heisenberg algebra. Let F_k be the matrix of the transformation $v_i \mapsto v_{i-k}$, that is the matrix $\|a_{ij} = \delta_{i-k,j}\|$. Let us compute $c(F_\ell, F_k)$.

First of all, obviously, $[F_\ell, F_k] = 0$, and hence $\phi[F_\ell, F_k] = 0$. Let us calculate, for $k > 0$, $\phi(F_{-k}), \phi(F_k)$, and $[\phi(F_{-k}), \phi(F_k)]$. The matrices of $F_{-k}, F_k, \phi(F_{-k}), \phi(F_k)$ are shown below.



Computations:

$\phi(F_{-k})$	$\phi(F_k)$	$\phi(F_{-k})\phi(F_k)$	$\phi(F_k)\phi(F_{-k})$	$[\phi(F_{-k}),\phi(F_k)]$
$v_{-k-1}\mapsto v_{-1}$	$v_{-1}\mapsto v_{-k-1}$	$v_{-1}\mapsto v_{-1}$	$v_{-k-1}\mapsto v_{-k-1}$	$v_{-1}\mapsto v_{-1}$
$v_{-k-2}\mapsto v_{-2}$	$v_{-2}\mapsto v_{-k-2}$	$v_{-2}\mapsto v_{-2}$	$v_{-k-2}\mapsto v_{-k-2}$	$v_{-2}\mapsto v_{-2}$
$v_{-k-3}\mapsto v_{-3}$	$v_{-3}\mapsto v_{-k-3}$	$v_{-3}\mapsto v_{-3}$	$v_{-k-3}\mapsto v_{-k-3}$	....
....	....	....	....	$v_{-k}\mapsto v_{-k}$

Thus, $\operatorname{tr}[\phi(F_{-k}),\phi(F_k)]=k$ and $c(F_{-k},F_k)=k$. A similar computation shows that if $\ell+k\neq 0$, then $\operatorname{tr}[F_\ell,F_k]=0$ (the matrix of $[F_\ell,F_k]$ does not have non-zero diagonal entries), and hence $c(F_\ell,F_k)=0$. Thus, elements F_k ($k\neq 0$) and z span a subalgebra of $\mathfrak{GL}$ isomorphic to the Heisenberg algebra.

By the way, this construction shows that the restriction of the cocycle c to the commutative subalgebra of $\mathfrak{GL}_0$ spanned by F_k, $k\neq 0$ is not cohomologous to zero (since the Heisenberg algebra is a non-trivial central extension of the commutative algebra). Hence, c is not cohomologous to zero as a cocycle of $\mathfrak{GL}_0$, and $\mathfrak{GL}$ is a non-trivial central extension of $\mathfrak{GL}_0$.

5.5.2.2. The Virasoro algebra. There is a two-parameter family of embeddings of $\mathfrak{Vir}$ into $\mathfrak{GL}$ with the parameters $\lambda,\mu\in\mathbb{C}$, and they arise from the $\mathfrak{Witt}$-module structure in $\mathcal{F}_{\lambda\mu}$. Namely, we denote by $\overline{e}_k$ ($k\in\mathbb{Z}$) an element of $\mathfrak{GL}_0$ which acts in V as $v_j\mapsto(\mu+j-\lambda(k+1))v_{j+k}$ (a familiar formula, isn't it?). We know only too well that $[\overline{e}_\ell,\overline{e}_k]=(k-\ell)\overline{e}_{k+\ell}$. Now, let us calculate $c(\overline{e}_\ell,\overline{e}_k)$. It is easy to see (precisely as in the Heisenberg case), that if $k+\ell\neq 0$, then the matrix of $\psi(\overline{e}_\ell,\overline{e}_k)$ does not have non-zero diagonal entries, so in this case $c(\overline{e}_\ell,\overline{e}_k)=0$.

It remains to calculate $c(\overline{e}_k,\overline{e}_{-k})$. Similarly to the previous calculation, the matrix of $\psi(\overline{e}_k,\overline{e}_{-k})$ has k non-trivial diagonal entries $\psi_{-j,-j}$, $j=1,\dots,k$. The contribution of $[\phi(\overline{e}_{-k}),\phi(\overline{e}_k)]$ into $\psi_{-j,-j}$ is the coefficient at v_{-j} in $\overline{e}_k\overline{e}_{-k}v_{-j}$. But there is a difference with the previous case: the commutator $[\overline{e}_{-k},\overline{e}_k]$ is not zero: it is equal to $2k\overline{e}_0$. This makes an additional contribution into $\psi_{-j,-j}$ the coefficient at v_{-j} in $2k\overline{e}_0v_{-j}$. From this, we obtain a formula for $c(\overline{e}_{-k},\overline{e}_k)=\operatorname{tr}\psi(\overline{e}_{-k},\overline{e}_k)$:

$$c(\overline{e}_{-k},\overline{e}_k)=\sum_{j=1}^{k}(\mu-j-\lambda(1-k))(\mu-(j+k)-\lambda(1+k))-2k\sum_{j=1}^{k}(\mu-j-\lambda).$$

A direct further calculation gives:

$$c(\overline{e}_{-k},\overline{e}_k)=k\mu(\mu-2\lambda-1)-\frac{k^3-k}{6}(6\lambda^2+6\lambda+1),$$

and this is our final result.

This shows that the formulas $e_i\mapsto\overline{e}_i$ for $i\neq 0$, $e_0\mapsto\overline{e}_0+\frac{1}{2}\mu(\mu-2\lambda-1)z$, $z\mapsto -2(6\lambda^2+6\lambda+1)z$ define an embedding $\mathfrak{Vir}\to\mathfrak{GL}$. We denote this embedding as $\eta_{\lambda\mu}$. (If λ and μ are known and fixed, we will consider $\mathfrak{Vir}$ as a subalgebra of $\mathfrak{GL}$; in particular, we will never use again the notation $\overline{e}_i$.)

5.5.2.3. The orthogonal and symplectic algebras. The space V possesses a symmetric inner product, $\langle v_i, v_j\rangle = \delta_{i+j,1}$, and a skew symmetric inner product, $\{v_i, v_j\} = \delta_{i+j,1}\frac{j-i}{2}$. Both are defined in the space $\widehat{V}$. Accordingly, we define two subalgebras of $\mathfrak{GL}_0$:

$$\mathfrak{O}_0 = \{g \in \mathfrak{GL}_0 \mid \langle gv', v''\rangle + \langle v', gv''\rangle \ \forall v, v'' \in \widehat{V}\}$$
$$\mathfrak{SP}_0 = \{g \in \mathfrak{GL}_0 \mid \{gv', v''\} + \{v', gv''\} \ \forall v, v'' \in \widehat{V}\}$$

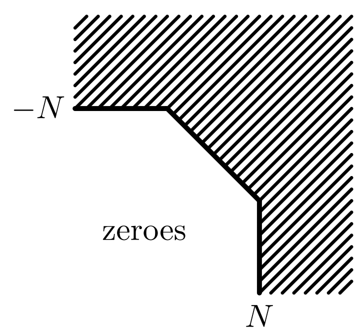
Both $\mathfrak{O}_0$ and $\mathfrak{SP}_0$ can be described in the terms of matrix entries. Namely,

$$g = \|g_{ij}\| \in \mathfrak{O}_0, \text{ if and only if } g_{1-i,j} = -g_{1-j,i} \ \forall i, j,$$
$$g = \|g_{ij}\| \in \mathfrak{SP}_0, \text{ if and only if } g_{1-i,j}(1-2i) = g_{1-j,i}(1-2j) \ \forall i, j.$$

Notice that the last two conditions describe symmetries in diagonals parallel to the main diagonal, so they do not conflict with the definition of $\mathfrak{GL}_0$. Also both can be applied to generalized Jacobi matrices.

A calculation (which we skip) shows that the restrictions of the cocycle c to $\mathfrak{O}_0$ and $\mathfrak{SP}_0$ are not cohomologous to zero, so they give rise to non-trivial one-dimensional central extensions $\mathfrak{O}$ and $\mathfrak{SP}$ of $\mathfrak{O}_0$ and $\mathfrak{SP}_0$.

5.5.2.4. The substitutes for the algebras of upper triangular matrices. Certainly, upper triangular matrices form a subalgebra of $\mathfrak{GL}_0$, but it does not play any serious role in the representation theory. This role is played by a whole family of subalgebras of $\mathfrak{GL}$. For a positive integer N, denote by $\mathfrak{P}_0(N)$ the subalgebra of $\mathfrak{GL}_0$ which consists of matrices $\|g_{ij}\|$ such that $g_{ij} = 0$, if $i > j, i > -N, j < N$ (see diagram below).



The inverse image of $\mathfrak{P}_0(N)$ is the trivial center extension of $\mathfrak{P}_0(N)$; we denote it by $\mathfrak{P}(N)$.

Also we will use the notation $\mathfrak{diag}$ for the algebra of *diagonal matrices*. The matrix with the only non-zero entry $E_{11} = 1$ will be denoted as h_i.

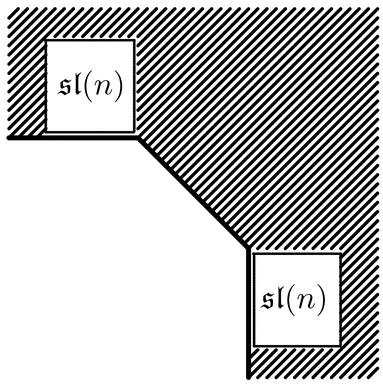
5.5.3. Representation theory for $\mathfrak{GL}$. There is no indisputable similarity between $\mathfrak{GL}$ and Kac-Moody and Virasoro algebras, because $\mathfrak{GL}$ has "infinite rank." In particular, there are no *Verma modules*. Still some basic notions of the representation theory work for the case of $\mathfrak{GL}$.

5.5.3.1. Bernstein-Gelfand-Gelfand modules and their weight vectors. A $\mathfrak{GL}$-module M is called a *Bernstein-Gelfand-Gelfand module of type* $c \in \mathbb{C}$, if (i) $zm = cm$ for every $m \in M$; (ii) the restriction of M to the algebra $\mathfrak{diag}$ of diagonal matrices decomposes into the sum of one dimensional $\mathfrak{diag}$-modules; (iii) for any $m \in M$ there exists an N such that $\dim U(\mathfrak{P}(N)) < \infty$ (where U stands for the universal enveloping algebra).

A *weight vector* of a $\mathfrak{GL}$-vector is, by definition, a common eigenvector of all elements of $\mathfrak{diag}$. The *weight* of a weight vector $m \in M$ is the sequence $\{\lambda_i\}$ of complex numbers such that $h_i m = \lambda_i m$ (where h_i is a one-entry matrix with $h_i v_j = \delta_{ij} v_j$).

PROPOSITION 5.25. *If $\{\lambda_i\}$ is the weight of some weight vector in a Bernstein-Gelfand-Gelfand module, then $\lambda_N = \lambda_{N+1} = \dots$ and $\lambda_{-N} = \lambda_{-N-1} = \dots$ for some N.*

Proof. Let $m \in M$ be a weight vector of weight $\{\lambda_i\}$. Then, for some N, the space $U(\mathfrak{P}(N)$.is finite-dimensional. But $\mathfrak{P}(N)$ contains subalgebras isomorphic to $\mathfrak{sl}(n)$ with arbitrary large n,



and it is well known that $\mathfrak{sl}(n)$ has no non-trivial representations of dimension $< n$. For all $i \geq N$, the differences $h_{i+1} - h_i$ and $h_{-i} - h_{-i-1}$ belong to one of these subalgebras $\mathfrak{sl}(n), n > \dim U(\mathfrak{P}(N)$ of $\mathfrak{P}(N)$. Hence their action on m is trivial, which means that $\lambda_i = \lambda_{i+1}$ and $\lambda_{-i} = \lambda_{-i-1}$.

We will call weights, which satisfy the conclusion of Proposition 5.25, *admissible*. For an admissible weight $\{\lambda_i\}$, we will denote λ_i and λ_{-i} with i large as λ_∞ and $\lambda_{-\infty}$.

5.5.3.2. Highest weight vectors. A weight vector in a $\mathfrak{GL}$- or $\mathfrak{gl}(\infty)$-module M is called a *highest weight vector*, if here exists an N such that (i) $gv \in \mathbb{C}v$ for any $g \in \mathfrak{P}(N)$; (ii) $gv = 0$, if $g \in \mathfrak{P}(n)$ and all the diagonal entries of g are zeroes. Actually, it is true that any irreducible Bernstein-Gelfand-Gelfand module has a unique up to a factor highest weight vector. We will not prove it here which makes it necessary to impose an additional assumption on modules considered in next Proposition; for our purposes, this assumption is quite harmless.

The weight of the highest weight vector of an irreducible Bernstein-Gelfand-Gelfand module M will be called the *highest weight* of M.

Notice that if M is an irreducible Bernstein-Gelfand-Gelfand $\mathfrak{gl}(\infty)$-module with an admissible highest weight then for all weights of weight vectors in M, λ_∞ and $\lambda_{-\infty}$ are the same.

PROPOSITION 5.26. (a) *Let M be an irreducible Bernstein-Gelfand-Gelfand $\mathfrak{GL}$-module possessing a highest weight vector v. Then M is generated by v over $U(\mathfrak{gl}(\infty))$.*

(b) *Let M be an irreducible Bernstein-Gelfand-Gelfand $\mathfrak{gl}(\infty)$-module with admissible highest weight $\{\lambda_i\}$. Then the structure of a $\mathfrak{gl}(\infty)$-module can be uniquely extended to a* *structure of a $\mathfrak{GL}$-module. Moreover, $z \in \mathfrak{GL}$ will act in M as $\lambda_{-\infty} - \lambda_\infty$.*

Proof of (a) is similar to the proof of similar statement for Verma modules. It is obvious that M is generated by v over $U(\mathfrak{GL})$. But we can reorder the factors in elements of $\mathfrak{GL}$ in such a way that elements of $\mathfrak{P}(N)$ go last. These elements take $\mathbb{C}v$ into $\mathbb{C}v$. Then we apply the matrices whose entries are in the domain $i > -N, j < N, i > j$, but a matrix from $\mathfrak{GL}$ has only a finite sets of non-zero entries in this domain, so these matrices are all in $\mathfrak{gl}(\infty)$.

To prove (b), let us first describe a canonical way to extend the structure of a Bernstein-Gelfand-Gelfand module over $\mathfrak{gl}(\infty)$ to a structure of a $\mathfrak{GL}$-module.

Let $g = \|a_{ij}\| \in \mathfrak{GL}_0$. Consider first the case when all the diagonal entries of g are zeroes. Let P is the set of all pairs (i, j) with $a_{ij} \neq 0$. Then $g = \sum\limits_{(i,j)\in P} a_{ij}E_{ij}$, $a_{ij} \in \mathbb{C} - 0$. Let m be a weight vector of M. Then only finitely many of vectors $E_{ij}m$ can be non-zero. Indeed, for some N, $W = U(\mathfrak{P}(N) \cap \mathfrak{gl}(\infty))m$ is finite-dimensional. Since M is the sum of one-dimensional $\mathfrak{diag} \cap \mathfrak{gl}(\infty)$-modules, the same is true for W. Hence, vectors from W can have finitely many different weights, but all $E_{ij}m$ with $i \neq j$ are weight vectors with different weights. Denote by P_0 the set of pairs (i, j) with non-trivial $E_{ij}m$ and set $gm = \sum\limits_{(i,j)\in P_0} a_{ij}E_{ij}m$.

Now, let g be a diagonal matrix, $g = \sum d_ih_i$ Let $\{\lambda'_i\}$, $\{\lambda''_i\}$ be weights of weight vectors $v', v'' \in M$. The series $\sum d_i\lambda'_i$ and $\sum d_i\lambda''_i$ diverge (in general), but their difference $\sum d_i(\lambda'_i - \lambda"_i)$ converges (is, actually, finite). Indeed, if $v \in M$ is a highest weight vector, then both v' and v'' are obtained from v by applying several matrices from $\mathfrak{gl}(\infty)$. Hence the weights of both v' and v'' are obtained from the weight of v by changes in finitely many positions, so there are only finitely many differences between them. Hence we may regard the infinite sums $d_i\lambda_i$ as well-defined up to a common summand which gives rise to an extension of the representation of $\mathfrak{gl}(\infty)$ to a *projective* representation of $\mathfrak{GL}_0$, that is, to a representation of $\mathfrak{GL}$, since the latter is a unique central extension of $\mathfrak{GL}_0$, according to the Feigin-Tsygan theorem 5.15 (see Section 5.3.1).

Next, let us prove that the $\mathfrak{GL}$-extension of the $\mathfrak{gl}(\infty)$-structure constructed above is unique. For this, it is enough to prove that for any weight vector $m \in M$ there exists such $N \in \mathbb{Z}$ that $gm = 0$ for any matrix $g = \|g_{ij}\| \in \mathfrak{GL}$ such that $g_{ij} = 0$ for $i = j$ or $i < -N$, or $j > N$. Let

$$m = E_{i_rj_r} \dots E_{i_1j_1}v,$$

where $i_s \neq j_s (s = 1, \dots, r)$ and let $N = \max(|i_1|, \dots, |i_r|, |j_1|, \dots, |j_r|)$. Obviously, $c(E_{i_s,j_s}, \mathfrak{GL}_0) = 0$, so $gm = gE_{i_rj_r} \dots E_{i_1j_1}v$ is the sum of 2^r terms of the form

$$E_{i'_1j'_1} \dots E_{i'_sj'_s} [\dots [g, E_{i''_1,j''_1}] \dots, E_{i''_tj''_t}]\, v,$$

where $\{(i'_1, j'_1), \dots, (i_s, j'_s)\}$ and $\{(i''_1, j''_i), \dots, (i''_t, j''_t)\}$ are two complimentary subsets of the set $\{i_1, j_1), \dots, (i_r, j_r)\}$. All these terms are zero by Condition (ii) in the definition of highest weight vectors in the beginning of this section.

It remains to prove that z acts in M as the multiplication by $\lambda_\infty - \lambda_{-\infty}$. But $z = [F_{-1}, F_1]$ where $F_{-1} = \sum_{j=-\infty}^{\infty} E_{j,j-1}$ and $F_1 = F^t_{-1}$ are generators of $\mathfrak{H} \subset \mathfrak{GL}$ described in Section 5.5.2.1. For the computation of the action of $[F_{-1}, F_1]$ in M, we can replace

F_{-1} and F_{-1}^t by their "approximations" $F_{-1}^N = \sum_{j=-N}^{N-1} E_{j,j-1}$ and $(F_{-1}^N)^t$ (with a large N). We have:

$$zv = [F_{-1}, F_{-1}^t]v = [F_{-1}^N, (F_{-1}^N)^t]v = (h_N - h_{-N})v = (\lambda_N - \lambda_{-N})v = (\lambda_\infty - \lambda_{-\infty})v.$$

This completes the proof of Proposition 5.26. Notice in conclusion that both the intersections $\mathfrak{diag} \cap \mathfrak{O}$ and $\mathfrak{diag} \cap \mathfrak{SP}$ consist of diagonal matrices $\sum d_i E_{ii}$ with $d_i = -d_{1-i}$ ($i \in \mathbb{Z}$), so in the cases of $\mathfrak{O}$ and $\mathfrak{SP}$ the weights of the Bernstein-Gelfand-Gelfand modules have the form $\lambda_i \mid i \geq 0$ with $\lambda_N = \lambda_{N+1} = \dots$ and Proposition 5.26 remains true with z acting as the multiplication by $-2\lambda_\infty$.

5.5.3.3. Finite Bernstein-Gelfand-Gelfand modules. A Bernstein-Gelfand-Gelfand $\mathfrak{GL}$-module M is called *finite*, if the action of E_{ij} with $j \neq i$ is locally nilpotent, that is, for every $m \in M$ there exists an r such that $E_{ij}^r m = 0$. It is easy to check that an irreducible Bernstein-Gelfand-Gelfand module is finite if and only if its highest weight $\{\lambda_i\}$ is *integral dominant*, that is, $\{\lambda_i\}$ is a non-decreasing sequence of integers. This makes finite modules similar to finite-dimensional modules in the classical representation theory.

We will consider some important examples in the next section.

5.5.4. Some important examples.

5.5.4.1. The $\mathfrak{GL}$-module of semi-infinite forms. A *semi-infinite form* on V is the "expression"

$$v_{i_1} \wedge v_{i_2} \wedge \dots, \ i_1 < i_2 < \dots \text{ and } i_n = n \text{ for } n \text{ large.}$$

For $i, j \in \mathbb{Z}$, we put

$$E_{ij}(v_{i_1} \wedge v_{i_2} \wedge \dots) = \begin{cases} v_{i_1} \wedge \dots \wedge v_{i_{m-1}} \wedge v_j \wedge v_{i_{m+1}} \wedge \dots & \text{if } i_m = i, \\ 0 & \text{if } i \notin \{i_1, i_2, \dots\}. \end{cases}$$

In particular, for $w = v_1 \wedge v_2 \wedge \dots$, $E_{ii}w = \begin{cases} w & \text{if } i > 0, \\ 0 & \text{if } i \leq 0. \end{cases}$ These formulas equip the space $\mathcal{V}$ of semi-infinite forms with the structure of an irreducible Bernstein-Gelfand-Gelfand $\mathfrak{gl}(\infty)$-module with the highest weight vector w, and the highest weight $(\dots, 0, 0, \overset{(1)}{1}, 1, 1 \dots)$. According to Proposition 5.26(b), this structure has a unique extension to a structure of a $\mathfrak{GL}$-module with z acting as id.

Notice that for any fixed $\lambda, \mu \in C$, the embedding $\eta_{\lambda\mu}\colon \mathfrak{Vir} \to \mathfrak{GL}$ (see Section 5.5.2.2) makes $\mathcal{V}$ a $\mathfrak{Vir}$-module, which is the same as $\mathcal{H}(\lambda, \mu)$; the isomorphism $\mathcal{V} \to \mathcal{H}(\lambda, \mu)$ acts as

$$v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \dots = f_{j_0} \wedge f_{j_1} \wedge f_{j_2} \wedge \dots, \ j_0 > j_1 > j_2 > \dots, \ \{j_\ell\} = \mathbb{Z} - \{i_k\}.$$

5.5.4.2. The spinor representations of $\mathfrak{O}$. We consider the space V with he inner product $\langle v_i, v_j \rangle = \delta_{i+j,1}$. The *Clifford algebra* $\mathrm{Cliff}(V)$ is multiplicatively generated by V (that is, $\{v_i\}$ is the system of generators) with the relation $v'v'' + v''v' = \langle v', v'' \rangle \cdot 1$. We denote as V_+ and V_- the subspaces of V spanned by, respectively, v_i with $i \geq 0$ and v_i

with $i < 0$. Then the subalgebra of $\mathrm{Cliff}(V)$ generated by $V_\pm$ is $\Lambda^* V_\pm$. Obviously, $\Lambda^* V_+$ is $\mathrm{Cliff}(V)/V_- \mathrm{Cliff}(V)$ which makes $\Lambda^* V_+$ a $\mathrm{Cliff}(V)$-module, obviously irreducible. The subspace of $\mathrm{Cliff}(V)$ spanned by 1 and monomials of degree 2 is, obviously, closed with respect to commutators, so it is a Lie algebra.

PROPOSITION 5.27. *This Lie algebra is isomorphic $\mathfrak{O}$.*

Proof. According to formulas in Section 5.2.3, $\mathfrak{O}_0$ consists of matrices $\|g_{ij}\|$ with $g_{1-i,j} = -g_{1-j,i}$, that is, is generated by 2-entries matrices G_{ij} nonzero entries $\delta_{i-i,j}$ and $-\delta_{1-j,i}$. A direct computation shows that the correspondence $G_{ij} \mapsto v_i v_j - \frac{1}{2}\delta_{i+j,1} \cdot 1$ establishes the isomorphism stated.

As a representation of $\mathfrak{O}$, $\Lambda^* V_+$ is the sum of two irreducible representations: $\Lambda^{\mathrm{even}} V_+$ and $\Lambda^{\mathrm{odd}} V_+$; these are called *spinor representations*. A construction similar to that in the proof of Proposition 5.26 makes $\Lambda^{\mathrm{even}} V_+$ and $\Lambda^{\mathrm{odd}} V_+$ $\mathfrak{O}$-modules.

The $\mathfrak{O}$-modules $\Lambda^{\mathrm{even}} V_+$ and $\Lambda^{\mathrm{odd}} V_+$ are closely related to the $\mathfrak{GL}$-modules of semi-infinite forms. Namely, the subalgebra of $\mathfrak{O}$ consisting of endomorphisms of V which preserve the decomposition $V = V_- \oplus V_+$ isomorphic to $\mathfrak{GL}$: an $\mathfrak{O}$-endomorphism of V, which preserve the decomposition $V = V_- \oplus V_+$, is determined by an automorphism of V_+, and we turn it into an endomorphism of V by means the renumeration of the basic elements: $\{v_0, v_1, v_2, v_3, v_4 \dots\} \to \{v_0, v_{-1}, v_1, v_{-2}, v_2, \dots\}$. The $\mathfrak{GL}$-module $\Lambda^* V_+$ is the same as the modules of "shifted" semiinfinite forms $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots,\ i_0 < i_1 < i_2 < \dots, v_{i_n} = n + k$ for fixed $k \in \mathbb{Z}$ and large n; the space of this shifted forms with a given k is denoted as $\mathcal{H}(k)$. This $\mathcal{H}(k)$ is a $\mathfrak{GL}$-module with the highest weight

$$\dots, 0, 0, 0, \overset{(k)}{1}, 1, 1, \dots$$

The isomorphism between exterior forms on $W = V_+$ and (shifted) semiinfimite forms on V is

$$w_{j_1} \wedge w_{j_2} \wedge \dots \wedge w_{j_r} \longleftrightarrow v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \dots,$$

where $\{i_0, i_1, i_2, \dots\}$ is the set of integers consisting of all negative integers of the form $-\frac{j_s + 1}{2}$ and all positive integers *not of the form* $\frac{j_s}{2}$. In the category of $\mathfrak{GL}$ modules, this isomorphism is

$$\Lambda^{\mathrm{even}} V_+ \cong \bigoplus_{k \text{ even}} \mathcal{H}(k),\ \Lambda^{\mathrm{odd}} V_+ \cong \bigoplus_{k \text{ odd}} \mathcal{H}(k).$$

5.5.4.3. The Weyl representation of $\mathfrak{SP}$. The space of this representation is the space $\mathcal{D}$ of polynomials of infinite set of variables $x_1, x_2, \dots$, which appeared in Section 5.1 as the space of the canonical representation of the Heisenberg algebra. It is easy to see that the operators $F_i F_j$, together with $\mathbb{1}$, span a subalgebra of the Lie algebra $\mathrm{End}\,\mathcal{D}$.

PROPOSITION 5.28. *This subalgebra is isomorphic to $\mathfrak{SP} \cap \mathfrak{gl}(\infty)$.*

Proof. According to the formulas in the Section 5.5.2.3, a matrix $\|g_{ij}\|$ belongs to $\mathfrak{SP}_0$, if $g_{1-i,j}(1-2i) = g_{1-j,i}(1-2j)$. Hence, $\mathfrak{SP} \cap \mathfrak{gl}(\infty)$ spanned by F_{ij} acting as

$$v_{1-i} \mapsto \frac{1-2j}{2} v_j, v_{1-j} \mapsto \frac{1-2i}{2} v_i, v_k \mapsto 0, \text{if } k \neq 1-i, 1-j$$

(obviously, $F_{ij} = F_{ji}$). It is easy to see that

$$[F_{ij}, F_{kl}] = \delta_{i+k,1}\frac{2k-1}{2}F_{j\ell} + \delta_{i+\ell,1}\frac{2k-1}{2}F_{jk} + \delta_{j+k,1}\frac{2k-1}{2}F_{i\ell} + \delta_{j+\ell,1}\frac{2k-1}{2}F_{ik},$$

which shows that the correspondence $F_{ij} \longleftrightarrow F_{i^*}F_{j^*}, k^* = \begin{cases} k, & \text{if } k > 0, \\ k-1, & \text{if } k \leq 0 \end{cases}$ is compatible with the commutators.

A construction similar to that in Proposition 5.26 may be used to extend the action of $\mathfrak{SP} \cap \mathfrak{gl}(\infty)$ to the action of $\mathfrak{SP}$. This representation of $\mathfrak{SP}$ is the right "Japanese version" of the classical Weyl representation.

5.5.5. A localization of $\mathfrak{H}$. The main idea of the construction below is to include the $\mathfrak{Vir}$-module $\mathcal{H}(\lambda, \mu)$ into a bigger module which will contain singular vectors not for some special values of λ and μ, but always. A similar construction exists for Verma modules over Virasoro (and, possibly, Kac-Moody algebras); see [25] .

We use the "bosonic version" $\mathcal{D}$ of the space $\mathcal{F}$ of semi-infinite forms. The extension consists in localization with respect to the variable x_1. We will consider even further extension allowing not only negative, but also complex powers of this variable.

For $\nu \in \mathbb{C}$, we denote by $\widehat{\mathcal{D}}_\nu$ the space of formal series

$$\sum_{n=0}^{\infty} p_n(x_2, x_3, \ldots, x_n)x_1^{\nu-n},$$

where p_n is a homogeneous polynomial of degree n (where we assume that $\deg x_j = j$). In particular, p_0 is a constant, and $p_1 = 0$. We use also notations $\widehat{\mathcal{D}}(\nu) = \bigoplus_{q\in\mathbb{Z}} \widehat{\mathcal{D}}_{\nu+q}$; this is a $\mathbb{Z}$-graded $\mathfrak{H}$-module.

Using the formulas

$$e_i = \sum_{\substack{r>s \\ r+s=i \\ r\neq 0,\, s\neq 0}} F_rF_s \left(+\frac{1}{2}F_{i/2}^2, \text{ if } i \text{ is even and} \neq 0\right) + \begin{cases} (\alpha i + \beta)f_i, & \text{if } i \neq 0, \\ \dfrac{\beta^2 - \alpha^2}{2}\mathbb{1}, & \text{if } i = 0, \end{cases}$$

$$z = (1 - 12\alpha^2)\mathbb{1}$$

and $\lambda = -\alpha - \dfrac{1}{2}$, $\mu = \beta - \alpha$ from Sections 5.1.2 and 5.1.3, we turn $\widehat{\mathcal{D}}(\nu)$ into a $\mathbb{Z}$-graded $\mathfrak{Vir}$-module. In this capacity, it will be denoted as $\widehat{\mathcal{H}}(\lambda, \mu, \nu)$. Obviously, there is a canonical isomorphism $\widehat{\mathcal{H}}(\lambda, \mu, \nu) \to \widehat{\mathcal{H}}(\lambda, \mu, \nu + q)$ of degree $-q$, and $\widehat{\mathcal{H}}(\lambda, \mu) = \widehat{\mathcal{H}}(\lambda, \mu, 0)$ contains $\mathcal{H}(\lambda, \mu)$.

PROPOSITION 5.29. *For any λ, μ, ν, q the module $\widehat{\mathcal{H}}(\lambda, \mu, \nu)$ contains a unique (up to a constant factor) singular vector of degree q.*

Proof. Let

$$w = \sum_{n=0}^{\infty} p_n(x_2, x_3, \ldots, x_n)x_1^{\nu+q-n}$$

and consider the equation

$$(e_{-1}+e_{-2}+\ldots)w=0.$$

If $(e_{-1}+e_{-2}+\ldots)w=\sum\limits_{n=0}^{\infty}p_n(x_2,x_3,\ldots,x_n)x_1^{\nu+q-n}=\sum\limits_{n=0}^{\infty}q_n(x_2,x_3,\ldots,x_n)x_1^{\nu+q-n}$, then this equation becomes the system $q_n=0$ for all n. Which p_m contribute to q_n?

Most of the summands in the formula for e_{-i} leave $x_1^{\nu+q-m}$ intact. The exceptions are $F_{-i-1}F_1, F_{-i+1}F_{-1}$, and $(-\alpha+\beta)F_{-1}$, if $i=1$. All the rest, will turn $p_m x_1^{\nu+q-m}$ into $(Dp_m)x_1^{\nu+q-m}$, where D is a differential operator of degree ≤ 2; so, for these parts of e_{-i}, p_m with $m\neq n$ do not contribute to q_n, and the contribution of p_n is Dp_n. Similarly, the contributions to q_n arising from $F_{-i+1}F_{-1}$, and $(-\alpha+\beta)F_{-1}$ are of the form $D'p_{n-1}$ and $D''p_{n-2}$, where D' and D'' are also differential operators of order ≤ 2.

And the most important thing: $F_{-i-1}F_1\left(p_{n+1}x_1^{\nu+q-n-1}\right)=(i+1)\dfrac{\partial p_{n+1}}{\partial x_{i+1}}x_1^{\nu+q-n}$.

Thus, our equation $q_n=0$ becomes

$$\sum_{j=2}^{\infty}j\frac{\partial p_{n+1}}{\partial x_j}+Dp_n+D'p_{n-1}+D''p_{n-2}=0$$

If we equate to zero every homogeneous component of the left hand part, we get the system

$$2\frac{\partial p_{n+1}}{\partial x_2}=-r_{n-1},\ 3\frac{\partial p_{n+1}}{\partial x_3}=-r_{n-2},\ 4\frac{\partial p_{n+1}}{\partial x_4}=-r_{n-3},\ldots, \tag{40}$$

where $r_{n-1},r_{n-2},\ldots$ are polynomials of degree $n-1,n-2,\ldots$ defined by the equality

$$(e_{-1}+e_{-2}+\ldots)\sum_{m=0}^{n}p_m(x_2,\ldots,x_m)x_1^{\nu+q-m}=(r_{n-1}+r_{n-2}+\ldots)x_1^{\nu+q-n}+\ldots,$$

where the last "$+\ldots$" means terms with $x_1^{\nu+q-n'}$, $n'>n$.

If we already know $p_0,p_1,\ldots,p_n$, we have the system (40) for p_{n+1}. Uniqueness of solution is obvious (since p_{n+1} is homogeneous of degree $n+1$), to prove the existence, we need to confirm the equality

$$i\frac{\partial r_{n-(j-1)}}{\partial x_i}=j\frac{\partial r_{n-(i-1)}}{\partial x_j}.$$

But the part of degree $-i-j+2$ of $(e_{-(i-1)}-e_{-(j-1)}-(j-i))(e_{-1}+e_{-2}+\ldots)$ is $[e_{-(i-1)},e_{-(j-1)}]-(j-1)e_{-(i+j-2)}=0$. Therefore, the part of degree $n-i-j+2$ of $(e_{-(i-1)}-e_{-(j-1)}-(j-i))(r_{n-1}+r_{n-2}+\ldots)$ is zero, and this part is precisely $(i-1)\frac{\partial r_{n-(j-1)}}{\partial x_{i-1}}=(j-1)\frac{\partial r_{n-(i-1)}}{\partial x_{j-1}}$. This completes the p[roof of Proposition 5.29.

The final result is: The module $\widehat{\mathcal{H}}(\lambda,\mu,\nu)$ has a canonical singular vector of the form

$$x_1^{\nu}+p_2(\lambda,\mu,\nu,x_2)x_1^{\nu-2}+p_3(\lambda,\mu,\nu,x_2,x_3)x_1^{\nu-3}+\ldots \tag{41}$$

COROLLARY 5.30. *The module $\mathcal{H}(\lambda,\mu)$ of semi-infinite forms has a singular vector of degree n if and only if $p_j(\lambda,\mu,n;x_2,\dots,x_j)=0$ for $j>n$. This vector has the form* (41).

This shows that if $\Psi_{-k,-\ell}=0$, then $p_j(\lambda,\mu,k\ell)=0$ for $j>k\ell$.

The explicit expressions for the polynomials p_j are not known, but they can be calculated from the equations given above. We exhibit the first three of these polynomials.

$$\begin{aligned}
p_2(\lambda,\mu,\nu;x_2) &= -\frac{1}{2}\nu x_2,\\
p_3(\lambda,\mu,\nu;x_2,x_3) &= \frac{1}{3}\left(\alpha_1\alpha_2-\frac{\nu-1}{2}\right)\nu x_3,\\
p_4(\lambda,\mu,\nu;x_2,x_3,x_4) &= \frac{1}{16}((\nu-3)\alpha_1^2-\alpha_1\alpha_3+(\nu-1))\nu x_2^2\\
&\quad+\frac{1}{8}((\nu-3)\alpha_1+2(\nu-1)\alpha_2-2\alpha_1\alpha_2\alpha_3)\nu x_4
\end{aligned}$$

where $\alpha_s=\mu+(s-1)\left(\lambda+\dfrac{1}{2}\right)$. Notice that $\Psi_{-k-\ell}=\alpha_k\alpha_\ell-\dfrac{(k-\ell)^2}{2}$.

6. Exercises

6.1. Prove that if $\mathfrak{g}=\mathbb{C}$ with zero $[\,,\,]$, then $U(\mathfrak{g})\cong\mathbf{C}[x]$; if $\mathfrak{g}=\mathbb{C}^n$ with zero $[\,,\,]$, then $U(\mathfrak{g})\cong\mathbf{C}[x_1,\dots,x_n]$.

6.2. Prove that the functor $\mathcal{A}ss\to\mathcal{L}ie$, which assigns to an associative algebra the Lie algebra with the same space and the operation $[A,B]=AB-BA$ possesses a right adjoint, and this right adjoint assigns to a Lie algebra its universal enveloping algebra (see Section 1.1.4).

6.3. Prove that $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{a}}V$ is an initial object of the category of $\mathfrak{a}$-modules, which contain V as a $\mathfrak{b}$-submodule. Prove the "dual statement": $\mathrm{Coind}_{\mathfrak{a}}^{\mathfrak{b}}V$ is a terminal object of the category of $\mathfrak{a}$-modules equipped with a $\mathfrak{b}$-projection onto V.

6.4. (about the modules $K(\lambda,\mu)$ and $J(\alpha,\beta)$ from Section 1.2.3). (a) Check that the formulas for the actions of f,h,e given in the end of Section 1.2.3 provide valid descriptions of $\mathfrak{sl}(2)$-modules. (b) For which λ,μ,α,β are the modules $K(\lambda,\mu)$ and $J(\alpha,\beta)$ isomorphic? (c) If the module $J(\alpha,\beta)$ is reducible, then list all their (proper, irreducible) submodules.

6.5. Prove that the quotient of a Kac-Moody algebra $\mathfrak{g}(A)$ over its center $\mathfrak{c}$ is *graded simple* in the sense that it has no proper ideals I compatible with the n-grading, $I=\bigoplus_{k_1,\dots,k_n}I\cap\mathfrak{g}(A)_{k_1,\dots k_n}$. Still $\mathfrak{g}(A)/\mathfrak{c}$ is not necessarily *simple*, it may have proper ideals; show this on the example A_1^1.

6.6. Prove that for $A=\begin{bmatrix}2&-1\\-2&2\end{bmatrix}$, the Kac-Moody algebra $\mathfrak{g}(A)$ is isomorphic to $\mathfrak{so}(5)$ (see Section 2.2.2).

6.7. Prove that for $A=\begin{bmatrix}2&-1\\-3&2\end{bmatrix}$, the Kac-Moody algebra $\mathfrak{g}(A)$ is isomorphic to G_2 (see Section 2.2.3).

6.8 Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, and let $\mathfrak{c}$ be the centrer of the Kac-Moody algebra $\mathfrak{g}(A)/\mathfrak{c}$. Prove that the homomorphism $\mathfrak{g}(A)/\mathfrak{c} \to \mathfrak{sl}(2) \otimes \mathbb{C}[[t, t^{-1}]$ constructed in Section 2.2.4 is an isomorphism. *Hint*. It has been proved in Section 2.2.4 that this homomorphism is onto, so it is sufficient to prove that it has no kernel. But this kernel would have to be a must be a graded ideal of $\mathfrak{g}(A)/\mathfrak{c}$, while $\mathfrak{g}(A)/\mathfrak{c}$ has no proper graded ideals (Exercise 6.5).

6.9. Let $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. Prove that the description of the Lie algebra $\mathfrak{g}(A)/\mathfrak{c}$ given in Section 2.2.5 is valid (the homomorphism constructed there is an isomorphism). *Hint*: the same as in Exercise 6.8.

6.10. Prove that there are precisely 6 isomorphism classes of indecomposable affine Lie algebras of rank 3. (Cartan matrices of these affine Lie algebras are listed in Section 2.4)

6.11. Prove that if $0 \neq k_1h_1 + \ldots + k_nh_n \in \mathfrak{c}$ and all k_i are non-negative integers, then $k_1\alpha_1 + \ldots + k_n\alpha_n$ is a root with $\langle \alpha, \alpha \rangle = 0$ (see Remark in Section 3.2.2.2).

6.12. Prove that the elements $b_i,\ i = 1, 2, 3, \ldots$ of the Lie algebra $\mathfrak{n}_+(A_1^1)$ constructed in Section 3.4.2 form a basis of this algebra and prove that $[b_i, b_j] = c_{ij}b_{i+j}$ where $b_{ij} = 0$ or ± 1 and $b_{i,j} \equiv i - j \bmod 3$. Actually, the whole algebra $A_1^1/\mathfrak{c}$ is generated by elements b_i, $i \in \mathbb{Z}$ with the same relation.

6.13. Prove that the Weyl group takes roots into roots.

6.14. Describe the action of the Weyl group on the roots of A_2^2.

6.15. Prove that the $\mathfrak{sl}(3)$-module $M(\lambda)$ with $\lambda_1 = m - u - 1$, $\lambda_2 = u - 1$ ($\lambda_1 + \lambda_2 = m - 2$, see Section 3.2.2.1) contains the following singular vector:

$$\sum_{s=0}^{m} u(u-1)\ldots(u-s+1)E_{31}^{s}E_{32}^{m-s}E_{21}^{m-s};$$

here E_{ij} is a one-entry matrix with $a_{ij} = 1$.

6.16. Determine, which of the modules $\mathcal{F}_{\lambda\mu}$ are reducible. For the reducible modules $\mathcal{F}_{\lambda\mu}$ find all their proper submodules.

6.17. Let M be a $\mathfrak{Witt}$-module with the basis f_j, $j \in \mathbb{Z}$ and $e_if_j = a_{ij}f_{i+j}$ with some $a_{ij} \in \mathbb{C}$. Prove that with exception of some degenerate case (for example in the case when all a_{ij} are different from zero) M is isomorphic to $\mathcal{F}_{\lambda\mu}$ for some λ, μ (find them).

6.18. (It is rather a riddle than an exercise.) The set of equivalence classes of central extensions of a Lie algebra (see Section 4.2.1) possesses a structure of a complex vectors space. Guess, what this structure is, and prove that the correspondence between classes of extensions and the two-dimensional cohomology is actually a vector space isomorphism.

6.19. Prove that the cocycle $c(e_i, e_j) = \frac{1}{12}\delta_{-i,j}(j^3 - j)$ of the Witt algebra (see Section 4.2.2) is unique in the sense that every other cocycle is cohomologous to $\lambda c, \lambda \in \mathbb{C}$.

6.20. (We use the notations from Section 5.1.2.) Prove that if the operators $\widetilde{e}_i = e_i + a_iF_i$ $(i \neq 0), \widetilde{e}_0 = e_0 + b\mathbb{I}, z = c\mathbb{I}$ satisfy the Virasoro commutator relations $[\widetilde{e}_i, \widetilde{e}_j] =$

$(j-i)\widetilde{e}_{i+j}+\frac{1}{12}\delta_{i,-j}(j^3-j)z$, then $a_i=\alpha i+\beta$, $b=\frac{\beta^2-\alpha^2}{2}$, $c=1-12\alpha^2$ for some complex numbers α,β.

6.21. Recall that the *Schur polynomial* $s_\rho(x_1,\ldots,x_k)$ where $\rho=(r_1,\ldots,r_k)$, $r_1\geq r_2\ldots\geq r_k\geq 0$, $r_1+\ldots+r_k=k$ is a partition of k (we denote the set of all partitions of k as $\mathcal{P}(k)$) is defined by the formula

$$s_\rho(x_1,\ldots,x_k)=\frac{\det\begin{bmatrix} x_1^{r_k} & x_2^{r_k} & \ldots & x_k^{r_k}\\ x_1^{1+r_{k-1}} & x_2^{1+r_{k-1}} & \ldots & x_k^{1+r_{k-1}}\\ \vdots & \vdots & \ddots & \vdots\\ x_1^{k-1+r_1} & x_2^{k-1+r_1} & \ldots & x_k^{k-1+r_1}\end{bmatrix}}{\det\begin{bmatrix} 1 & 1 & \ldots & 1\\ x_1 & x_2 & \ldots & x_k\\ \vdots & \vdots & \ddots & \vdots\\ x_1^{k-1} & x_2^{k-1} & \ldots & x_k^{k-1}\end{bmatrix}}.$$

This is a symmetric polynomial of degree k, hence

$$s_\rho=\sum_{\tau\in\mathcal{P}(k)} a_{\rho\tau}e_\tau$$

where for $\tau=(t_1,\ldots,t_k)$, $e_\tau=e_{t_1}\ldots e_{t_k}$ is a monomial of the elementary symmetric polynomials.

Below, we use the notations from Section 5.1.3. Prove that in $\mathcal{H}$

$$F_\sigma(\ldots f_{-3}\wedge f_{-2}\wedge f_{-1})=\sum_{\tau\in\mathcal{P}(k)} a_{\rho\tau}f_\tau$$

where for $\rho=(r_1,\ldots,r_k),\tau=(t_1,\ldots,t_k)$, $F_\rho=F_{r_1}\ldots F_{r_k}$ (**here** we assume that $F_0=1$) and $f_\tau=\ldots\wedge f_{-k-1}\wedge f_{-k+r_k}\wedge\ldots\wedge f_{-1+r_1}$.

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