MODULAR INVARIANT q-DEFORMED NUMBERS: FIRST STEPS

VALENTIN OVSIENKO

The polynomial

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q},$$

where q is a parameter, is commonly considered as a "quantum", or a q-analogue of a (positive) integer n. The expression goes back to Euler ($\approx$1760) who used it in the context of combinatorics and q-series. Consequently, Gauss ($\approx$1808) defined and studied polynomials based on $[n]_q$, called q-binomials. Quantum integers are extensively used in combinatorics, algebra, analysis, and mathematical physics.

The notion of "quantum rational" based on modular, or $\mathrm{PSL}(2, \mathbb{Z})$-invariance, was introduced in [14], and that of "quantum irrationals" in [15]. Properties of these q-numbers have since been studied and related to various subjects, such as combinatorics of posets [13, 5, 6, 18, 17], Markov numbers and Markov-Hurwitz approximation theory [7, 8, 11, 21], enumerative geometry, Grassmannians and triangulations of annuli [19, 10], triangulated categories and homological algebra [1, 22].

The goal of this short review is to explain the main ideas of the emerging new theory.

What is a q-analogue? A "q-deformation" or "quantization" of a mathematical quantity is usually a function, often a polynomial, or a power series in q. In physics, $q = e^{\hbar}$, where $\hbar$ is the Planck constant, while in mathematics q is a parameter. When $q \to 1$, one obtains the initial quantity. For instance, the Gaussian q-binomial coefficients are defined by

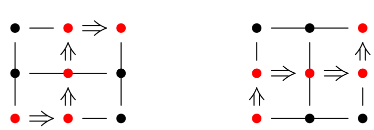
$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!},$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$. They are polynomials in q with positive integer coefficients.

A q-analogue must satisfy several requirements.

A. Given a sequence of integers that counts some objects, its q-analogue must count the same objects, but with more precision. For example, the binomial coefficient $\binom{n}{k}$ counts the number of *north-east lattice paths* (with no steps down or left) in the the $k \times (n-k)$-rectangle. The coefficient of q^m in the q-binomial $\binom{n}{k}_q$ is the number of such paths with exactly m boxes under the path.

Example 1. The polynomial $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$, counts lattice paths in a 2×2-square. There are $\binom{4}{2} = 6$ north-east lattice paths. Furthermore, there is exactly one such path with either $0, 1, 3,$ or 4 under boxes, and there are two paths with 2 under boxes:



This corresponds to the coefficients $1, 1, 2, 1, 1$ in the polynomial $\binom{4}{2}_q$.

In other words, a north-east lattice path is the boundary of a *Young diagram*, a fundamental notion of combinatorics and representation theory. The binomial $\binom{n}{k}$ thus counts the number of Young diagrams that fit into the $k \times (n-k)$-rectangle. The q-binomial $\binom{n}{k}_q$ also counts them, and the coefficient of q^m in this polynomial is the number of Young diagrams with m boxes.

B. As pointed out in the classical book [23], a q-analogue must count the number of points in a certain algebraic variety over the finite field $\mathbb{F}_q$. If $q = p^m$, where p is a prime integer, the Gaussian binomial $\binom{n}{k}_q$ is the number of points of the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{F}_q)$. This requirement is of a geometric nature.

C. A more analytic requirement: the same q-analogue appears in the so-called *q-calculus*. Assume that the non-commuting variables x and y satisfy the relation $yx = qxy$ of the "quantum plane"; the q-binomial theorem then states that

$$(x+y)^n = \sum_{0 \leqslant k \leqslant n} \binom{n}{k}_q x^k y^{n-k},$$

producing the q-binomials.

For mysterious reasons, exactly the same polynomials appear in three completely different situations! Let me quote Joseph Fourier: "Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them".

Why not $\frac{1-q^x}{1-q}$? When $x \in \mathbb{R}$ or $\mathbb{C}$ is not an integer, the expression $[x]_q := \frac{1-q^x}{1-q}$ is still the usual way of defining a q-analogue of x. An important property of this definition is the useful recurrence formula

$$[x+1]_q = q[x]_q + 1. \tag{1}$$

However, when x is rational this expression is not a rational function in q. Indeed, taking $x = \frac{n}{k}$, the above expression is the quotient

$$\frac{1 + q^{\frac{1}{k}} + q^{\frac{2}{k}} + \cdots + q^{\frac{n-1}{k}}}{1 + q^{\frac{1}{k}} + q^{\frac{2}{k}} + \cdots + q^{\frac{k-1}{k}}}.$$

Well, if we want a rational function in q, why don't we take $\left[\frac{n}{k}\right] := \frac{[n]_q}{[k]_q}$? Note that this is the above quotient with q replaced by q^k. But then we lose (1) and these expressions cannot have interesting properties analogous to the properties **A**, **B**, and **C** discussed above.

$\mathrm{PSL}(2,\mathbb{Z})$-invariant q-rationals. Given $x \in \mathbb{Q}$, is it possible to find a rational function with integer coefficients $[x]_q \in \mathbb{Z}(q)$ such that (1) still holds, as well as some analogs of the properties **A**, **B**, and **C**? The answer is "yes", and it is based on the geometric idea of invariance by a group action.

The set of rationals completed by one additional point, $\mathbb{Q} \cup \{\infty\}$, admits a transitive action of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ by fractional-linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d}, \qquad a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1. \tag{2}$$

Recall that $\mathrm{PSL}(2,\mathbb{Z})$ has two (standard) generators,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the translation $T(x) = x+1$, and inversion $S(x) = -\frac{1}{x}$. The only relations between T and S are: $S^2 = (TS)^3 = 1$. Whenever they are satisfied, T and S generate a $\mathrm{PSL}(2,\mathbb{Z})$-action.

It turns out that the group $\mathrm{PSL}(2,\mathbb{Z})$ also naturally acts on the field $\mathbb{Z}(q)$ of rational functions in q. We have the following theorem/definition.

Theorem 1 ([14, 9])**.** *(i) The matrices*

$$T_q := \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}, \qquad S_q := \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$$

generate a $\mathrm{PSL}(2,\mathbb{Z})$*-action on* $\mathbb{Z}(q)$ *by fractional-linear transformations* (2).

(ii) There exists a unique map $[\,]_q : \mathbb{Q} \to \mathbb{Z}(q)$ *commuting with the* $\mathrm{PSL}(2,\mathbb{Z})$*-action such that* $[0]_q = 0$.

Part (i) is immediate, one checks that the relations remain unchanged: $S_q^2 = (T_q S_q)^3 = 1$. Existence in part (ii) can be established by an explicit formula in terms of continued fractions; the uniqueness statement follows from the transitivity of the $\mathrm{PSL}(2,\mathbb{Z})$-action on $\mathbb{Q}$.

Note that if T_q is fixed then S_q is the unique matrix up to a scalar multiple satisfying these relations. Note also that, changing the parameter, $t = -q$, one obtains the classical *Burau representation* of the Braid group B_3; see [3] and references therein.

Theorem 1 can be reformulated as follows (see [9]): there exists a unique map $[\,]_q : \mathbb{Q} \to \mathbb{Z}(q)$ satisfying the recurrence (1) together with

$$\left[-\frac{1}{x}\right]_q = -\frac{1}{q[x]_q}. \tag{3}$$

The recurrence relations (1) and (3) suffice to calculate the image of any rational.

Example 2. Let us give several concrete examples that illustrate the general situation.

a) The quantization map $[\,]_q$ does not commute with arithmetic operations. Indeed,

$$\left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q}, \qquad \left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

give two different instances of "quantum 5". TUTORIAL VIDEO

b) Consider the classical Fibonacci numbers $(F_n) = 1, 1, 2, 3, 5, 8, 13, \ldots$ The consecutive quotients, $\frac{F_{n+1}}{F_n}$, lead to an interesting sequence of polynomials

$$\begin{array}{rcl} \left[\frac{8}{5}\right]_q &=& \dfrac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3}, \\[2ex] \left[\frac{13}{8}\right]_q &=& \dfrac{1+2q+3q^2+3q^3+3q^4+q^5}{1+2q+2q^2+2q^3+q^4}, \\[2ex] \left[\frac{21}{13}\right]_q &=& \dfrac{1+3q+4q^2+5q^3+4q^4+3q^5+q^6}{1+3q+3q^2+3q^3+2q^4+q^5}, \\ \cdots & \cdots & \cdots \end{array}$$

whose coefficients turn out to match the known triangular integer sequences A123245 and A079487 of OEIS.

Comment. The matrices T_q and S_q appeared in the literature separately. The action of T_q is equivalent to (1), and thus the matrix T_q is implicitly present already in q-integers (and everywhere in the q-calculus). The matrix S_q appeared in [24], under the name "spinor metric". It was also used in [2] for an elegant expression of quantum group relations. The connection to $\mathrm{PSL}(2,\mathbb{Z})$ was not used before [14].

The shape of polynomials. An important property is the following *total positivity*. Given a rational number $x = \frac{n}{k}$, its q-analogue is a quotient $[x]_q = \frac{N(q)}{K(q)}$ of two monic polynomials with positive integer coefficients, $N(q)$ and $K(q)$. This statement has the following stronger version.

Theorem 2 ([14])**.** *For every pair of rationals $\frac{n}{k} > \frac{n'}{k'}$ the polynomial $N(q)K'(q) - K(q)N'(q)$ has positive integer coefficients.*

In this sense, the quantization map $[\,]_q : \mathbb{Q} \to \mathbb{Z}(q)$ is order preserving.

The strongest result about the shape of the polynomials $N(q)$ and $K(q)$ is the following *unimodality property*. It was conjectured in [14], tackled by several authors [13, 5, 6], and eventually proved in [18].

Theorem 3 ([18])**.** *The sequences of coefficients of $N(q)$ and $K(q)$ are rank unimodal.*

This means that the sequence of coefficients of each polynomial form a "lonely mountain", with no oscillation. This is a very nice property because unimodal sequences are important in geometry and combinatorics. Recall that unimodality of Gaussian q-binomials is a celebrated theorem of Sylvester (1878), who solved Cayley's conjecture formulated over 20 years before.

We have seen in Example 2 that the polynomials $N(q)$ and $K(q)$ are not necessarily *palindromic*, i.e., symmetric with respect to the change $q \mapsto q^{-1}$ and renormalization. It turns out however, that this property holds for another notable family of polynomials. To every $A \in \mathrm{PSL}(2,\mathbb{Z})$ associate a matrix A_q with coefficients depending on q as follows. Express A in terms of T and S and replace T and S by T_q and S_q. The result does not depend on the expression.

Theorem 4 ([9])**.** *For every $A \in \mathrm{PSL}(2,\mathbb{Z})$ the trace polynomial $\mathrm{Tr}(A_q)$ is positive and palindromic.*

The role of the q-trace function on $\mathrm{PSL}(2,\mathbb{Z})$ is yet to be understood. So far, it was used to construct and study q-deformations of the Markov equation [7, 8, 9, 21]. A version of the unimodality theorem for $\mathrm{Tr}(A_q)$ is proved in [17].

Enumerative combinatorics of q-rationals. A similarity between q-rationals and q-binomials was observed in [14]. This similarity was reinforced in other works on the subject.

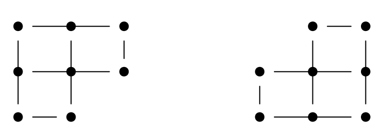
A. A combinatorial meaning of the coefficients of the polynomials $N(q)$ and $K(q)$ in a q-rational $\left[\frac{n}{k}\right]_q = \frac{N(q)}{K(q)}$ was suggested in [14] and beautifully reformulated in [19]. The statement is exactly the same as that in the case of q-binomials, but the rectangle is replaced by the so-called "snake graph".

Let $x = [a_1, \ldots, a_{2\ell}]$ be the standard continued fraction expansion of a rational $x \geqslant 1$. The *snake graph* associated with x is the collection of $a_1 + \cdots + a_{2\ell} - 1$ boxes in the square lattice, which the snake visites when crawling: $a_1 - 1$ steps up, a_2 steps to the right, a_3 steps up, etc., ending with $a_{2\ell} - 1$ steps to the right. As in the case of q-binomials, every north-east lattice path is the boundary of a Young diagram. But this time one only takes the paths with vertices in the snake graph.

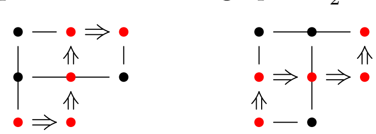
Theorem 5 ([14, 19])**.** *Given a rational $\frac{n}{k} \geqslant 1$ and its q-analogue $\left[\frac{n}{k}\right]_q = \frac{N(q)}{K(q)}$, the coefficient of q^m in $N(q)$ is the number of north-east lattice paths in the snake graph with m under boxes.*

The denominator $K(q)$ has a similar interpretation with a smaller snake graph.

Example 3. Consider again $\frac{5}{2} = [2,2]$ and $\frac{5}{3} = [1,1,1,1]$, the snake graphs of $\frac{5}{2}$ and $\frac{5}{3}$ are as follows



Counting north-east lattice paths in these snake graphs, one gets the polynomials from Example 2, a). For instance, there are exactly two paths in the snake graph of $\frac{5}{2}$ with one under box:



and this corresponds to the coefficient 2 in the numerator of $\left[\frac{5}{2}\right]_q$.

B. Nick Ovenhouse obtained the following beautiful geometric result which is also very similar to that of the q-binomials, when q is a power of a prime integer.

Theorem 6 ([19])**.** *$N(q)$ is the number of points in the collection of Schubert cells in the Grassmannian $\mathrm{Gr}_{m,\ell}(\mathbb{F}_q)$ with $m = a_1 + a_3 + \cdots + a_{2\ell-1}$ and $\ell = a_1 + a_2 + \cdots + a_{2\ell}$, that correspond to all of the Young diagrams that fit in the snake graph.*

Future work. Unimodal sequences appear in geometry and topology as Betti numbers, i.e., ranks of (co)homology groups of varieties and manifolds. Thanks to Poincaré duality, these sequences are often palindromic. Theorems 2–4 and 6 allow one to hope for existence of a cohomology theory in which the polynomials associated with q-rationals play the role of Hilbert-Poincaré polynomials. Bearing in mind the general principle of Deligne that counting over finite fields can be equivalent to computing cohomology over $\mathbb{C}$, Theorem 6 may be the key to answering this question.

Possible applications of q-rationals. Not much is known about it, but work has begun.

Relation of q-rationals to knot theory is very promising. According to John Conway, a rational $\frac{n}{k}$ parametrizes a certain class of "rational knots". Let $J_{\frac{n}{k}}(q)$ be its (normalized) Jones polynomial.

Theorem 7 ([14])**.** *The Jones polynomial of a rational knot can be expressed in terms of the numerator and denominator of the corresponding q-rational:* $J_{\frac{n}{k}}(q) = qN(q) + (1-q)K(q)$.

In [4] a somewhat "converse" approach is tasted: the Alexander polynomial was used to study q-rationals.

In [1] the authors study compactifications of the space of Bridgeland stability conditions of triangulated categories. q-rationals appear as braid group orbits in the boundary.

The fact that the representation (2) becomes the Burau representation of S_3 with $t = -q$ suggests that the results about q-rationals could be applied to study the Burau representation, which is an interesting subject of research; see [3].

A version of the q-binomial theorem is proved in the very recent preprint [12]. The main ingredient is the new notion of q-binomials $\binom{x}{k}_q$ for non-integer x. This work is a first step towards a $\mathrm{PSL}(2,\mathbb{Z})$-invariant q-calculus. A new version of q-Gamma function described in [12] deserves further study.

Stabilization of Taylor series: q-analogues of real numbers. Let $x \geqslant 1$ be an *irrational* number. Choose a sequence of rationals $(x_n)_{n\geqslant 1}$ converging to x and consider the corresponding sequence of q-rationals $[x_1]_q, [x_2]_q, \ldots$ Can we say that this sequence of rational functions converges in any sense?

Consider the Taylor expansions of the rational function $[x_n]_q$ at $q=0$, that by abusing of notation, will be dented by the same symbol $[x_n]_q = \sum_{k\geqslant 0} \varkappa_{n,k}\, q^k$. The following *stabilization phenomenon* is one of the most surprising properties of q-rationals.

Theorem 8 ([15])**.** *For every $k \geqslant 0$, the coefficients $\varkappa_{n,k}$ of the Taylor series of the functions $[x_n]_q$ stabilize, as n grows, and the limit coefficients, $\varkappa_k := \lim_{n\to\infty} \varkappa_{n,k}$, depend only on x.*

Note also that the recurrence (1) allows one to include the case $x < 1$, getting a Laurent series in q.

To illustrate this stabilization, consider once again the consecutive Fibonacci quotients that approximate the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2}$. The examples of the Taylor series are as follows

$$\begin{aligned}
\left[\tfrac{8}{5}\right]_q &= 1+q^2-q^3+2q^4-4q^5+7q^6-12q^7+21q^8-37q^9+65q^{10}-114q^{11}+200q^{12}\cdots\\
\left[\tfrac{21}{13}\right]_q &= 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+36q^8-75q^9+156q^{10}-325q^{11}+677q^{12}\cdots\\
\left[\tfrac{55}{34}\right]_q &= 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8-82q^9+184q^{10}-414q^{11}+932q^{12}\cdots
\end{aligned}$$

More and more terms coincide, and the series eventually stabilizes to

$$\begin{aligned}
[\varphi]_q &= 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8-82q^9+185q^{10}\\
&\quad -423q^{11}+978q^{12}-2283q^{13}+5373q^{14}-12735q^{15}+30372q^{16}\cdots
\end{aligned}$$

Quite remarkably, the sequence of coefficients in $[\varphi]_q$ coincides (up to an alternating sign) with the sequence A004148 of OEIS called the "generalized Catalan numbers". In general, the combinatorial meaning of the coefficients of the series $[x]_q$ is not understood.

Theorem 8 was first discovered by computer experimentation, and the stabilization phenomenon seemed like a miracle at first sight. The proof is based on the total positivity property (Theorem 2).

Analytic properties. Although we understand the stabilized Taylor series $[x]_q$ as a q-analogue of x, it is impossible to recover x from it substituting $q = 1$, as the series diverges at $q = 1$. It can be proved however that $x \neq x'$ implies $[x]_q \neq [x']_q$, so the power series $[x]_q$ contains the "full information" about x.

Assume $q \in \mathbb{C}$. The radius of convergence of the series $[x]_q$ was studied in [11, 21]. The main conjecture of [11] states that for every real $x > 0$ the radius of convergence of the series $[x]_q$ is greater or equal to $\frac{3-\sqrt{5}}{2} = 0.381966...$, and that equality holds only for x which is $\mathrm{PSL}(2,\mathbb{Z})$-equivalent to φ. This statement, provided it is true, is an analogue of the famous result due to Hurwitz claiming that the golden ratio is the most irrational number. A much weaker statement is proved.

Theorem 9 ([11])**.** *For every rational number $x > 0$ the radius of convergence of the series $[x]_q$ is at least* $3 - 2\sqrt{2} = 0.171572...$

This means that the polynomial $K(q)$ in $[x]_q = \frac{N(q)}{K(q)}$ has no zeros in the disc with this radius around 0.

Quadratic irrationals. When $x \in \mathbb{R}$ is a solution to a quadratic equation $ax^2 + bx + c = 0$ with integer coefficients a, b, c, it is called a quadratic irrational. This is the simplest class of irrationals, in this case $x = \frac{m+\sqrt{r}}{k}$, with integer m, k and $r \geqslant 0$. Elements of the theory of quantized quadratic irrationals were developed in [9].

Every quadratic irrational is a fixed point of an element of $\mathrm{PSL}(2,\mathbb{Z})$ under the action (2). Thanks to the $\mathrm{PSL}(2,\mathbb{Z})$-invariance, this property commutes with quantization.

Theorem 10 ([9])**.** *If x is a fixed point of $A \in \mathrm{PSL}(2,\mathbb{Z})$ then $[x]_q$ is a fixed point of A_q.*

It follows that $[x]_q = \frac{M(q)+\sqrt{R(q)}}{K(q)}$, where $M, K, R \in \mathbb{Z}[q]$ and, by Theorem 4, $R(q)$ is a palindrome.

Example 4 ([15, 9])**.** The series $[\varphi]_q$ and $\left[\sqrt{2}\right]_q$ are the Taylor series of the following functions

$$
\begin{aligned}
[\varphi]_q &= \frac{q^2 + q - 1 + \sqrt{(q^2 + 3q + 1)(q^2 - q + 1)}}{2q}, \\
\left[\sqrt{2}\right]_q &= \frac{q^3 - 1 + \sqrt{(q^4 + q^3 + 4q^2 + q + 1)(q^2 - q + 1)}}{2q^2}.
\end{aligned}
$$

What is the reason for the appearance of the "invisible" term $q^3 - 1$ that disappears when $q = 1$, and of $q^2 - q + 1$ which is a "quantum instance" of 1? These questions are unanswered.

The radius of convergence, R_x, of the series $[x]_q$ for a quadratic irrational x is the modulus of the smallest root of $R(q)$. For the above examples we get

$$
R_\varphi = \frac{3-\sqrt{5}}{2} \qquad \text{and} \qquad R_{\sqrt{2}} = \frac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2}.
$$

One last enigma. The theory of quantum numbers is still a baby making very first steps, and every step raises new questions. To end this review, I will mention one of them.

The Taylor series of $[\pi]_q$ starts as follows

$$
\begin{aligned}
[\pi]_q = \; & 1 + q + q^2 + q^{10} - q^{12} - q^{13} + q^{15} + q^{16} - q^{20} - 2q^{21} - q^{22} + 2q^{23} + 4q^{24} + q^{25} \\
& -4q^{27} - 4q^{28} - 2q^{29} + q^{30} + 5q^{31} + 8q^{32} + 3q^{33} - 3q^{34} - 10q^{35} - 12q^{36} - 5q^{37} \\
& +8q^{38} + 19q^{39} + 20q^{40} + 2q^{41} - 18q^{42} - 32q^{43} - 25q^{44} + 31q^{46} + 51q^{47} + 45q^{48} - 7q^{49} \cdots
\end{aligned}
$$

(observe, the coefficient of q^{45} vanishes for unknown reasons). This is the slowest growing series among other q-numbers that I know. It's tempting to conjecture that the radius of convergence of $[\pi]_q$ is 1. Does $[\pi]_q$ satisfy any equation? I calculated the first 666 terms trying to answer this question, but failed.

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References

[1] Bapat, A., Becker, L., and Licata, A. q-deformed rational numbers and the 2-Calabi–Yau category of type A_2. arXiv:2202.07613.
[2] Bernstein, J., Khovanova, T. *On the quantum group* $\mathrm{SL}_q(2)$, Comm. Math. Phys. 177 (1996), 691–708.
[3] Bharathram, V., Birman, J. On the Burau representation of B_3, arXiv:2208.12378.
[4] Bazier-Matte, V. and Schiffler, R. Knot theory and cluster algebras. *Adv. Math. 408* (2022).
[5] Elizalde, S., Plante, M., Roby, T., Sagan, B. Rowmotion on fences. *Sém. Lothar. Comb. 86* (2022).
[6] Elizalde, S. and Sagan, B. Partial rank symmetry of distributive lattices for fences. arXiv:2201.03044.
[7] Kogiso, T. q-Deformations and t-deformations of Markov triples. arXiv:2008.12913.
[8] Labbé, S., and Lapointe, M. The q-analog of the Markoff injectivity conjecture over the language of a balanced sequence. *Comb. Theory 2*, 1 (2022).
[9] Leclere, L., and Morier-Genoud, S. q-deformations in the modular group and of the real quadratic irrational numbers. *Adv. Appl. Math. 130* (2021).
[10] Leclere, L., and Morier-Genoud, S. Quantum continuants, quantum rotundus and triangulations of annuli. arXiv:2207.08906 .
[11] Leclere, L., Morier-Genoud, S., Ovsienko, V., and Veselov, A. On radius of convergence of q-deformed real numbers. arXiv:2102.00891.
[12] Machacek, J. and Ovenhouse, N. q-Rational and q-Real Binomial Coefficients. arXiv:2301.08185.
[13] McConville, T., Sagan, B. E., and Smyth, C. On a rank-unimodality conjecture of Morier-Genoud and Ovsienko. *Discrete Math. 344*, 8 (2021).
[14] Morier-Genoud, S., and Ovsienko, V. q-deformed rationals and q-continued fractions. *Forum Math. Sigma 8* (2020), e13, 55 pp.
[15] Morier-Genoud, S., and Ovsienko, V. On q-deformed real numbers. *Exp. Math. 31 (2022), 652–660.*
[16] Morier-Genoud, S., and Ovsienko, V. Quantum numbers and q-deformed Conway-Coxeter Friezes. *Math. Intelligencer 43 (2021), 61–70.*
[17] Oguz, E. K. Oriented posets and rank matrices. arXiv:2206.05517.
[18] Oguz, E. K., and Ravichandran, M. Rank polynomials of fence posets are unimodal. arXiv:2112.00518.
[19] Ovenhouse, N. q-rationals and finite Schubert varieties. arXiv:2111.07912.
[20] Ovsienko, V. Towards quantized complex numbers: q-deformed gaussian integers and the Picard group. *Open Communications in Nonlinear Mathematical Physics 1*, https://doi.org/10.46298 (2021).
[21] Ren, X. On radiuses of convergence of q-metallic numbers and related q-rational numbers. *Res. Number Theory 8* (2022).
[22] Ren, X. On q-deformed Farey sum and a homological interpretation of q-deformed real quadratic irrational numbers. arXiv:2210.06056.
[23] Stanley, R. Enumerative combinatorics. Volume 1. Second edition. Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, Cambridge, 2012. xiv+626 pp.
[24] Vokos, S.P., Zumino, B. Wess, J. *Analysis of the basic matrix representation of* $\mathrm{GL}_q(2,\mathbb{C})$, Z. Phys. C **48** (1990), no. 1, 65–74.