Aperiodic Monotiles

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"A longstanding open problem asks for ... a shape that admits tilings of the plane, but never periodic tilings. We answer this problem for topological disk tiles by exhibiting a continuum of combinatorially equivalent aperiodic polygons."

Thus begins the abstract of the preprint "An Aperiodic Monotile," first posted on arxiv.org on March 20, 2023 by David Smith, Joseph Myers, Craig Kaplan, and Chaim Goodman-Strauss ([1]). Propelled by social media, lively international Zoom gatherings, on-line science magazines, and *The New York Times*, the news streaked around the globe at (nearly) the speed of light.

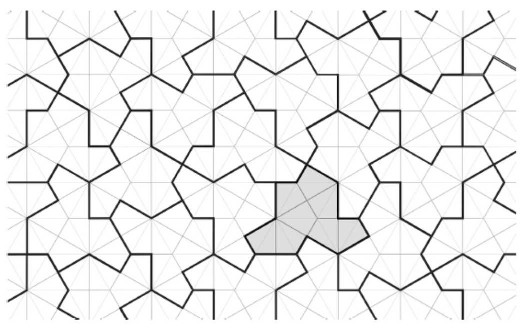

Figure 1: A portion of the aperiodic hat tiling. (Figure 1.1 in [1])

"Longstanding" looks back to Hao Wang's 1961 conjecture that any set of marked squares that can fill the plane without rotating or reflecting them can also do so periodically. Five years later, Wang's student Robert Berger disproved the conjecture with a set of 20,426 tiles. Berger himself later reduced that number, and others followed; it's now known that the minimum number is eleven ([2]), see Figure 2.

By 1972, Roger Penrose and others had found sets of two aperiodic tiles. The Penrose tilings were widely celebrated and widely studied, but left open the question of an aperiodic monotile. As Ludwig Danzer joked in hybrid German, is there *ein stein* in 2, 3, or any number of dimensions? Now, half a century after Penrose's first doodles ([3]), the einstein problem made headlines with a thirteen-sided polygon that the authors dubbed "the hat", see Figure 3.

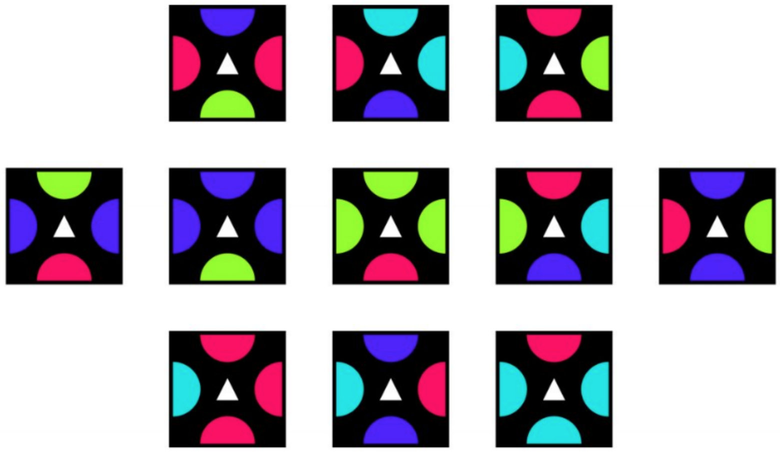

Figure 2: Eleven aperiodic Wang tiles (Figure 4 in [2])

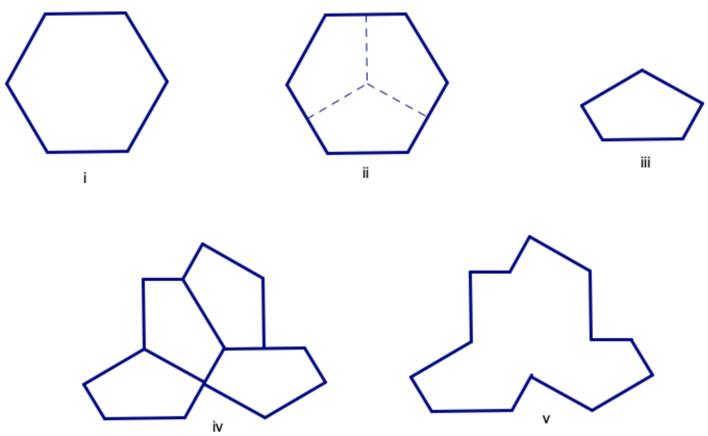

Figure 3: The hat tile is a union of eight congruent kites, each a sixth of a regular hexagon. If the edge lengths of the hexagons are 2, then the edges of the hat are 1 and $\sqrt{3}$ (we interpret the hat's long edge as two collinear edges of length 1.)

The hat was the third einstein to be proffered: a biprism (due to Martin Schmitt, John Conway, and Ludwig Danzer) was a focus of a Regional Geometry Institute held at Smith College in 1993 (see [5]), and Joshua Socolar and Joan Taylor proposed another in 2011 ([4]). These solutions sharpened rather than resolved the question. The biprism tiles in periodic layers rotated through an irrational angle, and the Socolar-Taylor tile is disconnected. These features, tilers agreed *ex post facto*, show that these tiles "are not what we are looking for." The hat, by contrast, is a simple polygon sitting in, or on, a tiling by regular hexagons, see Figure 1. As co-author Goodman-Strauss told *ScienceNews* ([6]), "Before this work, if youd asked what an einstein would look like, I wouldve drawn some crazy, squiggly, nasty thing.

Moreover, as Smith et al. showed, the hat belongs to a continuous family of aperiodic monotiles Tile(a, b), obtained from it by replacing its sides of length 1 with sides of length a and those of length $\sqrt{3}$ with sides of length b, where a and b are nonnegative real numbers, not both zero. (The hat itself is thus Tile(1, $\sqrt{3}$).) Morevover, all tilings by tile T(a, b) are combinatorially equivalent to those of the hat, see Figure 4.

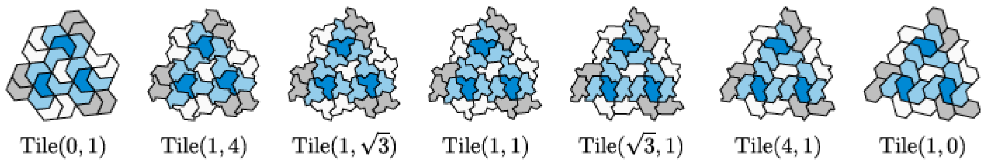

Figure 4: (Fig 2.3 of [1]). "The two edge lengths in the hat polykite can be manipulated independently, producing a continuum of shapes. A selection of those shapes is shown here, normalized for scale. Tile(0,1), Tile(1,1) and Tile(1,0) admit periodic tilings; all others are aperiodic."

To prove that the hat is aperiodic, Smith et al. first proved that it tiles the plane [1]. Their argument, a long-standing cornerstone of the aperiodic tiling theory, rests on the observation that hats can be grouped, uniquely, into clusters, and the clusters into larger clusters, ad infinitum. The cluster construction is subtle (in the hat case) because the hats group not into larger hats but into "metatiles", and the metatiles at the different levels are not precisely similar (for details, see the interactive browser-based visualization tool at `https://cs.uwaterloo.ca/~csk/hat/`). "We are not aware of other substitution systems that use rules like these," the authors note, "where successive generations are combinatorially but not geometrically compatible."

The authors gave two proofs, both by contradiction, that all hat tilings are nonperiodic. One rests on their hierarchical structure. Assume that a hat tiling T is periodic and let v be one of its translation vectors. Then $T + v = T$; that is, v maps T, including its hierarchical structure, onto itself. But, because the areas of the metatiles increase without bound as we move up the hierarchy, at a sufficiently high level v will lie entirely in a metatile's interior. Thus v does not map T to itself; i.e., T admits no translations.

Their second proof rests on the continuum discussed above. If a hat tiling T were periodic, then all the tilings in the continuum would be periodic too. In particular, the translation lattices of tilings with Tile(0,1) and with Tile(1,0) would be hexagonal, and they would be similar. But there is no similarlity mapping between these lattices that has the required scale factor $\sqrt{2/3}$.

"Elusive 'Einstein' solves a Longstanding Math Problem" announced the New York Times on March 28, 2023, and mathematicians around the world rejoiced. Yet some argued that the hat isn't what we're looking for either, because all its tilings include its mirror images too. One can argue that this is a matter of taste: the classic, encyclopedic textbook *Tilings and Patterns* ([7]) explicitly includes mirror images in its definition of "monohedral," and no one objected to rotated hats, though Wang had excluded rotations. Nevertheless Smith et al. took up the challenge and found that the equilateral Tile(a,a) is "weakly chiral," in the sense that all its tilings are nonperiodic if only rotated copies are allowed. Then, by suitably curving its edges, they modified Tile(a,a) into one that cannot be fitted to its mirror image. With this, the qualifier “weakly” becomes “strongly” and the tile becomes a “spectre”, see Figure 5.

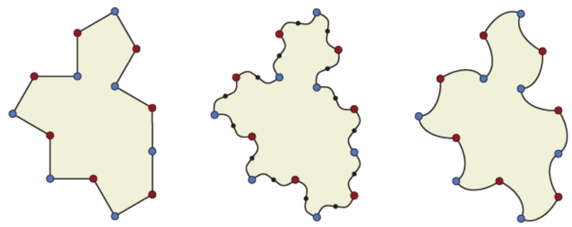

Figure 5: (Fig 1.1 of [8]). “The 14-sided polygon T(1,1), shown on the left, is a weakly chiral aperiodic monotile : if by fiat we forbid tilings that mix unreflected and reflected tiles, then it admits only non-periodic tilings. By modifying its edges, as shown in the centre and right for example, we obtain strictly chiral aperiodic monotiles called ‘Spectres’ that admit only non-periodic tilings even when reflections are permitted.”

Illustrators jumped right in, Figure 6.

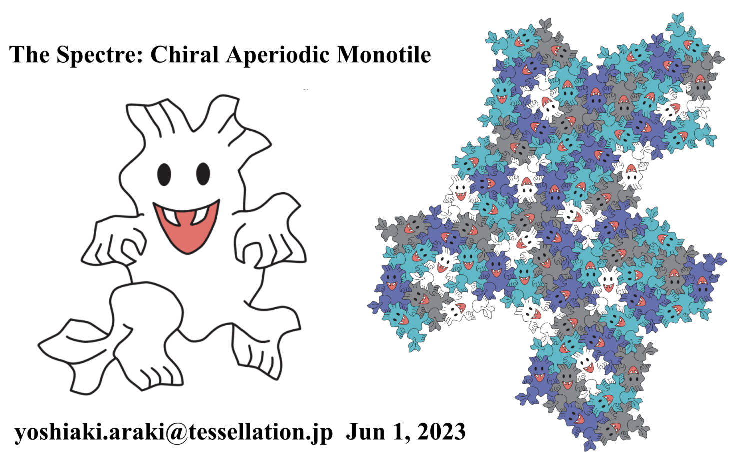

Figure 6: A spectre created by Yoshiaki Araki, shown here with his permission.

In addition to headlines, the hat and the spectre are generating a flood of papers on arXiv, Twitter, and listserves situating their tilings in the mathematical framework constructed for Penrose tilings and others. For example, we have already learned that the hat tilings can be constructed by the cut-and-project method (see, for example, [5]), that their diffraction spectrum is pure-point ([9] and [10]), see Figure 7, and that “worms” wriggle through them ([9]), see Figure 8.

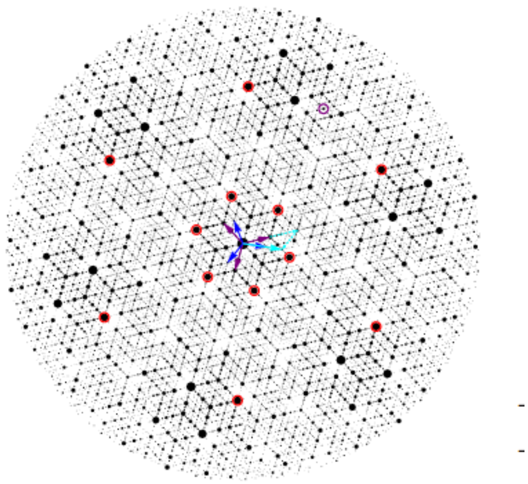

Figure 7: (Figure 9 of [9]). The diffraction pattern of unit masses placed at the vertices of a "key" tiling closely related to the hats.

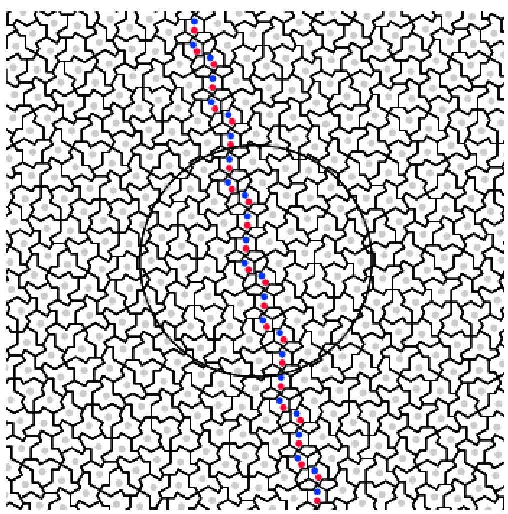

Figure 8: Figure 16 of [9]. A "worm" in a portion of a hat tiling.

And it all began less than three months ago! I expect this burgeoning research to lead to a richer understanding of the nature of order and of order in nature.

References

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