The Optimal Paper Moebius Band. arXiv:2308.12641
The Optimal Twisted Paper Cylinder. aarXiv:2309.14033
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Everyone knows how to make a Möbius band out of a paper rectangle: give it a 180° twist and attach the ends to each other. It is easy to do if the rectangle is long and narrow, but it is impossible if the ratio of the length to width is sufficiently small (say, equal to 1). Thus there exists a number λ such that if this ratio is greater than λ, a paper Möbius band can be made, and if it is smaller than λ, then a paper Möbius band does not exist.

To be precise, one is concerned with an isometric embedding of a flat Möbius band in 3-dimensional Euclidean space, the flat Möbius band being $[0,\ell] \times [0,1]$, factorized by $(0,y) \sim (\ell, 1-y)$.

This problem were studied in [3] (the question goes back to at least [6]), and the result obtained there was that

$$\frac{\pi}{2} \le \lambda \le \sqrt{3},$$

and that, conjecturally, $\lambda = \sqrt{3}$. Furthermore, if one replaces "isometric embedding" by "isometric immersion", then the optimal ratio equals $\pi/2$.

An expository account of these results can be found in Lecture 14 of [FT], and we use some figures from this book. Figure 1 shows that if the smoothness condition is abandoned, one can make a paper Möbius band from a rectangle with an arbitrary aspect ratio.

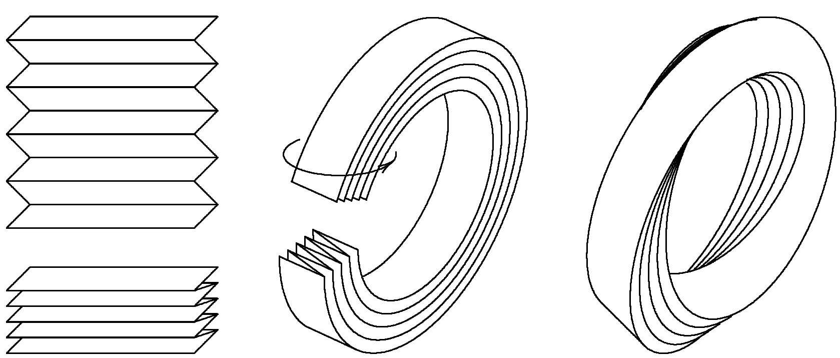

Figure 1

Figure 2 shows how to make an immersed "paper" (that is, developable) Möbius band from a rectangle with the aspect ratio greater than $\pi/2$.

And Figure 3 shows a paper Möbius band made of a rectangle with the aspect ratio greater than $\sqrt{3}$ (to maintain smoothness of these constructions, one needs to smooth the folds).

A breakthrough in this problem was recently make by Richard Schwartz who proved that, indeed, $\lambda = \sqrt{3}$, see [4]. This is a rare case when a good mathematical result could be explained to the general public: see [1]. We present the sketch of the proof below.

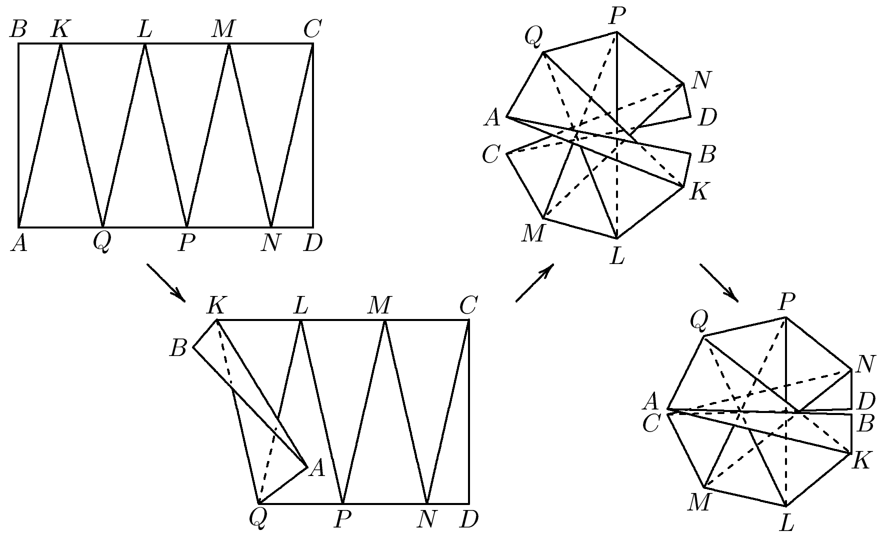

Figure 2

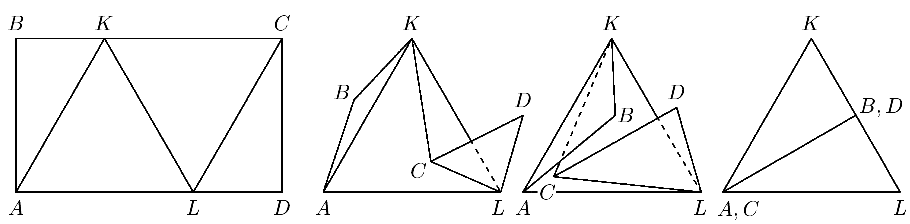

Figure 3

To start with, one assumes that paper is not stretchable, and one uses the classification of developable surfaces that goes back to Euler. Developable surfaces are ruled, and every point lies on a unique ruling, unless a neighborhood of the point is planar. See Figure 4. The lines containing the rulings of a generic non-planar developable surface are tangent to a space curve; the exceptional cases include cylinders and cones over space curves.

It follows that a paper Möbius band is ruled by segments of lengths at least 1 (we assume that the width of the rectangle equals 1). See Figure 5 for an infinite development of the band in the plane: this picture has a glide symmetry with the horizontal shift ℓ.

In particular, traversing the Möbius band, the line containing the ruling returns to the initial position, but with the reversed orientation. The first main lemma of the paper asserts that, in this continuous motion, there are two moments when the line intersects itself at the right angle. This topological result has a flavor of the Borsuk-Ulam theorem and, indeed, it can be deduced from it. The so inclined readers might try to prove this lemma themselves. Hint: define two functions on the space of pairs of distinct oriented lines, encoding the angle and the signed distance between them, whose simultaneous vanishing gives the desired intersecting lines.

Consider two rulings whose extensions intersect at the right angle. Cut the paper Möbius band along one of them and flatten it: one obtains an isosceles trapezoid whose lateral sides are two copies of this ruling. In space, these rulings are coplanar disjoint perpendicular segments, each of length at least 1. The boundary of the Möbius band is a closed curve that passes, alternating, through the end points of these segments. The second key lemma is a geometrical inequality that gives a lower bound of $\sqrt{3}$ to the length of such curve.

Another result of the paper says, informally, that if the aspect ratio is close to $\sqrt{3}$, then the respective paper Möbius band looks like the triangular example in Figure 3.

One can view a talk by Richard Schwartz on his result given on 09/27/2023 at the online Penn

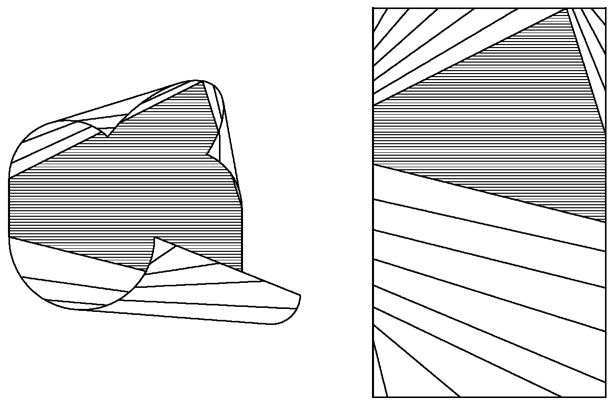

Figure 4

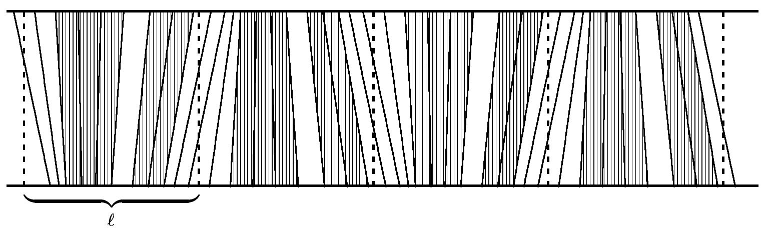

Figure 5

State Geometry Seminar `https://www.youtube.com/watch?v=Dy5ELJU-Zxg`.

The second paper by Schwartz tackles a similar problem: what is the optimal aspect ratio of a paper rectangle that is twisted twice before the ends are pasted together? The resulting surface is a developable twisted cylinder and, as with Möbius band, one has two versions of the question: embedding vs. immersion. Surprisingly, both questions have the same answer: the optimal number $\lambda = 2$. This is the main result of [5].

The proof of the necessity again consists of two lemmas: a topological statement and a geometrical inequality. Here is a sketch.

The topological lemma asserts that there exist two rulings whose endpoints divide both boundary components of the twisted cylinder into arcs of equal lengths (Schwartz calls them a balanced pair). This follows from the 1-dimensional version of the Borsuk-Ulam theorem, that is, from the intermediate value theorem. We take the liberty to leave it as an exercise to the reader.

Consider a balanced pair of rulings. They lie in parallel planes, say, horizontal ones. Their projections on the horizontal plane still have lengths at least 1. And since the two boundary components of the twisted cylinder are linked, their projections on the horizontal plane are intersecting curves. As before, these curves connect the end points of the projectied pair of balanced rulings, and the desired result follows from another planar geometric inequality.

Figure 6, borrowed from [5], shows how to fold a 1×2 rectangle to make a twisted cylinder (the ratio $\lambda > 2$, then the fold lines can be replaced by smooth narrow cylinders yielding a smooth developable surface).

Another result of [5] says, informally, that if the aspect ratio is close to 2, then the respective twisted cylinder looks like the one in Figure 6. This is an analog of a similar result concerning the Möbius band.

One naturally wonders what is the optimal aspect ratio of a paper rectangle whose ends are glued together after it is twisted n times (the Möbius band being the case of $n = 1$).

Another problem, also studied in [3], concerns the optimal ratio, say μ, of the perimeter to the

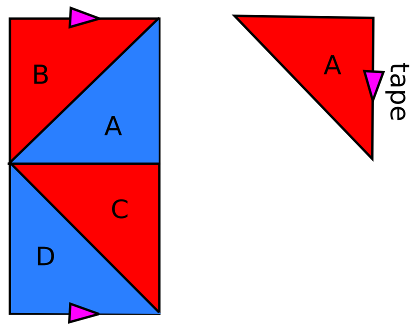

Figure 6

height of a paper cylinder that can be smoothly turned inside out. It is proved in [3] that

$$\pi \leq \mu \leq \pi + 2,$$

but the exact value is not known; conjecturally, it is $\pi + 2$.

References

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[6] W.Wunderlich, Über ein abwickelbares Möbiusband. Monatsh. Math. 66 (1962), 276–289.