

JSJ DECOMPOSITIONS OF *POINCARÉ* DUALITY PAIRS



LAWRENCE REEVES, PETER SCOTT
AND GADDE A. SWARUP



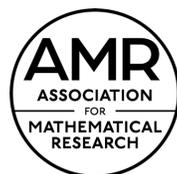
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JSJ Decompositions of Poincaré Duality Pairs

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Peter Scott and Gadde A. Swarup

Dedicated to Peter Scott
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Contents

1 Introduction	3
1.1 Introduction	3
2 Canonical decompositions for Poincaré duality pairs	11
2.1 Preliminaries	15
2.2 Poincaré duality pairs and essential annuli and tori	29
2.3 The main theorem	48
2.4 Torus Decompositions for $PD(n + 2)$ Groups and Pairs	56
2.5 Further properties of Torus Decompositions	65
2.6 Enclosing properties of Annulus-Torus Decompositions	75
2.7 Proof of the Main Theorem	88
2.8 Comparing the decompositions of a $PD(n + 2)$ pair and its double	111
2.9 Concluding remarks and problems	125
3 Comparing decompositions of Poincaré duality pairs	133
3.1 Introduction	133
3.2 Preliminaries	136
3.3 Examples of almost invariant sets	144
3.4 Proof of the main result	152
3.5 Comparisons	160
3.6 Results in dimension 3	173
3.7 Some related questions	178
4 A deformation theorem for Poincaré duality pairs in dimension 3	183
4.1 Introduction	183
4.2 Proof of the Main result	184
4.3 Discussion of Johansson's Deformation Theorem	196
Appendix	201

Chapter 1

Introduction

1.1 Introduction

The origins of this work go back to the seminal theorem of John Stallings on groups with infinitely many ends [St], which is an analogue of the Sphere Theorem for 3-manifolds. This suggested that there might be group theoretic analogues for some of the structural theorems from the theory of 3-manifolds. Soon after came an announcement by F. Waldhausen [Wa] which proposed annulus and torus theorems for orientable 3-manifolds and a structure theorem for Haken manifolds with incompressible boundary. This structure theorem gives a canonical decomposition of a Haken manifold with incompressible boundary into ‘rigid’ pieces and ‘known’ pieces. The rigid pieces are essentially hyperbolic. The known pieces are finitely covered by either a surface cross an interval (I -bundles) or a surface cross a circle (Seifert fibred). The union of known pieces is called the ‘characteristic submanifold’. Such decompositions were accomplished by Jaco and Shalen [JS] and Johannson [J]. These decompositions are now referred to as *JSh decompositions*. In addition, Johannson had some striking theorems such as a Haken 3-manifold with incompressible boundary and no essential annuli is determined up to homeomorphism by its fundamental group. Further, he showed that there are only finitely many Haken manifolds (up to homeomorphism) having a given fundamental group.

Our attempts in this direction started with an analogue of the annulus theorem for torsion-free hyperbolic groups [SS4]. Soon after Dunwoody and Swenson proved a complete analogue of the torus theorem for finitely generated groups [DS]. We briefly describe this theorem. Here, a torus group means a virtually

polycyclic group of some length n ($VPC(n)$). One considers the so-called almost invariant sets over such subgroups. Almost invariants over a subgroup H in G are the analogues of an immersion in the group G . The torus theorem asserts that if G has no almost invariant sets over VPC groups of length less than $n > 0$ and if G has almost invariant sets over a $VPC(n)$ groups then G splits over some $VPC(n)$ group. The case in which $n = 0$ is the famous theorem of Stallings. Their method also gives one part of a JSJ decomposition for groups, namely the I -bundle part. We used their techniques to obtain an analogue of the two-stage JSJ decompositions for finitely presented groups [SS2]. (See [S] for corrections. This set of corrections has been used by several people who used the results of [SS2]. It is added as an appendix to this paper.)

In this paper one considers almost invariant sets (a.i. sets) over $VPC(i)$ groups in an (almost) finitely presented group G . One considers a.i. sets over $VPC(n)$ groups (which will be called annuli) and a.i. sets over $VPC(n + 1)$ groups (which will be called tori) that do not cross the annuli. It is assumed that $n \geq 1$ and that G does not have any a.i. sets over $VPC(i)$ groups for $i < n$. The main theorem 13.12 of [SS2] shows that there is a bipartite graph of groups decomposition $\Gamma_{n,n+1}(G)$. One type of vertices, termed V_0 -vertices, correspond to fibred spaces in the 3-manifold case. The second type of vertices are termed V_1 -vertices. However, some of the V_0 -vertices have no fibering structure, they only commensurise some $VPC(n)$ or $VPC(n + 1)$ groups. We also showed that the arguments do not extend to three successive VPC groups. There is also another approach due to Guirardel and Levitt for one stage splittings [GL1]. The complete correspondence with the JSJ decomposition is obtained only for Poincaré duality pairs (PD pairs) which is the subject of the three papers in these notes.

A natural question is whether one can achieve the results of Theorem 13.12 from [SS2] by considering annuli first and tori second. This leads to the notion of an adapted a.i. set and carrying out the analogous work is difficult. This is carried out in another paper [GSS] which proves a relative torus theorem and also has another approach to regular neighbourhoods in group theory. We mention this in passing since Peter considered this a useful concept with other possible applications.

From the beginning Peter Scott's view was that JSJ decompositions can be viewed as a regular neighbourhood of cross-connected components of immersed annuli and tori. Here cross-connected component means that there is a chain of annuli (or tori) where two successive elements can not be homotoped apart.

In the following notes we show that the exact analogues of JSJ decompositions holds for Poincaré duality pairs.

We now briefly mention some other work in which the term JSJ decomposition is used. The first of these is the work of Rips and Sela [RS] who considered splittings of one ended torsion-free groups over infinite cyclic groups. If σ_1 and σ_2 are two such splittings over H_1 and H_2 (respectively), the splittings ‘cross’ if H_1 is hyperbolic with respect to σ_2 and H_2 is hyperbolic with respect to σ_1 . This corresponds to what were termed strong-strong crossings in [SS2]. Rips and Sela obtain a graph of groups decomposition of G with some vertex groups being surface groups (quadratically hanging groups in their terminology) which enclose all splittings over cyclic groups up to conjugacy. This has some spectacular applications described by Bestvina in [B]. Fujiwara and Papasoglu generalised this to splittings of one-ended finitely presented groups over all slender groups [FP]. This has found traction in the work of Girardel and Levitt [GL2] who describe much other work in their monograph. Again the concept of regular neighbourhood makes an appearance in this approach (see section 6 of [GL2]).

We next describe another technique of Peter Scott which he repeatedly used in the above papers and is called ‘good position’ for almost invariant sets. In [NSSS] this was extended to what is called ‘very good position’ which is the analogue of minimal surfaces for a.i. sets. This may have further application in group theory. It is natural to expect a proof of the torus theorem using very good position and regular neighbourhoods similar to that given by Andrew Casson in his China notes. There are many such concepts and techniques of Peter Scott scattered throughout these papers which may be of more general interest than the problems addressed here.

There are three papers in these notes:

[SS1] Canonical decompositions for Poincaré duality pairs

[RSS1] Comparing decompositions of Poincaré duality pairs

[RSS2] A deformation theorem for Poincaré duality pairs in dimension 3

All three of the papers deal with the analogues of JSJ decompositions of a compact orientable 3-manifold M with incompressible boundary ∂M considered by Jaco and Shalen [JS] and Johannson [J]. There are generalisations for (M, T) with T a finite union of compact incompressible surfaces in ∂M . The usual case is when $T = \partial M$ and this is what is generalised in the above papers.

Each of the above papers has a comprehensive introduction, so we give a short overall introduction. In all three papers the characteristic submanifold is conceived as a regular neighbourhood of annuli and tori.

In [SS1] the results of [SS2] are applied to the case of an orientable Poincaré duality pair of dimension $n + 2$, $n \geq 1$. In this case it is automatic that there are no a.i. sets over $VPC(i)$ subgroups for $i < n$. Thus, following [SS2], a.i. sets over $VPC(n)$ groups (annuli) and a.i. sets over $VPC(n + 1)$ groups (tori) which do not cross the previous ones are considered in forming regular neighbourhoods. This last condition on crossing, which looks somewhat arbitrary, ensures that tori do not cross the boundary. It turns out that one can also add all tori which do not cross the boundary. The theory of [SS2] gives us a bipartite graph of groups decomposition $\Gamma_{n,n+1}$ in which there are two types of vertices V_0 and V_1 with the V_0 -vertices corresponding to the CCCs of annuli and tori. The exact analogue of the JSJ decomposition is a completion $\Gamma_{n,n+1}^c$ of $\Gamma_{n,n+1}$. The difference is that some V_1 -vertices, which are small fibred, are transferred to V_0 -vertices and other changes are made to make $\Gamma_{n,n+1}^c$ bipartite. We will omit this change in the discussion below. Some parts of [SS1] about enclosing tori are relatively easy using the torus theorem. Other parts about enclosing annuli are somewhat harder. Part of the difficulty is due to the plague of special cases. After the completion of the work it turned out that the double of the decomposition for $(G, \partial G)$ is essentially the decomposition of the double DG of $(G, \partial G)$. This suggests that there may be other approaches to simplify the results of [SS1]. One clarification from [SS1] is that tori which do not cross annuli are among the a.i. sets which are relative to ∂G .

The results of [SS1] are greatly extended in [RSS1]. We start with an example which illustrates the necessity of some restriction on the tori considered in [SS2] and [SS1].

Example 1.1.1 (Scott's example). *This is Example 2.13 of [SS3]. Let F be an orientable surface with at least two boundary components and let C denote one of the boundary components. Thus $\pi_1(F)$ is free, and $\pi_1(C)$ is a free factor of $\pi_1(F)$. If the rank of $\pi_1(F)$ is at least 3, then it is easy to see that there is a nontrivial splitting of $\pi_1(F)$ as an amalgamated free product over $\pi_1(C)$. Similar considerations apply to express $\pi_1(F)$ as an HNN extension if it has rank 2.*

*We now take two copies F_1, F_2 of F and consider the two 3-manifolds $M_i = F_i \times S^1$, each with a boundary component T_i corresponding to $C_i \times S^1$. Form a 3-manifold M by gluing the M_i 's along T_i so that the fibrations do not match. The resulting torus T is a topologically canonical torus in the JSJ splitting of M . If each $\pi_1(F_i)$ has rank at least 3, we have $\pi_1(M_i) = A_i *_{H_i} B_i, i = 1, 2$, where $H_i = \pi_1(T_i)$. If G denotes $\pi_1(M)$, and H denotes the subgroup $H_1 = H_2$, and $A = A_1 *_{H} A_2$,*

$B = B_1 *_H B_2$, we have a splitting $G = A *_H B$ of G that crosses the splitting associated to T . Thus although T is topologically canonical, it is not algebraically canonical. Notice that embedded essential annuli in M_1 and M_2 , disjoint from T , yield splittings of G over the fibres of M_1 and M_2 , so that G also has splittings over incommensurable cyclic subgroups of H .

Other examples are discussed in [RSS1]. A common feature of these examples is that there are splittings over a $VPC(n+1)$ group H (i.e., a torus) and they do not cross any annuli and G splits over non commensurable $VPC(n)$ (i.e., annuli) subgroups of H . This is taken as the definition of a ‘special canonical torus’. All of them cross a.i. sets over finite index subgroups of H , and many of them cross splittings over H . The first main result of [RSS1] is the following.

Theorem 1.1.2. *The edge splittings of $\Gamma_{n,n+1}$ are either canonical or special canonical. (Here canonical refers to those splittings over annuli and tori which do not cross any a.i. sets over annuli and tori.)*

The arguments in the proofs lead to the next result.

Theorem 1.1.3. *The family $\mathcal{E}_{n,n+1}(G)$ of a.i. sets over $VPC(n)$ and $VPC(n+1)$ groups in G has a regular neighbourhood. It is obtained from $\Gamma_{n,n+1}(G)$ by collapsing the edges corresponding to special canonical tori.*

This finally leads us to the question which has been lurking so far. What happens if we consider only splittings over $VPC(n)$ groups (annuli) and $VPC(n+1)$ groups relative to the boundary and try to form a regular neighbourhood analogous to the Waldhausen decomposition of Neumann and Swarup [NS]? This is the main content of section 5 of [RSS1]. This too can be done and, as in [NS], it leads to a finite number of families of fibrations over small orbifolds and exceptional V_0 -vertices of $\Gamma_{n,n+1}$. The analogue of the Waldhausen decomposition is denoted $\Sigma_{n,n+1}$. The decomposition $\Sigma_{n,n+1}$ is obtained from $\Gamma_{n,n+1}$ by splitting the exceptional V_0 -vertices along an exceptional annulus. The theorem (5.10 of [RSS1]) describes $\Sigma_{n,n+1}$ in terms of special V_0 -vertices and special annuli contained in them.

In the case of PD(3) pairs it is shown in [RSS1] that the characteristic pieces are the same as in the 3-manifold case. This leads to the paper [RSS2] and an analogue of the Johansson Deformation Theorem for PD(3) pairs. As expected, the proof is quite algebraic and seems much simpler and shorter than the known proofs in the 3-manifold case. A consequence not mentioned there is the following.

Theorem 1.1.4. *For a given group G there are only finitely many PD(3) pairs $(G, \partial G)$ up to isomorphism.*

This completes our brief description of the three papers.

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Chapter 2

Canonical decompositions for Poincaré duality pairs

G. Peter Scott and Gadde A. Swarup

Abstract. The authors previously described an algebraic analogue of the JSJ-decomposition of a 3-manifold. This analogue is defined for any finitely presented, one-ended group. We study this analogue in the special case of Poincaré duality pairs.

Dedicated to Terry Wall

Introduction

In [22], as corrected in [23], we obtained canonical decompositions for almost finitely presented groups analogous to the *JSJ*-decomposition of a 3-manifold. In particular, for many almost finitely presented groups G , and any integer $n \geq 1$, we defined a decomposition $\Gamma_{n,n+1}(G)$, and we showed that when G is the fundamental group of an orientable Haken 3-manifold M with incompressible boundary, then $\Gamma_{1,2}(G)$ essentially yields the *JSJ*-decomposition of M . Further details are discussed in [24]. We recall that the *JSJ*-decomposition of M is given by a possibly disconnected compact submanifold $V(M)$ which is called the characteristic submanifold of M , such that the frontier of $V(M)$ consists of essential annuli and tori, and each component of $V(M)$ is an I -bundle or a Seifert fibre space. Further each component of the closure of $M - V(M)$ is simple. The frontier of $V(M)$ determines a graph of groups structure for $G = \pi_1(M)$, in

which all edge groups are free abelian of rank 1 or 2, and this is essentially the same as $\Gamma_{1,2}(G)$.

In this paper, we consider the structure of $\Gamma_{n,n+1}(G)$, in the case of Poincaré duality pairs $(G, \partial G)$ of dimension $n + 2$, where $n \geq 1$. The results we obtain are very closely analogous to the above description for 3-manifolds. This greatly generalises work of Kropholler in [11] and of Castel in [4]. Kropholler described a canonical decomposition of such Poincaré duality pairs in any dimension at least three, and Castel described a canonical decomposition of such Poincaré duality pairs in dimension three only, but their decompositions are analogous to the Torus Decomposition of an orientable Haken 3-manifold. See section 2.4 for a discussion.

A Poincaré duality pair is the algebraic analogue of an aspherical manifold with aspherical boundary components whose fundamental groups inject. It consists of a group G which corresponds to the fundamental group of the manifold, and a family ∂G of subgroups which correspond to the fundamental groups of the boundary components, and the whole setup satisfies an appropriate version of Poincaré duality. The main difficulty in establishing the results in this paper is that if one considers any of the decompositions of G described in [22], then a priori there is no connection between the decomposition and the boundary groups of the pair. In the topological case, if one considers the full characteristic submanifold $V(M)$ of an orientable Haken manifold M with incompressible boundary, one can double M along its boundary to obtain a closed Haken 3-manifold DM , and there is a natural submanifold DV of DM which is the double of $V(M)$. Further, in most cases, DV is the characteristic submanifold of DM . In the algebraic context, doubling a $(n + 2)$ -dimensional Poincaré duality pair $(G, \partial G)$ along its boundary yields a $(n + 2)$ -dimensional Poincaré duality group DG , but in general there is no natural way to double an algebraic decomposition of G to obtain a corresponding decomposition of DG . However, after establishing all the properties of the decomposition $\Gamma_{n,n+1}(G)$, when $(G, \partial G)$ is a Poincaré duality pair, we will show in section 2.8 that the algebraic situation is very similar to the topological one. In the topological setting, one can also reverse this process and construct the full characteristic submanifold of an orientable Haken manifold M with incompressible boundary by starting with the characteristic submanifold $V(DM)$ of DM and “undoubling” to obtain the required submanifold $V(M)$ of M . This greatly simplifies the construction of the characteristic submanifold of M . If we start with the Poincaré duality group DG , and the decomposition $\Gamma_{n+1}(DG)$, then the natural algebraic analogue of “undoubling” is simply to restrict this decomposition to G using the Subgroup Theorem. This determines a

graph of groups structure on G , and it follows from our results in section [2.8](#) that this decomposition of G is $\Gamma_{n,n+1}(G)$. However this does not simplify the proof that $\Gamma_{n,n+1}(G)$ has the properties we require. In fact, the proof of this "undoubling" result in section [2.8](#) depends on first establishing all the properties of the decomposition $\Gamma_{n,n+1}(G)$. The difficulty is that we are unable to show directly that the decomposition of G obtained by restricting $\Gamma_{n+1}(DG)$ to G has any of the enclosing properties required of $\Gamma_{n,n+1}(G)$. This is partly because our idea of enclosing is much stronger than simply requiring certain subgroups of G to be conjugate into certain vertex groups of a given decomposition of G . For example if a splitting of G is enclosed by a vertex v of a graph of groups decomposition Γ of G , then we can split Γ at v to obtain a refined graph of groups structure for G which has an extra edge associated to the given splitting.

An important point about our ideas in [\[22\]](#) and in this paper is that we consider all almost invariant subsets of a group G rather than just those which correspond to splittings. In the topological setting, this corresponds to considering essential maps of codimension-1 manifolds rather than just essential embeddings. We do not know how to carry out the program in this paper using only splittings. The main goal in [\[22\]](#) was to enclose the algebraic analogues of immersed annuli and tori (the analogues were almost invariant sets over virtually polycyclic groups), which is the natural generalization of the approaches in [\[18\]](#) and [\[20\]](#). It turns out that the analogy is stronger in the case of Poincaré duality pairs. The decomposition $\Gamma_{1,2}(G)$ which we obtained in [\[22\]](#) was constructed to enclose the analogues of immersions of annuli and tori whereas in *JSJ* theory [\[7, 8, 31\]](#), the aim was to enclose essential Seifert pairs. When G is the fundamental group of a 3-manifold, the difference turns out to be minor (consisting of small Seifert fibre spaces) and one can easily go from one decomposition to the other. In the case of groups it seems more natural to enclose almost invariant sets over virtually polycyclic subgroups, that is the analogues of immersions of annuli and tori, rather than Seifert pairs and to make the distinction clear, we will call the decompositions that we obtain Annulus-Torus decompositions. It should be pointed out that when doubling a manifold, the *JSJ*-decompositions behave better than the Annulus-Torus decompositions, and we will switch from one to the other when it is convenient. Similar comments apply to our algebraic decompositions of Poincaré duality pairs. This is made precise in Theorem [2.8.6](#) and Remark [2.8.7](#).

Our main result, Theorem [2.3.14](#) is a description of the decomposition $\Gamma_{n,n+1}(G)$, and its completion $\Gamma_{n,n+1}^c(G)$, for an orientable $PD(n+2)$ pair $(G, \partial G)$, when $n \geq 1$. We leave the precise statement till later because it requires the intro-

duction of a substantial amount of terminology. Some of the result is simply a restatement of our results from [22]. But an important part of the result is that all the edge splittings of $\Gamma_{n,n+1}(G)$ are induced by essential annuli and tori in $(G, \partial G)$, in a sense which we define in section 2.2. This means that, for each $n \geq 1$, the decomposition $\Gamma_{n,n+1}(G)$ of G is closely analogous to the *JSJ*-decomposition of an orientable 3-manifold.

In section 2.1 we recall the main definitions and results from our work in [22], as corrected in [23], and from our more recent paper with Guirardel [6]. In section 2.2 we discuss the definition of a Poincaré duality group and pair, and then we discuss essential annuli and tori (the terms ‘annulus’ and ‘torus’ are used in a generalised sense) in orientable Poincaré duality pairs. We show that each essential annulus and torus has a naturally associated almost invariant set, which we call its dual. We also show that all almost invariant sets over *VPCn* groups in an orientable $PD(n+2)$ pair are ‘generated’ by duals of essential annuli. In section 2.3 we give several more definitions which finally allow us to state our main theorem, Theorem 2.3.14. In section 2.4 we discuss the analogues of torus decompositions for orientable $PD(n+2)$ groups and pairs. These decompositions were already obtained by Kropholler [11] under the extra hypothesis that any *VPC* subgroup has finitely generated centraliser, a condition which he called Max-c. In [12] Kropholler showed that the Max-c condition holds in dimension three. However an example due to Mess [16] shows that the Max-c condition is not always satisfied. In dimension three, this decomposition was also obtained by Castel [4]. The comparison between our results and Kropholler’s results in the case of orientable $PD(n+2)$ groups is discussed briefly in [22], but we discuss this in more detail here. In section 2.5 we analyse further our torus decomposition of Poincaré duality pairs. In section 2.6 we continue studying orientable $PD(n+2)$ pairs, and consider the crossing of almost invariant subsets over *VPCn* groups with almost invariant subsets over *VPC(n+1)* groups. This is a new feature of our arguments in this paper, which could not be handled in the more general setting of [22]. In section 2.7 we bring together the various pieces and prove our main result, Theorem 2.3.14. In section 2.8 we are able to prove Theorem 2.8.6 which shows that, for an orientable $PD(n+2)$ pair $(G, \partial G)$, one can double the decomposition $\Gamma_{n,n+1}^c(G)$ to obtain $\Gamma_{n+1}^c(DG)$. Finally in section 2.9 we discuss some natural further questions.

This paper is a revised version of [25]. The main changes are that sections 2 and 3 of that paper have been removed, as the theory therein has now been developed more thoroughly and generally in [6], and a new section, numbered 2.8, has been added. In addition, there are several minor corrections and improvements

in the exposition.

2.1 Preliminaries

In this section we recall the main definitions and results from [22], as corrected in [23], which we will use. But we will start by briefly discussing some 3-manifold theory which motivates and guides all our work. Let M be an orientable Haken 3-manifold with incompressible boundary. Jaco and Shalen [7] and Johannson [8] proved the existence and uniqueness of the characteristic submanifold of M . We will denote this submanifold by $JSJ(M)$. Its frontier consists of essential annuli and tori in M , and each component of $JSJ(M)$ is a Seifert fibre space or I -bundle. Further any essential map of a Seifert fibre space into M can be properly homotoped to lie in $JSJ(M)$, and this condition characterizes $JSJ(M)$. In order to compare this with algebraic generalisations, we note that, in particular, any essential map of the annulus or torus into M can be properly homotoped to lie in $JSJ(M)$. This weaker condition does not characterize $JSJ(M)$, but does characterise a submanifold of M which we denote by $AT(M)$. The letters AT stand for Annulus-Torus. This is discussed in detail in chapter 1 of [22], but the notation $AT(M)$ is not used. Any essential map of the annulus or torus into M can be properly homotoped to lie in $AT(M)$, and $AT(M)$ is minimal, up to isotopy, among all essential submanifolds of M with this property. (A compact submanifold of M is essential if its frontier consists of essential embedded surfaces.) We will say that the family of all essential annuli and tori in M fills $AT(M)$, and we regard $AT(M)$ as a kind of regular neighbourhood of this family. The connection between $AT(M)$ and $JSJ(M)$ can be described as follows. The submanifold $JSJ(M)$ has certain exceptional components. These are of two types. One type is a solid torus W whose frontier consists of three annuli each of degree 1 in W , or of one annulus of degree 2 in W , or of one annulus of degree 3 in W . The other type lies in the interior of M and is homeomorphic to the twisted I -bundle over the Klein bottle. (Note that as M is orientable, only one such bundle can occur.) Then $AT(M)$ can be obtained from $JSJ(M)$ by discarding all these exceptional components, replacing each of them by a regular neighbourhood of its frontier, and finally discarding any redundant product components from the resulting submanifold.

Recall from the previous paragraph that the family of all essential annuli and tori in M fills $AT(M)$, and that we regard $AT(M)$ as a kind of regular neighbourhood of this family. However, $AT(M)$ is filled by a smaller family of essential

annuli and tori in M , and this turns out to be crucial for the algebraic analogues we are discussing in this paper. Let $AT_{\partial}(M)$ denote the union of those components of $AT(M)$ which meet ∂M , and let $AT_{int}(M)$ denote the union of the remaining components of $AT(M)$. Then any essential annulus in M can be properly homotoped to lie in $AT_{\partial}(M)$. Thus it is clear that any essential torus in M which is homotopic into $AT_{int}(M)$ cannot cross any such annulus, and that $AT_{int}(M)$ must be filled by tori which are homotopic into $AT_{int}(M)$. Further, it is easy to show that $AT_{\partial}(M)$ is filled by the family of all essential annuli in M . We conclude that $AT(M)$ is filled by the family of all essential annuli in M together with those essential tori in M which do not cross any essential annulus in M .

For future reference, we will also need to discuss the Torus Decomposition of M and its relationship with $AT(M)$. As above one can characterise a submanifold $T(M)$ of M by the property that any essential map of a torus into M can be homotoped into $T(M)$ and that $T(M)$ is minimal, up to isotopy, among all essential submanifolds of M with this property. Of course if M admits no essential annulus, then $AT(M)$ and $T(M)$ are equal. In general, $T(M)$ is obtained from $AT(M)$ as follows. Any component of $AT_{int}(M)$ is left unchanged. Now $AT_{\partial}(M)$ has three types of component. The first type is a Seifert fibre space which is not a solid torus and such that each boundary torus lies in the interior of M , or is contained in ∂M , or meets ∂M in vertical annuli. The second type is a solid torus which meets ∂M in annuli, and the third type is an I -bundle over a surface F which meets ∂M in the associated ∂I -bundle over F . As no essential torus in M can be homotopic into a component of $AT_{\partial}(M)$ of the second or third type, all such components are omitted when we form $T(M)$. Finally let W denote a component of $AT_{\partial}(M)$ which is of the first type. Thus W is a Seifert fibre space which is not a solid torus and $W \cap \partial M$ consists of tori and vertical annuli in ∂W . By pushing into the interior of W each torus component of ∂W which meets ∂M in annuli, we obtain a Seifert fibre space W' which is contained in and homeomorphic to W . Note that the components of the closure of $W - W'$ are homeomorphic to $T \times I$, and $W' \cap \partial M$ consists only of tori. Replacing W by W' for each such component of $AT_{\partial}(M)$ finally yields $T(M)$. Note that it is clear that $T(M) \subset AT(M)$ and that an essential torus in M is homotopic into $AT(M)$ if and only if it is homotopic into $T(M)$.

Next we recall the cohomological formulation of the theory of ends and of almost invariant subsets of a group. Let G be a group and let E be a set on which G acts on the right. Let PE denote the power set of E . Under Boolean addition ("symmetric difference") this is an additive group of exponent 2. Write

FE for the additive subgroup of finite subsets. We refer to two sets A and B whose symmetric difference lies in FE as *almost equal*, and write $A \stackrel{a}{=} B$. This amounts to equality in the quotient group PE/FE . Now define

$$QE = \{A \subset E : \forall g \in G, A \stackrel{a}{=} Ag\}.$$

The action of G on PE by right translation preserves the subgroups QE and FE , and QE/FE is the subgroup of elements of PE/FE fixed under the induced action. Elements of QE are said to be *almost invariant*. If we take E to be G with the action of G being right multiplication, then the number of ends of G is

$$e(G) = \dim_{\mathbb{Z}_2} (QG/FG).$$

If G is finite, all subsets are finite and clearly $e(G) = 0$. Otherwise, G is an infinite set which is invariant (not merely “almost”), so $e(G) \geq 1$.

If H is a subgroup of G , and we take E to be the coset space $H \backslash G$ of all cosets Hg , still with the action of G being right multiplication, then the number of ends of the pair (G, H) is

$$e(G, H) = \dim_{\mathbb{Z}_2} \left(\frac{Q(H \backslash G)}{F(H \backslash G)} \right).$$

When H is trivial, so that $e(G, H) = e(G)$, this can be formulated in terms of group cohomology as follows. The abelian group PG is naturally a (right) \mathbb{Z}_2G -module, and the submodule FG can be identified with the group ring \mathbb{Z}_2G . Thus the invariant subgroup QG/FG equals $H^0(G; PG/\mathbb{Z}_2G)$. Now the short exact sequence of coefficients

$$0 \rightarrow \mathbb{Z}_2G \rightarrow PG \rightarrow PG/\mathbb{Z}_2G \rightarrow 0$$

yields the following long exact cohomology sequence.

$$H^0(G; \mathbb{Z}_2G) \rightarrow H^0(G; PG) \rightarrow H^0(G; PG/\mathbb{Z}_2G) \xrightarrow{\delta} H^1(G; \mathbb{Z}_2G) \rightarrow H^1(G; PG) \rightarrow$$

For any group G , the group $H^n(G; PG)$ is zero if $n \neq 0$, and isomorphic to \mathbb{Z}_2 when $n = 0$. And if G is infinite, then $H^0(G; \mathbb{Z}_2G) = 0$. Also when G is infinite, the non-zero element of $H^0(G; PG)$ maps to the element of $H^0(G; PG/\mathbb{Z}_2G)$ which corresponds to the equivalence class of G in QG/FG , under the identification of these two groups. It follows that when G is infinite, the group

$H^1(G; \mathbb{Z}_2G)$ can be identified with the collection of all almost invariant subsets of G modulo almost equality and complementation.

If H is nontrivial, let E denote the coset space $H \backslash G$. The abelian group PE is naturally a (right) \mathbb{Z}_2G -module, and we denote the submodule FE by \mathbb{Z}_2E . Thus the invariant subgroup QE/FE equals $H^0(G; PE/\mathbb{Z}_2E)$. Now the short exact sequence of coefficients

$$0 \rightarrow \mathbb{Z}_2E \rightarrow PE \rightarrow PE/\mathbb{Z}_2E \rightarrow 0$$

yields the following long exact cohomology sequence.

$$H^0(G; \mathbb{Z}_2E) \rightarrow H^0(G; PE) \rightarrow H^0(G; PE/\mathbb{Z}_2E) \xrightarrow{\delta} H^1(G; \mathbb{Z}_2E) \rightarrow H^1(G; PE) \rightarrow$$

For any group G and any subgroup H , the group $H^n(G; PE)$ is isomorphic to $H^n(H; \mathbb{Z}_2)$, and if the index of H in G is infinite, then $H^0(G; \mathbb{Z}_2E) = 0$. Thus, as above, when H has infinite index in G , the image of the coboundary map $H^0(G; PE/\mathbb{Z}_2E) \rightarrow H^1(G; \mathbb{Z}_2E)$ can be identified with the collection of all almost invariant subsets of $H \backslash G$ modulo almost equality and complementation.

If G is finitely presented, this description connects very nicely with topology. Let X be an Eilenberg-MacLane space $K(G, 1)$ with finite 2-skeleton, let \tilde{X} denote its universal cover, and let X_H denote the cover of X with fundamental group H . Thus X_H is the quotient of \tilde{X} by the (left) action of H acting as a covering group. Because X has finite 2-skeleton, the part of the long exact cohomology sequence shown above is isomorphic to the corresponding part of the following long exact cohomology sequence for X_H ,

$$H_f^0(X_H; \mathbb{Z}_2) \rightarrow H^0(X_H; \mathbb{Z}_2) \rightarrow H_e^0(X_H; \mathbb{Z}_2) \xrightarrow{\delta} H_f^1(X_H; \mathbb{Z}_2) \rightarrow H^1(X_H; \mathbb{Z}_2) \rightarrow$$

where $H_f^i(X_H; \mathbb{Z}_2)$ denotes cellular cohomology with finite supports. Thus when H has infinite index in G , the image of the coboundary map $H_e^0(X_H; \mathbb{Z}_2) \rightarrow H_f^1(X_H; \mathbb{Z}_2)$ can be identified with the collection of all almost invariant subsets of $H \backslash G$ modulo almost equality and complementation. If G is finitely generated but not finitely presented, we can take X to be a $K(G, 1)$ with finite 1-skeleton but it is no longer correct to identify $H^1(G, \mathbb{Z}_2E)$ with $H_f^1(X_H; \mathbb{Z}_2)$. In fact, cellular cohomology with finite supports for a cell complex which is not locally finite is an unreasonable idea, as the coboundary of a finite cochain need not be finite. However it is easy to define a modified version of this theory in our particular setting. For future reference we set this out as a remark.

Remark 2.1.1. *Let G be a group which is finitely generated but need not be finitely presented, and let X be a $K(G, 1)$ with finite 1–skeleton. Let H be a subgroup of G , and let X_H denote the cover of X with fundamental group H . We replace the use of cochains on X_H with finite support by cochains whose support consists of only finitely many cells above each cell of X . Note that as X has only finitely many 1–cells, such 1–cochains are finite. In this paper, $H_f^1(X_H; \mathbb{Z}_2)$ will denote the appropriate cohomology group of this cochain complex. This enables us to identify $H^1(G, \mathbb{Z}_2 E)$ with $H_f^1(X_H; \mathbb{Z}_2)$.*

This is the topological formulation of Lemma 7.4 in [3].

Recall that the invariant subgroup QE/FE equals $H^0(G; PE/\mathbb{Z}_2 E)$. Thus the elements of $H^0(G; PE/\mathbb{Z}_2 E)$ are equivalence classes of almost invariant subsets of $H \setminus G$ under the equivalence relation of almost equality. Also the elements of $H_e^0(X_H; \mathbb{Z}_2)$ are equivalence classes of cellular 0–cochains on X_H which have finite coboundary. The support of such a cochain is a subset of the vertex set of X_H . Thus whether or not G is finitely presented, the isomorphism between $H^0(G; PE/\mathbb{Z}_2 E)$ and $H_e^0(X_H; \mathbb{Z}_2)$ associates to an almost invariant subset Y of $H \setminus G$ a subset Z of the vertex set of X_H with finite coboundary, where Z is unique up to almost equality. This is a convenient fact which we will use on several occasions. Note that Y is trivial, i.e. finite or co-finite, if and only if Z is finite or co-finite. Also if Y and Y' are almost invariant subsets of $H \setminus G$ with corresponding subsets Z and Z' of the vertex set of X_H , then the intersections $Y \cap Y'$ and $Z \cap Z'$ also correspond.

Next we recall some more basic facts about almost invariant sets. If X and Y are subsets of G , the four sets $X \cap Y$, $X \cap Y^*$, $X^* \cap Y$ and $X^* \cap Y^*$ are called the *corners* of the pair (X, Y) . If X is H –almost invariant and Y is K –almost invariant, a corner of the pair (X, Y) is *small* if it is H –finite or K –finite. (These two conditions are equivalent so long as X and Y are both nontrivial and G is finitely generated.) We say that X *crosses* Y if all four corners of the pair (X, Y) are K –infinite. The preceding parenthetical comment shows that if X and Y are nontrivial and G is finitely generated, then X crosses Y if and only if Y crosses X . In [21] and [22], we defined a partial order \leq on certain families of almost invariant subsets of a finitely generated group G as follows. The idea of the definition is that $Y \leq X$ means that Y is “almost” contained in X . If E is a family of almost invariant subsets of G , we say that the elements of E are in *good position* if whenever U and V are elements of E such that two of the corners of the pair are small, then one corner is empty. If the elements of E are in good position, then we defined $Y \leq X$ to mean that either $Y \cap X^*$ is empty or it is the

only small corner of the pair (X, Y) . We showed that \leq is a partial order on E . Note that if $Y \subset X$, then automatically $Y \leq X$. Note also that the requirement of good position was needed to avoid the possibility of having distinct sets X and Y such that $Y \leq X$ and $X \leq Y$. In [19], we defined an even stronger condition. If E is a family of almost invariant subsets of G , we say that the elements of E are in *very good position* if whenever U and V are elements of E , either none of the four corners of the pair is small or one is empty. This is equivalent to the partial orders on E induced by inclusion and by \leq being the same. We also showed that one can often arrange that families of almost invariant sets are in very good position by replacing the given sets by equivalent ones.

Next we recall the theory of algebraic regular neighbourhoods. In [22], we defined an algebraic regular neighbourhood (Definition 6.1) and a reduced algebraic regular neighbourhood (Definition 6.18) of a family of almost invariant subsets of a finitely generated group G . See also Definition 9.1 in [6]. We discuss the difference between these objects immediately after Definition 2.1.4 below. Each is a bipartite graph of groups structure Γ for G with certain properties. The basic property of Γ is that the V_0 -vertices enclose the given almost invariant sets. See chapters 4 and 5 of [22], or section 3 of [6], for the definition and basic properties of enclosing. Algebraic regular neighbourhoods need not exist, but we showed that when they exist they are unique up to isomorphism of bipartite graphs of groups. We also showed that if one has a finite family of almost invariant sets each over a finitely generated subgroup of G , then it always has an algebraic regular neighbourhood and a reduced algebraic regular neighbourhood. The main results of [22] were existence results for algebraic regular neighbourhoods of infinite families in several special cases. In this paper, we will use the existence results for reduced algebraic regular neighbourhoods.

Before stating these existence results, we briefly discuss how the topological and algebraic situations are related. Groups which are virtually polycyclic (*VPC*) play an important role in this paper. The Hirsch length of such a group will simply be called the length for brevity. A group which is *VPC* of length n will be called *VPC* n . We will often need to refer to a group which is *VPC* of length at most n . Such a group will be called *VPC*($\leq n$). We will also use the notation *VPC*($< n$) in a similar way. Almost invariant sets which do not cross other almost invariant sets play a special role. We will need the following definition from [22].

Definition 2.1.2. *Let G be a one-ended finitely generated group and let X be a nontrivial almost invariant subset over a subgroup H of G .*

For $n \geq 1$, we will say that X is n -canonical if X crosses no nontrivial K -almost invariant subset of G , for which K is $VPC(\leq n)$.

Let G denote any one-ended almost finitely presented group. The natural algebraic analogue of an essential annulus in a 3-manifold M is a nontrivial almost invariant subset of G over a $VPC1$ subgroup, and the natural analogue of an essential torus in M is a nontrivial almost invariant subset of G over a $VPC2$ subgroup. As $AT(M)$ is a kind of regular neighbourhood of the family of all essential annuli and tori in M , it would seem natural to consider the algebraic regular neighbourhood in G of the family of all nontrivial almost invariant subsets of G which are over VPC subgroups of length 1 or 2. However, we showed in Example 11.7 of [22] that, even when G is the fundamental group of a 3-manifold, such a family need not possess an algebraic regular neighbourhood with the right properties. But recall from our discussion on page 16 that $AT(M)$ is filled by the family of all essential annuli in M together with those essential tori in M which do not cross any essential annulus in M . The algebraic analogue of this family is the family $\mathcal{F}_{1,2}$ of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPC1$ subgroups and of equivalence classes of all 1-canonical almost invariant subsets of G which are over $VPC2$ subgroups. In [22], as corrected in [23], we showed that $\mathcal{F}_{1,2}$ has an algebraic regular neighbourhood which is precisely analogous to the decomposition given by $AT(M)$. We also described analogous constructions for VPC subgroups of G of higher length. This is contained in Theorem 2.1.16 below.

Next we need to introduce some more definitions which we used in [22].

Definition 2.1.3. *If E is a G -invariant family of nontrivial almost invariant subsets of a group G , we will say that an element of E which crosses no element of E is isolated in E .*

When forming an algebraic regular neighbourhood of a family E of almost invariant sets, isolated elements yield special vertices which we also call isolated.

Definition 2.1.4. *A vertex of a graph of groups Γ is isolated if it has exactly two incident edges for each of which the inclusion of the associated edge group into the vertex group is an isomorphism.*

Remark 2.1.5. *If Γ consists of a single vertex v and a single edge, then v is not isolated, as only one edge is incident to v .*

Note that the two edges incident to an isolated vertex have the same associated edge splitting. Conversely if two distinct edges e and e' of a minimal graph

Γ of groups have associated edge splittings which are conjugate, there is an edge path λ in Γ which starts with e and ends with e' such that all the interior vertices of λ are isolated, and all the edges in λ have associated edge splittings which are conjugate. When one forms the algebraic regular neighbourhood Γ of a family E of nontrivial almost invariant subsets of a group G , it may happen that Γ has such edge paths with more than two edges. One could reduce Γ by simply collapsing each maximal such edge path in Γ to a single edge. However Γ is bipartite and one wants to preserve this property, so one may instead need to collapse such a maximal edge path to two edges. The resulting bipartite graph of groups is the reduced algebraic regular neighbourhood of E . It never has three distinct edges such that the associated splittings of G are all conjugate. We formalise this in the following definition.

Definition 2.1.6. *A minimal bipartite graph of groups Γ is called reduced bipartite if it does not have three distinct edges such that the associated splittings of G are all conjugate.*

Some other special types of vertices may occur when one forms an algebraic regular neighbourhood.

Definition 2.1.7. *Let Γ be a minimal graph of groups decomposition of a group G . A vertex v of Γ is of VPC -by-Fuchsian type if $G(v)$ is a VPC -by-Fuchsian group, where the Fuchsian group is finitely generated and is not finite nor two-ended, and there is exactly one edge of Γ which is incident to v for each peripheral subgroup K of $G(v)$, and this edge carries K .*

If the length of the normal VPC subgroup of $G(v)$ is n , we will say that v is of $VPCn$ -by-Fuchsian type.

Remark 2.1.8. *It is possible that a single edge of Γ can have both ends incident to v . In this case, the two inclusions of the associated edge group into $G(v)$ must have images which are distinct peripheral subgroups of $G(v)$ up to conjugacy.*

Note that if $G = G(v)$, then Γ must consist of v alone, and the Fuchsian quotient group of G corresponds to a closed orbifold. Conversely if the Fuchsian quotient group of $G(v)$ corresponds to a closed orbifold, then Γ must consist of v alone, and $G = G(v)$. Note also that if v is of $VPCn$ -by-Fuchsian type, then each peripheral subgroup of $G(v)$ is $VPC(n+1)$.

The assumption in Definition [2.1.7](#) that the Fuchsian quotient of $G(v)$ not be finite nor two-ended is made to ensure the uniqueness of the $VPCn$ normal subgroup of $G(v)$ with Fuchsian quotient. This is immediate from the following result.

Lemma 2.1.9. *Let G be a group with a normal $VPCk$ subgroup L with Fuchsian quotient Φ . Suppose that Φ is not finite nor two-ended. If L' is a $VPCk$ normal subgroup of G with Fuchsian quotient, then L' must equal L .*

Proof. If L' is not contained in L , the image of L' in Φ is a nontrivial normal VPC subgroup, which we denote by N . As no Fuchsian group can be $VPC2$, it follows that N must be $VPC0$ or $VPC1$. If N is $VPC0$, i.e. finite, then Φ must also be finite. If N is $VPC1$, then it must be of finite index in Φ . As we are assuming that Φ is not finite nor two-ended, it follows that L' must be contained in L . Similarly L must be contained in L' , so that L and L' are equal, as required. \square

We note that in [22] we used the word Fuchsian to include discrete groups of isometries of the Euclidean plane, as well as the hyperbolic plane. The additional groups were all virtually $\mathbb{Z} \times \mathbb{Z}$. The reason for this abuse of language was that we wanted to include the case of VPC groups in the statements of our results. However in all of the main results of this paper, it will be convenient to exclude the case of VPC groups.

Next we prove two simple results about VPC groups which will be needed on several occasions.

Lemma 2.1.10. *Let G be a $VPC(n + 1)$ group which splits over a subgroup L . Then L is $VPCn$, and is normal in G with quotient which is isomorphic to \mathbb{Z} or $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Proof. The result is equivalent to asserting that, for a $VPC(n + 1)$ group G , any minimal G -tree must be a point or a line. We will prove this by induction on the length of G . The induction starts when G has length 1. Then G has two ends and the result is standard. Now suppose that G has length $n + 1 \geq 2$, and that the result is known for $VPC(\leq n)$ groups. Let T be a minimal G -tree. There is a subgroup G' of finite index in G which normalises some $VPCn$ subgroup L' . By our induction assumption, the action of L' on T must fix a point or have a minimal subtree T' which is a line. In the second case, the minimal subtree of T left invariant by G' must also be T' . In the first case, we let T' denote the fixed subtree of L' , i.e. T' consists of all vertices and edges fixed by L' . The action of G' must preserve T' , so that the quotient group $L' \backslash G'$ acts on T' . As this quotient group has two ends, it follows that the minimal subtree of T' left invariant by G' is a point or a line. Thus in either case, the minimal subtree of T left invariant by G' is a point or a line. As G' has finite index in G , this minimal subtree must equal T , so that T itself is a point or a line as required. \square

Lemma 2.1.11. *Let G be a $VPC(n+1)$ group with a normal $VPCn$ subgroup L such that $L \backslash G$ is isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$. Let K be a normal $VPCn$ subgroup of G which is commensurable with L . Then the following hold:*

1. K is contained in L .
2. If $K \backslash G$ is isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, then $K = L$.

Proof. 1) If K is not contained in L , then the image of K in the quotient $L \backslash G$ is a nontrivial finite normal subgroup. As neither \mathbb{Z} nor $\mathbb{Z}_2 * \mathbb{Z}_2$ possesses such a subgroup, it follows that K must be contained in L as required.

2) If $K \backslash G$ is isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, we can apply the first part with the roles of K and L reversed. We deduce that L is contained in K , so that $K = L$ as required. \square

We also prove the following useful technical results about VPC -by-Fuchsian groups.

Lemma 2.1.12. *Let G be a group with a normal $VPCn$ subgroup L with Fuchsian quotient Φ , and let K be a $VPC(n+1)$ subgroup of G .*

1. *Then $L \cap K$ is a normal $VPCn$ subgroup of K with quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$.*
2. *If H is a normal $VPCn$ subgroup of K with quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, and if H is commensurable with L , then H equals $L \cap K$.*

Proof. 1) As L is $VPCn$ and normal in G , the intersection $L \cap K$ must be $VPC(\leq n)$, and normal in K . Hence the quotient of K by $L \cap K$ is a VPC subgroup of Φ of length at least 1. As a Fuchsian group can have no $VPC2$ subgroups, it follows that k must equal n and the quotient of K by $L \cap K$ must be $VPC1$. As the only $VPC1$ subgroups of a Fuchsian group are isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, the result follows.

2) This follows from Lemma [2.1.11](#). \square

Next we have a uniqueness result for a VPC -by-Fuchsian structure on a group.

Lemma 2.1.13. *Let G be a group with a normal $VPCn$ subgroup L with Fuchsian quotient Φ which is not virtually cyclic, and suppose G also has a normal $VPCm$ subgroup L' with Fuchsian quotient Φ' which is not virtually cyclic. Then $L = L'$.*

Proof. Suppose L' is not contained in L . Then the image of L' in Φ is a nontrivial normal VPC subgroup. But a Fuchsian group which is not virtually cyclic cannot contain such a subgroup. It follows that $L' \subset L$. Similarly, it follows that $L \subset L'$ so that $L = L'$, as required. \square

Now we can state the results from [22] which play a basic role in this paper. Recall that if H is a subgroup of a group G , the *commensuriser*, $Comm_G(H)$, of H in G is the subgroup of G consisting of all elements g such that the conjugate of H by g is commensurable with H . Trivially, $Comm_G(H)$ contains H . We will say that $Comm_G(H)$ is *large* if it contains H with infinite index, and is *small* otherwise.

The following existence result is essentially the statement of Theorem 12.3 of [22]. We have made one slight modification in the last sentence of part 3), where we refer to the number of coends of a subgroup in a group. See page 33 of [22] for a brief discussion of this concept. It was introduced independently by Bowditch [2] under the name of coends, by Geoghegan [5] under the name of filtered coends, and by Kropholler and Roller [13] under the name of relative ends.

Theorem 2.1.14. *Let $n \geq 1$, and let G be a one-ended, almost finitely presented group which is not VPC and does not admit any nontrivial almost invariant subsets over $VPC(< n)$ subgroups, and let \mathcal{F}_n denote the collection of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPCn$ subgroups.*

Then \mathcal{F}_n has an unreduced and a reduced algebraic regular neighbourhood in G . Let $\Gamma_n = \Gamma(\mathcal{F}_n : G)$ denote the reduced algebraic regular neighbourhood of \mathcal{F}_n in G .

Then Γ_n is a minimal, reduced bipartite, graph of groups decomposition of G . Each V_0 -vertex v of Γ_n satisfies one of the following conditions:

1. *v is isolated, and $G(v)$ is $VPCn$.*
2. *v is of $VPC(n - 1)$ -by-Fuchsian type, and elements of \mathcal{F}_n enclosed by v cross strongly if at all.*
3. *$G(v)$ is the full commensuriser $Comm_G(H)$ for some $VPCn$ subgroup H , such that $e(G, H) \geq 2$, and elements of \mathcal{F}_n enclosed by v cross weakly if at all.*

Further, if H is a $VPCn$ subgroup of G such that $e(G, H) \geq 2$, then Γ_n will have a non-isolated V_0 -vertex v such that $G(v) = Comm_G(H)$ if and only if H has at least 4 coends in G .

Γ_n consists of a single vertex if and only if \mathcal{F}_n is empty, or G itself satisfies condition 2) or 3). In the first case, Γ_n consists of a single V_1 -vertex. In the second case, Γ_n consists of a single V_0 -vertex.

Remark 2.1.15. *The assumption that G is not VPC is made to simplify the statement of this result. If G is $VPC(n+1)$, then Γ_n consists of a single V_0 -vertex v with associated group G . In this case, v may not satisfy any of the conditions in the above theorem.*

We will say that a V_0 -vertex in case 3) is of commensuriser type if v is not isolated, and is of large commensuriser type if, in addition, H has large commensuriser.

Note that we showed in Example 11.1 of [22] that, even if G is finitely presented, the group associated to a V_0 -vertex of commensuriser type need not be finitely generated.

The statement at the end of part 3) with the assumption that H has at least 4 coends in G comes from the proofs of Propositions 7.16, 7.17, 8.1 and 8.6 of [22]. Note that if H has large commensuriser in G , the proof of Proposition 8.1 of [22] shows that H has infinitely many coends in G . This uses the fact that we have excluded the case when G is VPC .

When $n = 2$, this result is the algebraic analogue of the torus decomposition $T(M)$ of a closed orientable Haken 3-manifold M . See the start of this section for a discussion of $T(M)$. As M is compact, its fundamental group G is finitely presented. As M is irreducible, it follows that G is also one-ended. As M is closed, our discussion in [24] implies that G does not admit any nontrivial almost invariant subsets over $VPC1$ subgroups. Now the above result asserts that $\Gamma_2(G)$ exists, and we showed in [22] that it is the graph of groups structure for G determined by the frontier of $T(M)$ in M . The V_0 -vertices of $\Gamma_2(G)$ correspond to the components of $T(M)$. In this case, $\Gamma_2(G)$ has no V_0 -vertices of commensuriser type. An isolated V_0 -vertex of $\Gamma_2(G)$ corresponds to a component of $T(M)$ homeomorphic to $T \times I$. A V_0 -vertex of $\Gamma_2(G)$ which is of $VPC1$ -by-Fuchsian type corresponds to a component of $T(M)$ which is homeomorphic to a Seifert fibre space.

Next we come to the following more general result which is essentially the statement of Theorem 13.12 of [22]. Again we assume that G is not VPC in order to simplify the statement.

Theorem 2.1.16. *Let $n \geq 1$, and let G be a one-ended, almost finitely presented group which is not VPC and does not admit any nontrivial almost invariant subsets over $VPC(< n)$ subgroups, and let $\mathcal{F}_{n,n+1}$ denote the collection of equivalence*

classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup, together with the equivalence classes of all n -canonical almost invariant subsets of G which are over a $VPC(n+1)$ subgroup.

Then $\mathcal{F}_{n,n+1}$ has an unreduced and a reduced algebraic regular neighbourhood in G . Let $\Gamma_{n,n+1} = \Gamma(\mathcal{F}_{n,n+1} : G)$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G .

Then $\Gamma_{n,n+1}$ is a minimal, reduced bipartite, graph of groups decomposition of G . Each V_0 -vertex v of $\Gamma_{n,n+1}$ satisfies one of the following conditions:

1. v is isolated, and $G(v)$ is VPC of length n or $n+1$.
2. v is of $VPCk$ -by-Fuchsian type, where k equals $n-1$ or n , and elements of $\mathcal{F}_{n,n+1}$ enclosed by v cross strongly if at all.
3. $G(v)$ is the full commensuriser $Comm_G(H)$ for some VPC subgroup H of length n or $n+1$, such that $e(G, H) \geq 2$, and elements of $\mathcal{F}_{n,n+1}$ which are enclosed by v and are over groups commensurable with H cross weakly if at all.

Further, if H is a VPC subgroup of G of length n or $n+1$, such that $e(G, H) \geq 2$, then $\Gamma_{n,n+1}$ will have a non-isolated V_0 -vertex v such that $G(v) = Comm_G(H)$ if and only if H has at least 4 coends in G .

$\Gamma_{n,n+1}$ consists of a single vertex if and only if $\mathcal{F}_{n,n+1}$ is empty, or G itself satisfies condition 2) or 3). In the first case, $\Gamma_{n,n+1}$ consists of a single V_1 -vertex. In the second case, $\Gamma_{n,n+1}$ consists of a single V_0 -vertex.

When $n = 1$, this result is the algebraic analogue of the Annulus-Torus decomposition $AT(M)$ of an orientable Haken 3-manifold M with incompressible boundary. See the start of this section for a discussion of $AT(M)$. As M is compact, its fundamental group G is finitely presented. As M is irreducible and has incompressible boundary, it follows that G is also one-ended. Now the above result asserts that $\Gamma_{1,2}(G)$ exists, and we showed in [22] that it is the graph of groups structure for G determined by the frontier of $AT(M)$ in M . The V_0 -vertices of $\Gamma_{1,2}(G)$ correspond to the components of $AT(M)$. An isolated V_0 -vertex of $\Gamma_{1,2}(G)$ corresponds to a component of $AT(M)$ homeomorphic to $A \times I$ or $T \times I$. A V_0 -vertex of $VPC1$ -by-Fuchsian type corresponds to a component of $AT(M)$ which is a Seifert fibre space not meeting ∂M . A V_0 -vertex of $VPC0$ -by-Fuchsian type corresponds to a component of $AT(M)$ which is an I -bundle and meets ∂M in the associated S^0 -bundle. A V_0 -vertex v of commensuriser

type, where $G(v)$ is the full commensuriser $Comm_G(H)$ for some $VPC1$ subgroup H , corresponds to a component of $AT(M)$ which is a Seifert fibre space and meets ∂M . Finally, in this case, $\Gamma_{1,2}(G)$ has no V_0 -vertex v of commensuriser type, where $G(v)$ is the full commensuriser of a $VPC2$ subgroup of G .

In the special case when M is closed or admits no essential annuli, then G has no nontrivial almost invariant subsets over any $VPC1$ subgroup, so that $\Gamma_2(G)$ is defined and equals $\Gamma_{1,2}(G)$.

Recall that $JSJ(M)$ can be obtained from $AT(M)$ by adding certain exceptional submanifolds of M . We will describe analogous algebraic constructions for any n and any group G for which $\Gamma_{n+1}(G)$ or $\Gamma_{n,n+1}(G)$ exist. The results of these constructions are graphs of groups structures $\Gamma_{n+1}^c(G)$ and $\Gamma_{n,n+1}^c(G)$ for G , which we call the completions of $\Gamma_{n+1}(G)$ and $\Gamma_{n,n+1}(G)$ respectively.

Definition 2.1.17. *Let G be a group for which the decompositions $\Gamma_{n+1}(G)$ or $\Gamma_{n,n+1}(G)$ exist. The completions of these decompositions, denoted $\Gamma_{n+1}^c(G)$ and $\Gamma_{n,n+1}^c(G)$ respectively are graphs of groups structures for G obtained as follows:*

If $\Gamma_{n+1}(G)$ or $\Gamma_{n,n+1}(G)$ has a V_1 -vertex w such that $G(w)$ is $VPC(n+1)$, and if w has a single incident edge e with $G(e)$ of index 2 in $G(w)$, then we subdivide e into two edges. The new vertex is a V_1 -vertex and w becomes a V_0 -vertex. If the original V_0 -vertex of e is isolated, then in addition we collapse e to a point, which becomes a new V_0 -vertex.

If $\Gamma_{n,n+1}(G)$ has a V_1 -vertex w such that $G(w)$ is $VPCn$, and if w has a single incident edge e with $G(e)$ of index 2 or 3 in $G(w)$, or if w has exactly three incident edges each carrying $G(w)$, then we subdivide each of the incident edges into two edges. The new vertices are V_1 -vertices and w becomes a V_0 -vertex. If the original V_0 -vertex of any of the edges incident to w is isolated, then in addition we collapse that edge.

Making all these changes for every such V_1 -vertex of $\Gamma_{n+1}(G)$ and $\Gamma_{n,n+1}(G)$ yields $\Gamma_{n+1}^c(G)$ and $\Gamma_{n,n+1}^c(G)$ respectively.

Remark 2.1.18. *If $n = 1$, and G is the fundamental group of an orientable Haken 3-manifold M with incompressible boundary, then the discussion in [22] shows that $\Gamma_{1,2}^c(G)$ is the graph of groups determined by the frontier of $JSJ(M)$ in M . The V_0 -vertices of $\Gamma_{1,2}^c(G)$ correspond to the components of $JSJ(M)$. Those V_0 -vertices of $\Gamma_{1,2}^c(G)$ which are obtained from V_1 -vertices of $\Gamma_{1,2}(G)$ correspond to the exceptional components of $JSJ(M)$. If M is closed or admits no essential annuli, then $\Gamma_2^c(G)$ is defined and equals $\Gamma_{1,2}^c(G)$. Also those V_0 -vertices of $\Gamma_2^c(G)$ which are obtained from V_1 -vertices of $\Gamma_2(G)$ correspond to the exceptional components of $JSJ(M)$.*

2.2 Poincaré duality pairs and essential annuli and tori

We refer to Brown [3], Bieri and Eckmann [1], Kapovich and Kleiner [9], and Wall [34] for various definitions of Poincaré duality groups and pairs. The definition of Bieri and Eckmann is that G is a Poincaré duality group of dimension $n + 2$ if $H^i(G; \mathbb{Z}G)$ is 0, when $i \neq n + 2$, and is isomorphic to \mathbb{Z} when $i = n + 2$. Further G is orientable if the action of G on $H^{n+2}(G; \mathbb{Z}G)$ is trivial. In the following, we will be mostly concerned with orientable Poincaré duality groups and pairs. A Poincaré duality pair is a pair $(G, \partial G)$, where $\partial G = \{S_1, \dots, S_m\}$ is a system of subgroups of G , such that the double of G along ∂G is a Poincaré duality group. Theorem 8.1 of [1] shows that each S_i must be a $PD(n + 1)$ group. Note that the order of the S_i 's is irrelevant, and that repetitions are allowed. However, if any repetition occurs, or even if two distinct S_i 's are conjugate, there is a $PD(n + 1)$ group H , such that the pair $(G, \partial G)$ equals $(H, \{H, H\})$, which is a trivial $PD(n + 2)$ pair analogous to the product of a closed $(n + 1)$ -manifold with the unit interval. We will usually assume that $n \geq 1$, so that our Poincaré duality groups and pairs are at least 3-dimensional. Bieri and Eckmann [1] show that their definition implies that G is almost finitely presented, which suffices for the accessibility results that we use. If G is finitely presented then the corresponding $K(G, 1)$ space is dominated by a finite complex (see Theorem 7.1 in Chapter 8 of [3]) and then G is a Poincaré duality group in the sense of Wall [34].

A Poincaré duality pair is a special case of what the authors of [6] call a group system, but it seems natural to use the language of pairs in the setting of this paper. In [6], many of our results require that the group system be of finite type, but this condition is automatic for Poincaré duality pairs. An important idea which we first introduced in [25], and is worked out in more detail in [6], is that of an almost invariant subset of a group G being adapted to a family of subgroups. As this idea will play an important role in this paper, we reproduce the definition from [6], and some of the following remarks. Lemma 2.2.5 summarises the properties of adapted almost invariant sets which we will need in this paper.

Definition 2.2.1. (Definition 5.1 of [6]) *Let G be a group and let H and S be subgroups. Let $\mathcal{S} = \{S_i\}_{i \in I}$ be a family of subgroups of G , with repetitions allowed.*

A H -almost invariant subset X of G is strictly adapted to the subgroup S if, for all $g \in G$, the coset gS is contained in X or in X^ .*

A H -almost invariant subset X of G is adapted to S , or is S -adapted, if it is equivalent to a H' -almost invariant subset X' of G such that X' is strictly adapted

to S .

A H -almost invariant subset X of G is adapted to the family \mathcal{S} , or is \mathcal{S} -adapted, if it is adapted to each S_i .

Remark 2.2.2. If X is \mathcal{S} -adapted, then so is any almost invariant subset of G which is equivalent to X .

If X is adapted to the family \mathcal{S} , and we replace each S_i by some conjugate, then X is also adapted to the new family. For if X is strictly adapted to a subgroup S , and $k \in G$, then Xk is H -almost invariant and equivalent to X , and is strictly adapted to $k^{-1}Sk$.

Note that if X is adapted to the family \mathcal{S} , then, for each i , there is a K_i -almost invariant subset X_i of G which is equivalent to X and is strictly adapted to S_i , but the X_i 's may all be different. In general, it is difficult for an almost invariant set to be strictly adapted to more than one subgroup of G .

The following definition, due to Müller [17], is natural when one considers splittings of a group.

Definition 2.2.3. Let K be a group with a splitting σ over a subgroup H , and let $\mathcal{S} = \{S_i\}$ be a family of subgroups of K . The splitting σ of K is adapted to \mathcal{S} , or is \mathcal{S} -adapted, if each S_i is conjugate into a vertex group of σ .

There is also a natural generalisation of Definition 2.2.3 to graphs of groups.

Definition 2.2.4. Let K be a group with a graph of groups structure Γ , and let $\mathcal{S} = \{S_i\}$ be a family of subgroups of K . Then Γ is adapted to \mathcal{S} , or is \mathcal{S} -adapted, if each S_i is conjugate into a vertex group of Γ .

In [6], it is shown that these terminologies are all compatible.

In topological terms, this concept seems very natural when one considers manifolds with boundary, and this is how Muller's definition arose. Consider a manifold M , and a codimension-1 embedded submanifold F in the interior of M , such that F is two-sided, closed and π_1 -injective. Then clearly the splitting of $\pi_1(M)$ over $\pi_1(F)$ determined by F is adapted to the family \mathcal{S} of subgroups of M carried by the components of ∂M . On the other hand, if F has boundary and is properly embedded, then the corresponding splitting of $\pi_1(M)$ over $\pi_1(F)$ is likely not to be adapted to \mathcal{S} .

For the purposes of this paper we need to recall from [6] some properties of adapted almost invariant subsets of a group, which we summarize in the following lemma.

Lemma 2.2.5. *Let \overline{G} be a group with subgroups H and G , such that $H \subset G$, and suppose that \overline{G} has a minimal graph of groups decomposition Γ with a vertex V whose associated group is G . Let \mathcal{S} denote the family of subgroups of G associated to the edges of Γ which are incident to V .*

1. *If \overline{X} is a H -almost invariant subset of \overline{G} which is enclosed by V , and if X denotes the H -almost invariant subset $\overline{X} \cap G$ of G , then X is adapted to the family \mathcal{S} .*
2. *Suppose that the group system (G, \mathcal{S}) is of finite type. If X is a nontrivial \mathcal{S} -adapted H -almost invariant subset of G , then X has an extension \overline{X} to \overline{G} , i.e. there is a H -almost invariant subset \overline{X} of \overline{G} which is enclosed by V , such that $\overline{X} \cap G = X$. Further if X is associated to a splitting of G over H , then \overline{X} is associated to a splitting of \overline{G} over H .*

Remark 2.2.6. *As Poincaré duality pairs are automatically group systems of finite type, this condition in part 2) of the lemma can be ignored in this paper.*

We will collect here some useful basic results about Poincaré duality groups and pairs. The following result due to Kropholler and Roller is Lemma 2.2 of [14]. It will be needed at several points in this paper. Similar results are well known in the topology of 3-manifolds. In this setting, cases 1) and 2) of the lemma below occur when one has an I -bundle over a closed surface.

Lemma 2.2.7. *(Kropholler and Roller) Let $(G, \partial G)$ be a $PD(n+2)$ pair with ∂G non-empty. Then one of the following holds:*

1. *G is a $PD(n+1)$ group and ∂G consists of a single group S which has index 2 in G .*
2. *G is a $PD(n+1)$ group and the pair $(G, \partial G)$ is the trivial pair $(G, \{G, G\})$.*
3. *For each group S in ∂G , the index of S in G is infinite, and $\text{Comm}_G(S) = S$. Further if S and S' are distinct groups in ∂G , they are not conjugate commensurable.*

It will also be convenient to state separately the following easy consequences.

Corollary 2.2.8. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair. Then*

1. If ∂G consists of a single group S which has index 2 in G , then G itself must be a non-orientable $PD(n+1)$ group.
2. If S is a group in ∂G , and K is an orientable $PD(n+1)$ subgroup of G commensurable with S , then K is contained in S .
3. If $(G, \partial G)$ splits, adapted to ∂G , over a $PD(n+1)$ subgroup H , and if H is commensurable with an orientable $PD(n+1)$ subgroup K of G , then K is contained in H .

Proof. 1) As $(G, \partial G)$ is an orientable $PD(n+2)$ pair, S must be an orientable $PD(n+1)$ group. Thus G is also a $PD(n+1)$ group. If G were orientable, the fact that S has index 2 in G would imply that the induced map $\mathbb{Z} \cong H_{n+1}(S) \rightarrow H_{n+1}(G) \cong \mathbb{Z}$ would be multiplication by 2. But this map must be zero as $S = \partial G$. This contradiction shows that G must be non-orientable.

2) We apply Lemma 2.2.7. If case 3) of that lemma holds, the fact that K must commensurise S implies that K is contained in S . If case 2) of that lemma holds, the result is trivial. If case 1) of that lemma holds, then part 1) of this lemma tells us that G is a non-orientable $PD(n+1)$ group. Hence S is the orientation subgroup of G , so that any orientable $PD(n+1)$ subgroup of G must be contained in S .

3) First suppose that ∂G is empty so that G is an orientable $PD(n+2)$ group. As G splits over H we have G equal to $A *_H B$ or to $A *_H$ for some subgroups A or B . In the first case, Theorem 8.1 of [1] tells us that each of the pairs (A, H) and (B, H) is an orientable $PD(n+2)$ pair. In the second case, there are two inclusions of H into A whose images we denote by H_1 and H_2 , and Theorem 8.1 of [1] tells us that the pair $(A, \{H_1, H_2\})$ is an orientable $PD(n+2)$ pair. As K is commensurable with H , it must be conjugate into A or B . Now part 2) shows that K is contained in H , as required.

If ∂G is not empty, we recall that the given splitting of G over H is adapted to ∂G . Thus if we consider DG , the double of G over ∂G , and apply part 2) of Lemma 2.2.5, the given splitting of G over H induces a splitting of DG over H . Now we can apply the above argument to the orientable $PD(n+2)$ group DG to deduce that K is contained in H , as required. \square

The following observation will also be useful at several points in this paper. In the setting of 3-manifolds, the corresponding result is that if a Haken manifold M with non-empty incompressible boundary has VPC fundamental group, then M must be an I -bundle over the torus or Klein bottle.

Corollary 2.2.9. *Let $(G, \partial G)$ be a $PD(n+2)$ pair, and suppose that G is VPC . Then one of the following holds:*

1. ∂G is empty and G is $VPC(n+2)$.
2. ∂G is non-empty, G is $VPC(n+1)$, and either $(G, \partial G)$ is the trivial pair $(G, \{G, G\})$, or ∂G is a single group S , and G contains S with index 2.

Proof. If ∂G is empty, then G has cohomological dimension $n+2$, and so G must be $VPC(n+2)$. Thus we have case 1) of the corollary.

If ∂G is non-empty, each group in ∂G is $PD(n+1)$, and is also VPC . Thus each group in ∂G is $VPC(n+1)$. Now G must have cohomological dimension $n+1$, and so G is also $VPC(n+1)$. As this implies that any group in ∂G has finite index in G , Lemma 2.2.7 implies that we must have case 2) of the corollary. \square

Now we will begin our discussion of almost invariant sets in Poincaré duality groups and pairs. But first here are two important facts about such sets which follow immediately from Lemma 4.3 of Kropholler in [11].

Lemma 2.2.10. *(Kropholler and Roller) Let $n \geq 1$, and let $(G, \partial G)$ be a $PD(n+2)$ pair. Then*

1. G has no nontrivial almost invariant subset over a $VPC(< n)$ subgroup.
2. If ∂G is empty, so that G is a $PD(n+2)$ group, then G has no nontrivial almost invariant subset over a $VPC(\leq n)$ subgroup.

Remark 2.2.11. *In the 3-manifold setting, so that $n = 1$, part 1) corresponds to the fact that a Haken manifold M with incompressible boundary must have a one-ended fundamental group, and part 2) corresponds to the additional fact that when M is closed it cannot admit essential annuli.*

The analogy between π_1 -injective maps of surfaces into 3-manifolds and almost invariant subsets of groups was one of the guiding principles in [21] and [22]. In particular, essential maps of annuli and tori into orientable 3-manifolds have corresponding nontrivial almost invariant sets.

We start by describing the analogous correspondence when one considers an orientable $PD(n+1)$ subgroup H of an orientable $PD(n+2)$ group G . As G and H are orientable, it follows that $e(G, H) = 2$. Thus G has a nontrivial H -almost invariant subset X_H which is unique up to equivalence and complementation.

This is the almost invariant subset of G which we associate to H . We call it the *dual* of H . The restriction that H and G be orientable is not crucial to ensure that $e(G, H) = 2$, but it does simplify the statements somewhat. What is crucial is that when we consider the inclusion of H into G , it should commute with the orientation homomorphisms. For otherwise $e(G, H) = 1$, so that G has no nontrivial H -almost invariant subsets. In topological terms, we want our codimension-1 manifolds to have trivial normal bundle or equivalently to be two-sided. Our higher dimensional algebraic analogue of a torus is an orientable $PD(n+1)$ group H which is also $VPC(n+1)$, and then a torus in an orientable $PD(n+2)$ group G is an injective homomorphism $\Phi : H \rightarrow G$. Note that a $PD(n+2)$ group G is torsion free, so that a $VPC(n+1)$ subgroup of G is always $PD(n+1)$.

Next we consider $PD(n+2)$ pairs. If $(G, \partial G)$ is an orientable $PD(n+2)$ pair with nonempty boundary, and if H is an orientable $PD(n+1)$ subgroup of G , then $e(G, H)$ may not equal 2. Thus it is no longer clear how to naturally associate a H -almost invariant subset of G to H . However in the topological context, a map of a codimension-1 closed orientable manifold into an orientable manifold determines a corresponding almost invariant set in a natural way. For simplicity, consider the case when $n = 1$, and G is the fundamental group of an orientable 3-manifold M with incompressible boundary, and H is isomorphic to the fundamental group of a closed orientable surface F . Pick a map of F into the interior of M so that $\pi_1(F)$ maps to H by the given isomorphism, and consider its lift to the cover M_F of M with fundamental group H . Then M_F need not have two ends, so that the group H does not determine a unique H -almost invariant subset of G . But, as F is two-sided in M , the lift of F into M_F separates M_F into two pieces, and so F *does* determine a unique H -almost invariant subset Y of G . Note that Y will be trivial if and only if F is homotopic into a component of ∂M . Note also that as F does not meet ∂M , the associated almost invariant set Y is adapted to ∂G , the family of subgroups of $G = \pi_1(M)$ carried by the components of ∂M .

In the algebraic setting, where $(G, \partial G)$ is an orientable $PD(n+2)$ pair with nonempty boundary and H is an orientable $PD(n+1)$ subgroup of G , we will also associate to H a H -almost invariant subset of G which is adapted to ∂G . We do this by considering the double DG of G along ∂G . Since DG is orientable, H determines a nontrivial H -almost invariant subset X_H of DG . The H -almost invariant subset of G which we associate to H is the intersection $Y = X_H \cap G$. It is clear that Y is a trivial H -almost invariant subset of G if H is conjugate to a subgroup of one of the S_i 's. The following result shows that, as in the topological

situation, this is the only way in which Y can be trivial.

Lemma 2.2.12. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair with nonempty boundary, and let H be an orientable $PD(n+1)$ subgroup of G . Let X_H denote the H -almost invariant subset of DG determined by H , and let Y denote the H -almost invariant subset $X_H \cap G$ of G .*

Then Y is adapted to ∂G . Further, if Y is trivial, then H is conjugate to a subgroup of one of the S_i 's.

Proof. To prove this result, we need to use some techniques from [22]. Let Δ denote the graph of groups structure for DG corresponding to its construction by doubling. Thus Δ has two vertices w and \bar{w} , and edges corresponding to the groups in ∂G , each joining the two vertices. We identify G with the vertex group $G(w)$. As DG is an orientable $PD(n+2)$ group, each S_i determines, up to complementation and equivalence, a unique S_i -almost invariant subset X_i of DG . Thus X_i is associated to the edge splitting of Δ which is over S_i . Since each of H and S_i has two coends in DG , Proposition 7.4 of [22] shows that the associated almost invariant sets X_H and X_i must cross strongly if they cross at all. As $H \subset G = G(w)$, it is clear that X_H cannot cross any X_i strongly, and so does not cross any X_i at all. Thus X_H must be enclosed by one of the vertices of Δ .

If X_H is enclosed by \bar{w} , then H must be a subgroup of $G(\bar{w})$. As H is also a subgroup of $G(w)$, this implies that H is conjugate into some S_i , as required. It also implies that Y is trivial, and so is automatically adapted to ∂G , completing the proof in this case.

For the rest of this proof, we will assume that X_H is enclosed by w . Thus Lemma 2.2.5 shows that $Y = X_H \cap G$ is adapted to ∂G , which proves the first part of the lemma.

Now suppose that Y is trivial, so that one of Y or Y^* is H -finite. We consider the action of DG on the universal covering DG -tree T of Δ . There is a vertex v of T with stabiliser $G = G(v)$ which lies above w such that X_H is enclosed by v . Corollary 4.16 of [22] tells us that X_H determines a nontrivial partition of the edges of T which are incident to v . Suppose that Y is H -finite and let e be an edge of T which is incident to v and on the X_H -side of v . Then X_H is equivalent to an almost invariant subset W of G which contains $G(e)$. The fact that $Y = X_H \cap G(v)$ is H -finite implies that $W \cap G(v)$ is also H -finite, and hence in particular that $G(e)$ itself is H -finite. Thus $H \cap G(e)$ has finite index in $G(e)$. Now $G(e)$ is a conjugate of some S_i . As H and S_i are both $PD(n+1)$, a subgroup of $G(e)$ of finite index is also $PD(n+1)$ and hence of finite index in H .

It follows that H is commensurable with a conjugate of S_i . As H is orientable, part 2) of Corollary 2.2.8 shows that H is conjugate to a subgroup of S_i . This completes the proof of the lemma. \square

The conclusion of the above discussion is that if $(G, \partial G)$ is an orientable $PD(n+2)$ pair with non-empty boundary, and H is an orientable $PD(n+1)$ subgroup of G , we can associate to H a H -almost invariant subset Y of G , and Y will be nontrivial so long as H is not conjugate into some S_i . Further Y is adapted to ∂G . We will say that Y is *dual* to H . When H is $VPC(n+1)$ and Y is nontrivial, we will say that H is an *essential* torus in $(G, \partial G)$. If G is an orientable $PD(n+2)$ group, then, as above, we associate X_H itself to H . In this case, we may also refer to H as an essential torus in G , though the word ‘essential’ is redundant in this case.

For later reference, we briefly consider the situation where $(G, \partial G)$ is an orientable $PD(n+2)$ pair with non-empty boundary, and H is a non-orientable $PD(n+1)$ subgroup of G . Unlike the case when ∂G is empty, G may possess nontrivial H -almost invariant subsets. However the following result says that no such subset of G can be adapted to ∂G .

Lemma 2.2.13. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair with non-empty boundary, let H be a non-orientable $PD(n+1)$ subgroup of G , and let X be a nontrivial H -almost invariant subset of G . Then X is not adapted to ∂G .*

Proof. As usual we let DG denote the double of G along ∂G , so that DG is an orientable $PD(n+2)$ group. We recall that if K is a $PD(n+1)$ subgroup of DG , then $e(DG, K)$ equals 2 if K is orientable and equals 1 otherwise. In particular, DG has no nontrivial H -almost invariant subset.

Suppose that X is adapted to ∂G . Then Lemma 2.2.5 tells us that there is a H -almost invariant subset \bar{X} of DG such that $\bar{X} \cap G$ equals X . As X is a nontrivial almost invariant set, so is \bar{X} . This contradiction shows that X cannot be adapted to ∂G , as required. \square

For the rest of this section, we will discuss essential maps of higher dimensional ‘annuli’ into orientable $PD(n+2)$ pairs. It turns out that we need two types of higher dimensional analogue of an annulus. The first and most obvious type is a trivial orientable $PD(n+1)$ pair $A_H = (H; \{H, H\})$, where $n \geq 1$, and H is an orientable PDn group which is also $VPCn$. We call this an *untwisted annulus*. The second type is an orientable $PD(n+1)$ pair $\Lambda_H = (H, H_0)$, where H is a non-orientable PDn group which is also $VPCn$, and H_0 is the orientation subgroup of H . We call this a *twisted annulus*. When $n = 2$, an example

of a 3-manifold of this type is the twisted I -bundle over the Klein bottle with orientable total space. Note that there are no twisted annuli when $n = 1$, as any $PD1$ group is orientable. In particular, there are no twisted annuli in a 3-manifold. Algebraically, an annulus in a $PD(n + 2)$ pair $(G, \partial G)$ is an injective homomorphism of group pairs $\Theta : A_H \rightarrow (G, \partial G)$, or $\Theta : \Lambda_H \rightarrow (G, \partial G)$. This means that Θ maps H to G and also maps each group in $\partial\Lambda_H$ to a conjugate of some group in ∂G . Again the above orientation restrictions are not crucial, but they do simplify the statements. What is crucial is that Θ should commute with the orientation homomorphisms, so that, in topological terms, our annuli have trivial normal bundle or equivalently are two-sided. The reason for this is that we need to associate a nontrivial H -almost invariant subset of G to each essential annulus. We will do this in Definition [2.2.19](#).

It will be very helpful to consider a map θ of aspherical spaces such that the induced map on fundamental groups is Θ . For this we need to choose a $K(H, 1)$, and use a mapping cylinder construction to make a $K(G, 1)$ with the $K(S_i, 1)$'s as disjoint subcomplexes, for $i \geq 1$. To simplify the notation for an untwisted annulus, we write A for $K(H, 1) \times I$, and ∂A for $K(H, 1) \times \partial I$. The two components of ∂A will be denoted by $\partial_0 A$ and $\partial_1 A$. If $n = 1$, then A can be chosen to be the usual annulus $S^1 \times I$. For a twisted annulus, we write A for the twisted I -bundle over $K(H, 1)$ determined by the orientation homomorphism of H , and write ∂A for the induced S^0 -bundle. Finally we write M for $K(G, 1)$, and ∂M for the union of the $K(S_i, 1)$'s, for $i \geq 1$. Then Θ is induced by a map $\theta : (A, \partial A) \rightarrow (M, \partial M)$. Note that in the untwisted case, such a map θ is determined up to homotopy by choosing a copy of H in two conjugates of groups in ∂G , such that the two copies of H are conjugate in G . And in the twisted case, θ is determined up to homotopy by choosing a copy of H in G and a conjugate of some group in ∂G such that the intersection of H with this conjugate contains H_0 . Thus an annulus can be thought of purely algebraically. We will say that Θ is *essential* if θ cannot be homotoped relative to ∂A into ∂M . It is clear that the essentiality of an annulus is also a purely algebraic property. An untwisted annulus is essential if and only if the images of the two boundary groups are not conjugate in a group in ∂G . And a twisted annulus is essential if and only if H_0 lies in a boundary group K in ∂G , and $H \cap K = H_0$. Note that as G is finitely generated, we can choose M to have finite 1-skeleton. If G is finitely presented, we can also choose M to have finite 2-skeleton, as each S_i is finitely generated.

Now suppose that θ is an essential map of an untwisted annulus A into M , where $\pi_1(A)$ is equal to H , and identify H with its image in G under θ_* . Let

M_H denote the cover of M with fundamental group H , and let \widetilde{M} denote the universal cover of M . Let $\theta_H : (A, \partial A) \rightarrow (M_H, \partial M_H)$ be the lift of θ . The induced map on homology sends the fundamental class $[A] \in H_{n+1}(A, \partial A; \mathbb{Z})$ to an element α of $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$. Let Σ denote the component of ∂M_H which contains $\theta_H(\partial_0 A)$. We will assume that the base point of A lies in $\partial_0 A$. As M_H and Σ are aspherical and the inclusion of Σ in M_H induces an isomorphism of fundamental groups, Σ is a deformation retract of M_H . As θ is essential, it follows that $\theta_H(\partial_1 A)$ must lie in a different component Σ' of ∂M_H . This implies that α is a nontrivial element of $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$.

If θ is an essential map into M of a twisted annulus A with $\pi_1(A)$ equal to H , we again identify H with its image in G under θ_* . Recall that $\pi_1(\partial A)$ equals the orientation subgroup H_0 of H . Let M_H denote the cover of M with fundamental group H , and let $\theta_H : (A, \partial A) \rightarrow (M_H, \partial M_H)$ be the lift of θ . Again we let α denote the image in $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$ of the fundamental class $[A] \in H_{n+1}(A, \partial A; \mathbb{Z})$. Let Σ denote the component of ∂M_H which contains $\theta_H(\partial A)$, and consider the inclusions $H_0 = \pi_1(\partial A) \subset \pi_1(\Sigma) \subset \pi_1(M_H) = H$. If $\pi_1(\Sigma)$ equals H , then M_H deformation retracts to Σ , which contradicts the hypothesis that θ is essential. It follows that we must have $\pi_1(\Sigma) = H_0$. In turn this implies that α is a nontrivial element of $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$. We note that the double cover A_0 of A with $\pi_1(A_0)$ equal to H_0 is an untwisted annulus, and the induced map $\theta_0 : A_0 \rightarrow M_0$ is also essential, where M_0 is the cover of M with fundamental group H_0 .

Remark 2.2.14. *The above discussion shows that if A is an annulus, twisted or untwisted, and $\theta : (A, \partial A) \rightarrow (M, \partial M)$ is an essential map, then in either case the image of the fundamental cycle $[A]$ with \mathbb{Z}_2 -coefficients is also nontrivial in $H_{n+1}(M_H, \partial M_H; \mathbb{Z}_2)$ and is the specialization of α .*

We also note that, whether or not A is twisted, the induced action of H on the universal cover \widetilde{M} of M preserves the union of two distinct components of $\partial \widetilde{M}$. If some element of H interchanges these two components, then A is twisted. Otherwise, A is untwisted.

Conversely, suppose that two distinct components Σ and T of $\partial \widetilde{M}$ are each stabilised by an orientable PDn subgroup H of G . Then there is an essential map θ of an untwisted annulus A to M with $\pi_1(A)$ equal to H , and a lift $\theta_H : A \rightarrow M_H$ which maps $\partial_0 A$ to $H \setminus \Sigma$ and maps $\partial_1 A$ to $H \setminus T$. We denote this essential untwisted annulus in $(M, \partial M)$ by $H_{\Sigma, T}$.

If the union of Σ and T is stabilised by a PDn subgroup H of G and if some element of H interchanges Σ and T , let H_0 denote the subgroup of H of index

2 which stabilises both Σ and T . If H_0 is the orientation subgroup of H , there is an essential map θ of a twisted annulus A to M with $\pi_1(A)$ equal to H , and the induced essential map θ_0 of the double cover A_0 of A with $\pi_1(A_0)$ equal to H_0 has a lift to M_{H_0} which maps $\partial_0 A_0$ to $H \setminus \Sigma$ and maps $\partial_1 A_0$ to $H \setminus T$. We denote this essential twisted annulus in $(M, \partial M)$ by $H_{\Sigma, T}$.

In order to associate an almost invariant subset of G to an essential annulus, we will need the following result, which was proved by Swarup in [30]. See [15] for a purely algebraic proof. The statement we give here is equivalent to Theorem 2 of [30].

Lemma 2.2.15. (Swarup) *Let G be a finitely generated group, and let H be a subgroup of infinite index in G . If φ is a homomorphism from H to \mathbb{Z} , denote the kernel of φ by N . Suppose that whenever φ is non-zero, we have $e(G, N) = 1$. Then the restriction map $r : H^1(G; \mathbb{Z}[H \setminus G]) \rightarrow H^1(H; \mathbb{Z})$ is trivial.*

Proof. As before, we let M denote a $K(G, 1)$ with the $K(S_i, 1)$'s as disjoint sub-complexes, for $i \geq 1$. Recall from Remark 2.1.1 that we can identify $H^1(G, \mathbb{Z}[H \setminus G])$ with $H_f^1(M_H; \mathbb{Z})$. If G is finitely generated but not finitely presented, we need to modify the usual definition of $H_f^1(M_H; \mathbb{Z})$ as discussed there. In any case, M need not be locally finite, so any reference we make to the number of ends of a cover of M really refers to the number of ends of the 1-skeleton of the cover. We also let r denote the natural map $H_f^1(M_H; \mathbb{Z}) \rightarrow H^1(M_H; \mathbb{Z})$.

Let β denote an element of $H_f^1(M_H; \mathbb{Z})$, and let $\gamma \in H^1(M_H; \mathbb{Z})$ denote $r(\beta)$. Thus γ can be represented by a map $g : M_H \rightarrow S^1$. Represent β by a finite cocycle $c : M_H^{(1)} \rightarrow \mathbb{Z}$ on the 1-skeleton $M_H^{(1)}$ of M_H , and let Σ denote the support of c . Thus Σ is a finite subcomplex of $M_H^{(1)}$, and c restricted to any simplex of $M_H^{(1)} - \Sigma$ is trivial. As c also represents γ , for any component L of $M_H^{(1)} - \Sigma$, the map $\pi_1(L) \rightarrow \pi_1(S^1)$ induced by g is trivial. Since Σ is finite, $M_H^{(1)} - \Sigma$ has at least one unbounded component L whose coboundary δL must be finite, as $\delta L \subset \delta \Sigma$.

Now suppose that γ is non-zero. We consider the induced map $g_* : H \rightarrow \mathbb{Z}$, let N denote the kernel of g_* , and consider the cover $q_N : M_N \rightarrow M_H$. As the map $\pi_1(L) \rightarrow \pi_1(S^1)$ induced by g is trivial, L lifts to M_N . As the infinite quotient group H/N acts on M_N , it follows that L has infinitely many disjoint lifts to M_N . This implies that M_N has infinitely many ends, which contradicts the hypothesis that $e(G, N) = 1$. It follows that γ must be zero which completes the proof of the lemma. \square

We will want to apply this result to Poincaré duality pairs. The result we obtain is the following.

Lemma 2.2.16. *Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair, and let H be a PDn subgroup of G . Then the restriction map $r : H^1(G; \mathbb{Z}[H \setminus G]) \rightarrow H^1(H; \mathbb{Z})$ is trivial.*

Remark 2.2.17. *The PDn subgroup H need not be orientable.*

Proof. As before, we let M denote a $K(G, 1)$ with the $K(S_i, 1)$'s as disjoint sub-complexes, for $i \geq 1$. We need to check all the hypotheses of Lemma 2.2.15. Certainly G is finitely generated and H has infinite index in G . Now suppose that N is the kernel of a non-zero homomorphism from H to \mathbb{Z} . We need to show that $e(G, N) = 1$. As N has infinite index in the PDn group H , a theorem of Strebel [29] tells us that N has cohomological dimension $\leq n - 1$. Hence $H_k(N; \mathbb{Z}) = 0$, for any $k \geq n$, and the same holds for any subgroup of N . In particular, $H_{n+1}(M_N; \mathbb{Z})$ and $H_n(\partial M_N; \mathbb{Z})$ are both zero. It follows from the exact sequence of the pair $(M_N, \partial M_N)$ that $H_{n+1}(M_N, \partial M_N; \mathbb{Z})$ is zero. By Poincaré duality, this implies that $H_f^1(M_N; \mathbb{Z})$ is zero, so that M_N has only one end. Hence $e(G, N) = 1$ as required. \square

Our real interest lies in the corresponding restriction map with \mathbb{Z}_2 coefficients in place of \mathbb{Z} . This map need not be trivial, but the fact that r is trivial yields enough information about the case of \mathbb{Z}_2 coefficients for our purposes.

Corollary 2.2.18. *Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair, and let H be a PDn subgroup of G . Let ρ denote the map of cohomology groups given by reduction of the coefficients modulo 2. Then the image of $\rho : H^1(G; \mathbb{Z}[H \setminus G]) \rightarrow H^1(G; \mathbb{Z}_2[H \setminus G])$ is contained in the image of the coboundary map $\delta : H^0(G; P[H \setminus G]/\mathbb{Z}_2[H \setminus G]) \rightarrow H^1(G; \mathbb{Z}_2[H \setminus G])$, given on page 18.*

Proof. As before, we let M denote a $K(G, 1)$ with finite 1-skeleton, and let M_H denote the cover of M with fundamental group H .

Consider the diagram

$$\begin{array}{ccccc}
 & & H^1(G; \mathbb{Z}[H \setminus G]) & \xrightarrow{r} & H^1(H; \mathbb{Z}) \\
 & & \downarrow \rho & & \downarrow \rho \\
 H^0(G; P[H \setminus G]/\mathbb{Z}_2[H \setminus G]) & \xrightarrow{\delta} & H^1(G; \mathbb{Z}_2[H \setminus G]) & \xrightarrow{\bar{r}} & H^1(H; \mathbb{Z}_2) \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 H_e^0(M_H; \mathbb{Z}_2) & \xrightarrow{\delta} & H_f^1(M_H; \mathbb{Z}_2) & \xrightarrow{\bar{r}} & H^1(M_H; \mathbb{Z}_2)
 \end{array}$$

where the bottom two rows come from the long exact cohomology sequences given before Remark 2.1.1. Lemma 2.2.16 tells us that the map r is zero. As the image of δ equals the kernel of the map \bar{r} , it follows immediately that the image of ρ is contained in the image of δ . \square

Now we can associate an almost invariant set to an essential annulus as promised.

Definition 2.2.19. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let θ be an essential annulus in $(M, \partial M)$ with fundamental group H .*

As discussed before Lemma 2.2.15, the essential annulus θ determines a non-zero element $\alpha \in H_{n+1}(M_H, \partial M_H; \mathbb{Z}) \cong H_{n+1}(G, \partial G; \mathbb{Z}[H \setminus G])$. The Poincaré dual of α , regarded as an element of this second group, is a nontrivial element β of $H^1(G; \mathbb{Z}[H \setminus G])$. Corollary 2.2.18 shows that $\rho(\beta)$ is contained in the image of δ , and so, as discussed on page 18, determines an almost invariant subset of $H \setminus G$ modulo almost equality and complementation. The pre-image in G of such a set is a H -almost invariant subset X_θ of G .

We will say that X_θ is dual to the essential annulus θ . On occasion, it will also be convenient to say that the almost invariant subset $H \setminus X$ of $H \setminus G$ is dual to θ .

If $(G, \partial G)$ is an orientable $PD(n+2)$ pair, and the almost invariant subset of G associated to a splitting is dual to an essential annulus or torus, we will say that the splitting itself is dual to an essential annulus or torus, as appropriate.

If $(G, \partial G)$ is an orientable $PD(n+2)$ pair which admits an essential annulus, the dual almost invariant subset of G is nontrivial and is over a $VPCn$ subgroup. The converse is not true. In general G will have many nontrivial such subsets which are not dual to any essential annulus. In the following two results, we consider this in more detail. In particular, we show that if G has a nontrivial almost invariant subset over a $VPCn$ subgroup, then $(G, \partial G)$ admits an essential annulus.

Proposition 2.2.20. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair and let H be a $VPCn$ subgroup of G . Let α be an element of $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$, so that $\partial\alpha \in H_n(\partial M_H; \mathbb{Z})$ is supported by some finite number k of components of ∂M_H . Then the following statements hold:*

1. *If α is non-zero, then k is non-zero.*
2. *If α is non-zero, then each of the k components of ∂M_H which support $\partial\alpha$ carries a subgroup of finite index in H . If H is orientable then $k \geq 2$.*

3. If G has a nontrivial H -almost invariant subset, then $(G, \partial G)$ admits an essential map of an annulus whose fundamental group is a subgroup of finite index in H .

Proof. 1) Consider the exact sequence.

$$H_{n+1}(M_H; \mathbb{Z}) \rightarrow H_{n+1}(M_H, \partial M_H; \mathbb{Z}) \xrightarrow{\partial} H_n(\partial M_H; \mathbb{Z}) \rightarrow H_n(M_H; \mathbb{Z})$$

Since H is a PDn group, it follows that $H_{n+1}(M_H; \mathbb{Z})$ is zero. Thus the map ∂ in this sequence is injective. If α is non-zero, it follows that $\partial\alpha$ is non-zero, and hence that k is non-zero.

2) If Σ is one of the components of ∂M_H which supports $\partial\alpha$, then $H_n(\Sigma; \mathbb{Z})$ must be nontrivial. Note that the fundamental group of any component of ∂M_H is a subgroup of H . As H is PDn , Strebel's result in [29] implies that a subgroup of H of infinite index has cohomological dimension less than n . Hence $\pi_1(\Sigma)$ must have finite index d in H , so that $H_n(\Sigma; \mathbb{Z})$ is infinite cyclic. Now suppose that H is orientable. Then $H_n(M_H; \mathbb{Z})$ must also be infinite cyclic. Further the map $H_n(\Sigma; \mathbb{Z}) \rightarrow H_n(M_H; \mathbb{Z})$ is multiplication by d , and so is injective. We know that the map $H_n(\partial M_H; \mathbb{Z}) \rightarrow H_n(M_H; \mathbb{Z})$ has nontrivial kernel as it contains $\partial\alpha$. It follows that ∂M_H has a second boundary component Σ' which carries $\partial\alpha$. Thus when H is orientable, we must have $k \geq 2$.

3) If G has a nontrivial H -almost invariant subset, Corollary 2.2.18 implies that $H^1(G; \mathbb{Z}[H \setminus G])$, and hence $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$, is nontrivial. Suppose first that H is orientable. Then part 2) shows that there are two distinct components Σ and Σ' of ∂M_H such that each of $\pi_1(\Sigma)$ and $\pi_1(\Sigma')$ is a subgroup of finite index in H . It follows that $(G, \partial G)$ admits an essential map of an untwisted annulus whose fundamental group is a subgroup of finite index in H . If H is non-orientable, we consider the orientation subgroup H_0 of H which is of index 2. As H_0 is a subgroup of H of finite index, G has a nontrivial H_0 -almost invariant subset. Now we apply the above discussion to H_0 in place of H and obtain an essential map of an untwisted annulus whose fundamental group is a subgroup of finite index in H_0 , and hence of finite index in H , as required. This completes the proof of the proposition. \square

In the next result, we prove more.

Proposition 2.2.21. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair and let H be a $VPCn$ subgroup of G . Then we have the following results.*

1. No nontrivial H -almost invariant subset of G can be adapted to ∂G .

2. If G has a nontrivial H -almost invariant subset X , there is an orientable subgroup H' of finite index in H , such that X is equivalent to a sum of H' -almost invariant subsets of G each dual to an untwisted annulus.
3. If ∂M_H has k components each of which carries a subgroup of finite index in H , then the number of coends of H in G is at least k .
4. If the number of coends of H in G is at least k , then H has a subgroup L of finite index such that ∂M_L has k components each of which carries L .
5. If G has a nontrivial H -almost invariant subset X which crosses no nontrivial almost invariant subset of G over any finite index subgroup of H , then X is dual to an annulus.

Proof. 1) Suppose there is a nontrivial H -almost invariant subset Y of G which is adapted to ∂G . We will consider the $PD(n+2)$ group DG obtained by doubling G along ∂G , and the corresponding graph of groups decomposition Δ of DG , which has a vertex w with associated group G . Lemma 2.2.5 tells us that there is a nontrivial H -almost invariant subset X of G which is enclosed by w such that $X \cap G$ equals Y . But part 2) of Lemma 2.2.10 tells us that DG has no nontrivial almost invariant subsets over $VPCn$ subgroups. This contradiction shows that no nontrivial H -almost invariant subset of G can be adapted to ∂G , as required.

2) Let X be a nontrivial H -almost invariant subset of G , and let Y denote the almost invariant subset $H \setminus X$ of $H \setminus G$. As discussed before Remark 2.1.1, the equivalence class of Y under almost equality is an element of $H^0(G; P[H \setminus G]/\mathbb{Z}_2[H \setminus G])$. We let $[Y]$ denote the image of this equivalence class in $H^1(G; \mathbb{Z}_2[H \setminus G])$ under the coboundary map δ given on page 18. Thus $[Y]$ is represented by any almost invariant subset of $H \setminus G$ which is almost equal to Y or to Y^* . As H is $VPCn$, and torsion free, it is PDn . Thus Corollary 2.2.18 tells us that there is an element β of $H^1(G; \mathbb{Z}[H \setminus G])$ such that $\rho(\beta) = [Y]$. Let α denote the element of $H_{n+1}(G, \partial G; \mathbb{Z}[H \setminus G])$ which is Poincaré dual to β . Regard α as an element of $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$, and consider $\partial\alpha \in H_n(\partial M_H; \mathbb{Z})$. From Proposition 2.2.20, $\partial\alpha$ is non-zero, so there is at least one component of ∂M_H which lies in its support. Each component Σ of ∂M_H which supports $\partial\alpha$ carries a subgroup of H of finite index. Let H'' denote the intersection of all conjugates in H of these subgroups, let H_0 denote the maximal orientable subgroup of H of index at most 2, and let H' denote the intersection $H'' \cap H_0$. Thus H' is an orientable normal subgroup of H of finite index. By replacing H by H' , we can assume that H

is orientable and that the finitely many components $\Sigma_1, \dots, \Sigma_k$ of ∂M_H which support $\partial\alpha$ all carry H . As H is orientable, we must have $k \geq 2$.

If i and j are distinct integers, there is an essential untwisted annulus A_{ij} in M_H with fundamental group H and whose boundary components lie in Σ_i and Σ_j . Let α_{ij} denote the image of the fundamental class of A_{ij} in $H_{n+1}(M_H, \partial M_H; \mathbb{Z})$. Then $\partial\alpha_{ij} \in H_n(\partial M_H; \mathbb{Z})$ is supported by Σ_i and Σ_j . Regard α_{ij} as an element of $H_{n+1}(G, \partial G; \mathbb{Z}[H \setminus G])$, and let $\beta_{ij} \in H^1(G; \mathbb{Z}[H \setminus G])$ be Poincaré dual to α_{ij} . Recall that ρ denotes reduction of coefficients mod 2, and that Corollary 2.2.18 tells us that $\rho(\beta_{ij}) \in H^1(G; \mathbb{Z}_2[H \setminus G])$ equals $[Y_{ij}]$, for some almost invariant subset Y_{ij} of $H \setminus G$ which is said to be dual to A_{ij} .

Let \bar{Y} denote $\rho\partial\alpha \in H_n(\partial M_H; \mathbb{Z}_2)$, and let \bar{Y}_{ij} denote $\rho\partial\alpha_{ij} \in H_n(\partial M_H; \mathbb{Z}_2)$. Then \bar{Y}_{ij} is supported by Σ_i and Σ_j . Let γ denote any nontrivial element of $H_{n+1}(M_H, \partial M_H; \mathbb{Z}_2)$ whose image in $H_n(\partial M_H; \mathbb{Z}_2)$ is supported by some subset of the Σ_i 's. As each Σ_i carries H , this image must be supported by at least two components of ∂M_H . Thus a simple induction argument on k shows that \bar{Y} must be equal to a sum of \bar{Y}_{ij} 's.

Now consider the following commutative diagram. The vertical maps are Poincaré duality isomorphisms, the top horizontal map is the boundary map in the long exact homology sequence of the pair $(M_H, \partial M_H)$, and the bottom horizontal map is induced by the inclusion of ∂M_H into M_H .

$$\begin{array}{ccc} H_{n+1}(M_H, \partial M_H; \mathbb{Z}_2) & \xrightarrow{\partial} & H_n(\partial M_H; \mathbb{Z}_2) \\ \downarrow \cong & & \downarrow \cong \\ H_f^1(M_H; \mathbb{Z}_2) & \xrightarrow{i^*} & H_f^1(\partial M_H; \mathbb{Z}_2) \end{array}$$

As H is PDn , it follows that $H_{n+1}(M_H; \mathbb{Z}_2)$ is zero. Thus the map ∂ in this diagram must be injective, so that i^* is also injective.

Recall that $H^1(G; \mathbb{Z}_2[H \setminus G])$ and $H_f^1(M_H; \mathbb{Z}_2)$ are naturally isomorphic, so that we can identify $[Y]$ and $[Y_{ij}]$ with elements of $H_f^1(M_H; \mathbb{Z}_2)$. Now, under Poincaré duality, $i^*[Y]$ corresponds to $\bar{Y} \in H_n(\partial M_H; \mathbb{Z}_2)$, and $i^*[Y_{ij}]$ corresponds to \bar{Y}_{ij} . As \bar{Y} is equal to a sum of \bar{Y}_{ij} 's, and i^* is injective, it follows that $[Y]$ is equal to a sum of $[Y_{ij}]$'s. Thus Y is equivalent to a sum of Y_{ij} 's and their complements. Hence X is equivalent to a sum of H -almost invariant subsets of G each dual to an untwisted annulus in M_H , as required.

3) Suppose that ∂M_H has k components each of which carries a subgroup of finite index in H . In order to show that the number of coends of H in G is at least k , it suffices to show there is a subgroup H' of H of finite index such that $e(G, H')$ is at least k . As in part 2), by replacing H by a suitable subgroup of finite

index, we can suppose that ∂M_H has components $\Sigma_1, \dots, \Sigma_k$ each carrying H . Consider the composite map

$$H_e^0(M_H; \mathbb{Z}_2) \xrightarrow{\delta} H_f^1(M_H; \mathbb{Z}_2) \xrightarrow{i^*} H_f^1(\partial M_H; \mathbb{Z}_2) \xrightarrow{\cong} H_n(\partial M_H; \mathbb{Z}_2).$$

If $1 \leq i < j \leq k$, the almost invariant subset Y_{ij} of $H \setminus G$ (in part 2) determines an element of $H_e^0(M_H; \mathbb{Z}_2)$ whose image $\overline{Y_{ij}}$ in $H_n(\partial M_H; \mathbb{Z}_2)$ under this composite map is supported by Σ_i and Σ_j . As the $\overline{Y_{ij}}$'s span a $(k-1)$ -dimensional subgroup of $H_n(\partial M_H; \mathbb{Z}_2)$, and the kernel of δ is nontrivial, it follows that $H_e^0(M_H; \mathbb{Z}_2)$ has dimension at least k , so that $e(G, H)$ is at least k , as required.

4) If the number of coends of H in G is at least k , then H has a subgroup H_1 of finite index such that $e(G, H_1) \geq k$. It follows from part 2) of this proposition that H_1 has a subgroup L of finite index such that the space of L -almost invariant subsets of G spanned by such sets which are dual to an untwisted annulus has dimension at least k . Pick a finite family of L -almost invariant subsets of G , each dual to an untwisted annulus, which together span a space with dimension at least k . The corresponding annuli in M_L have boundaries in a finite family of components of ∂M_L . As in part 2), we can replace L by a subgroup of finite index so that all these boundary components carry L . It follows immediately that there must be at least k such components of ∂M_L , as required.

5) Let X be a nontrivial H -almost invariant subset of G which crosses no nontrivial almost invariant subset of G over any finite index subgroup of H . In part 2) of this lemma, we showed that, after replacing H by a suitable subgroup of finite index, the almost invariant subset $Y = H \setminus X$ of $H \setminus G$ is equivalent to a sum of almost invariant subsets Y_{ij} , $i \neq j$, of $H \setminus G$ where Y_{ij} is dual to an untwisted annulus in M_H with boundary in $\Sigma_i \cup \Sigma_j$. By re-labelling the Σ_i 's if needed, we can assume that $1 \leq i, j \leq m$ and that each index between 1 and m occurs.

If $m = 2$, then Y is equivalent to Y_{12} , and we are done. We will show that no other case is possible.

If $m = 3$, then, after renumbering, Y must be equivalent to $Y_{12} + Y_{23}$. But this implies that \overline{Y} is supported on Σ_1 and Σ_3 which contradicts our assumption that $m = 3$.

Now suppose that $m \geq 4$. Recall that an almost invariant subset Y of $H \setminus G$ determines an element $[Y]$ of $H_e^0(M_H; \mathbb{Z}_2)$ which can be represented by a 0-cochain with finite coboundary. The support Z of this cochain is an infinite subset of vertices in M_H , with infinite complement Z^* . Similarly each Y_{ij} yields a corresponding infinite subset of vertices Z_{ij} in M_H , with infinite complement

Z_{ij}^* . It will be convenient to consider Z and the Z_{ij} 's rather than Y and the Y_{ij} 's. Lemma 2.2.22 below shows that there are distinct integers i, j, k and l such that the vertex set of Σ_k is almost contained in Z_{ij} , and the vertex set of Σ_l is almost contained in Z_{ij}^* . This implies that the four corners of the pair (Z, Z_{ij}) are infinite, as Z and Z^* each meet both Σ_k and Σ_l in an infinite set of vertices. Hence the four corners of the pair (Y, Y_{ij}) are infinite, so that Y crosses Y_{ij} . This contradicts our hypothesis that X crosses no nontrivial almost invariant subset of G over any finite index subgroup of H , which completes the proof of the lemma. \square

In the next lemma we consider how two untwisted annuli can cross, by which we mean that the dual almost invariant sets cross. The corresponding picture in the 3-manifold setting is very simple. Start with a 2-disc D with four disjoint open intervals in its boundary. Then remove the rest of ∂D . The resulting surface has four boundary components, and it is trivial that of the six arcs which join pairs of distinct boundary components, there are two which cross. The product of this manifold with S^1 is a 3-manifold M with four annulus boundary components, and there are two annuli in M which cross.

Lemma 2.2.22. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair and let H be an orientable $VPCn$ subgroup of G . Suppose that $\Sigma_1, \dots, \Sigma_4$ are distinct components of ∂M_H each with fundamental group H . Let Y_{ij} denote the almost invariant subset of $H \setminus G$ dual to the annulus A_{ij} in M_H which has fundamental group H and has boundary in Σ_i and Σ_j . Then there are distinct integers i, j, k and l such that Y_{ij} crosses Y_{kl} .*

Proof. As at the end of the previous lemma, it will be convenient to consider Z_{ij} rather than Y_{ij} , where Z_{ij} is the support of a 0-cochain on M_H with finite coboundary which represents the element of $H_e^0(M_H; \mathbb{Z}_2)$ determined by Y_{ij} .

For three distinct integers i, j and k , the vertex set of Σ_k must be almost contained in Z_{ij} or Z_{ij}^* . For simplicity we will say that Σ_k is almost contained in Z_{ij} to mean that the vertex set of Σ_k is almost contained in Z_{ij} . Note that Σ_i and Σ_j are not almost contained in Z_{ij} or in Z_{ij}^* .

Now let i, j, k and l be distinct integers. We will say that Z_{ij} separates Σ_k and Σ_l if Σ_k is almost contained in Z_{ij} , and Σ_l is almost contained in Z_{ij}^* , or vice versa.

Claim: Y_{ij} crosses Y_{kl} if and only if Z_{ij} separates Σ_k and Σ_l , and Z_{kl} separates Σ_i and Σ_j .

Suppose first that Z_{ij} separates Σ_k and Σ_l . Then Y_{ij} must cross Y_{kl} , because all four corners of the pair (Z_{ij}, Z_{kl}) will be infinite as each has infinite intersection with Σ_k or Σ_l .

Next suppose that Z_{ij} does not separate Σ_k and Σ_l . Without loss of generality, we can assume that Σ_k and Σ_l are both almost contained in Z_{ij} . Thus each of the corners $Z_{ij}^* \cap Z_{kl}$ and $Z_{ij}^* \cap Z_{kl}^*$ intersects Σ_k and Σ_l in a finite set. As each of Σ_i and Σ_j is almost contained in one of Z_{kl} or Z_{kl}^* , it follows that one of these two corners intersects at most one of Σ_i and Σ_j in an infinite set. Without loss of generality, we can suppose that this corner is $Z_{ij}^* \cap Z_{kl}$, which we denote by V . Let W denote $Y_{ij}^* \cap Y_{kl}$. Thus W is an almost invariant subset of $H \setminus G$ and the corresponding element $[W]$ of $H_f^1(M_H; \mathbb{Z}_2)$ is represented by a 1-cocycle equal to the coboundary δV . Our choice of V means that at most one component of ∂M_H is not almost contained in V^* . This implies that the element \overline{W} of $H_n(\partial M_H; \mathbb{Z}_2)$ is supported on at most one component of ∂M_H . As H is orientable, part 2) of Proposition 2.2.20 implies that W must be trivial. It follows that Y_{ij} and Y_{kl} do not cross, which completes the proof of the claim.

Now suppose that the lemma is false. Then the above claim shows that, for any four distinct integers i, j, k and l , the boundary components Σ_k and Σ_l must both be almost contained in Z_{ij} or both in Z_{ij}^* . By replacing each of Z_{12} and Z_{23} by its complement if needed, we can arrange that Z_{12} meets each of Σ_3 and Σ_4 in a finite set and that Z_{23} meets each of Σ_1 and Σ_4 in a finite set. Note that we cannot have $Y_{12} \leq Y_{23}$ as Z_{12} meets Σ_1 in an infinite set, and Z_{23} meets Σ_1 in a finite set. And we cannot have $Y_{23} \leq Y_{12}$ as Z_{23} meets Σ_3 in an infinite set and Z_{12} meets Σ_3 in a finite set. It follows that $Y_{12} \cup Y_{23}$ is equivalent to Y_{13} or its complement. Thus $Z_{12} \cup Z_{23}$ is almost equal to Z_{13} or Z_{13}^* . Now $Z_{12} \cup Z_{23}$ meets Σ_4 in a finite set, and has infinite intersection with Σ_2 . Hence it is not the case that Σ_2 and Σ_4 are both almost contained in Z_{13} or in Z_{13}^* , so that Z_{13} separates Σ_2 and Σ_4 . Now the above claim shows that Y_{13} and Y_{24} must cross. This contradicts our supposition, which completes the proof of the lemma. \square

Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair and, as usual, let M be an aspherical space with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Let DM denote the space obtained by doubling M along ∂M , and let DG denote the fundamental group of DM . If M is a 3-manifold, an annulus in M can be doubled to yield a torus in DM . If the annulus is inessential, it can be homotoped to have image a loop in ∂M . Thus the torus can be homotoped to have the same image and so is not π_1 -injective. However, if the annulus is essential, then the torus will be π_1 -injective. The same

construction works in the more general setting of this section. Doubling a topological annulus $f : (A, \partial A) \rightarrow (M, \partial M)$ yields a map $Df : DA \rightarrow DM$, where DA denotes the double of A along its boundary. Thus Df is a map of a torus into DM . Again, if f is inessential, Df will not be π_1 -injective, but if f is essential, then Df will be π_1 -injective. To see this, first suppose that A is untwisted, so that $(A, \partial A)$ is of the form $(C \times I, C \times \partial I)$, let $H = \pi_1(A)$, and consider the lift f_H of f to the cover M_H of M such that $\pi_1(M_H) = H$. As f is essential in $(M, \partial M)$, the images $f_H(C \times \{0\})$ and $f_H(C \times \{1\})$ must lie in distinct components of ∂M_H . It follows that a component of the pre-image of $Df(DA)$ in $(DM)_H$ is homeomorphic to $C \times \mathbb{R}$, so that Df must be π_1 -injective. If A is twisted, we simply apply the above argument to the untwisted double cover of A . Now part 3) of Proposition [2.2.20](#) implies that if G has any nontrivial almost invariant subset over a $VPCn$ subgroup, then there is an essential torus in DG .

In terms of almost invariant sets, the preceding discussion shows that if Y is a nontrivial H -almost invariant set dual to an essential annulus in an orientable $PD(n+2)$ pair $(G, \partial G)$, then there is a natural way to double Y . One obtains a nontrivial almost invariant subset X of DG , such that $X \cap G$ equals Y , and X is over the double of H , i.e. an essential torus in DG . At first sight, this result sounds somewhat similar to that in Lemma [2.2.5](#), as both results are about constructing an almost invariant subset X of DG from an almost invariant subset Y of G . However they are completely different as Lemma [2.2.5](#) requires that Y be adapted to ∂G , whereas a nontrivial H -almost invariant set over a $VPCn$ subgroup of G is never adapted to ∂G , by part 1) of Proposition [2.2.21](#).

2.3 The main theorem

In the previous section, we discussed the analogues in an orientable $PD(n+2)$ pair of annuli and tori in a 3-manifold. In this section we will finally state our main theorem, but first we need to discuss the analogues of the various types of component of the characteristic submanifold of a 3-manifold. Recall that if $(G, \partial G)$ is a Poincaré duality pair our aim is to produce a bipartite graph of groups structure for G in which V_0 -vertices are analogous to components of the characteristic submanifold. In particular if G is the fundamental group of a Haken 3-manifold M , this graph of groups structure is dual to the frontier of the characteristic submanifold of M . In our earlier discussion in section [2.1](#) we described only two types of such component, namely I -bundles and Seifert fibre spaces. But in order to describe the algebraic analogues correctly, we will

need to subdivide into several cases. There are special cases when an I -bundle has infinite cyclic fundamental group, and we also need to distinguish between Seifert fibre spaces depending on how they meet ∂M .

We start by considering a component W of the characteristic submanifold of an orientable Haken 3-manifold M such that W is an I -bundle over a surface F , and F is not an annulus or Moebius band. Thus $\pi_1(W)$ equals $\pi_1(F)$, so is not finite nor two-ended, and the frontier of W in M consists of the restriction of the I -bundle to ∂F and so consists of essential annuli. In addition, if F is orientable, the I -bundle is trivial and W meets ∂M in two copies of F , and if F is non-orientable, then the I -bundle is nontrivial and W meets ∂M in one copy of the orientable double cover of F . Let \tilde{F} , \tilde{W} and \tilde{M} denote the universal covers of F , W and M respectively. Thus \tilde{W} is homeomorphic to $\tilde{F} \times I$. Further each component of the pre-image in \tilde{M} of W consists of a copy of \tilde{W} such that $\tilde{W} \cap \partial\tilde{M} = \tilde{F} \times \{0, 1\}$, and $\tilde{F} \times \{0\}$ and $\tilde{F} \times \{1\}$ lie in distinct components of $\partial\tilde{M}$. Thus the induced action of $\pi_1(W)$ on \tilde{M} preserves the union of two distinct components of $\partial\tilde{M}$. If some element of $\pi_1(W)$ interchanges these two components, then W is a twisted I -bundle. Otherwise, W is untwisted. This leads to the following definition.

Definition 2.3.1. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let M be an aspherical space with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Let Γ be a minimal graph of groups decomposition of G , and let v be a vertex of Γ which is of $VPC(n-1)$ -by-Fuchsian type. (See Definition [2.1.7](#). Note that each peripheral subgroup of $G(v)$ is $VPCn$.)*

Then v is of I -bundle type if there are two distinct components Σ and T of $\partial\tilde{M}$ such that

1. *the induced action of $G(v)$ on \tilde{M} preserves the union of Σ and T , and*
2. *for each peripheral subgroup K of $G(v)$, if e_K denotes the edge of Γ which is incident to v and carries K , then the edge splitting associated to e_K is given by the essential annulus $K_{\Sigma, T}$. (See the discussion just before Lemma [2.2.15](#).)*

Next we consider a component W of the characteristic submanifold of M such that W is a Seifert fibre space, and the orbifold fundamental group of the base orbifold of W is not finite nor two-ended. Thus $\pi_1(W)$ is $VPC1$ -by-Fuchsian, the frontier of W in M consists of boundary torus components or of vertical annuli in its boundary, and W meets ∂M in boundary torus components

or vertical annuli in its boundary. We distinguish three types of such components W .

We will say that W is an *interior* component if it lies in the interior of M . Otherwise W is a *peripheral* component. Our first definition is the algebraic analogue of an interior component of the characteristic submanifold of a 3-manifold.

Definition 2.3.2. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let Γ be a minimal graph of groups decomposition of G . Let v be a vertex of Γ which is of $VPCn$ -by-Fuchsian type. (See Definition 2.1.7)*

Then v is of interior Seifert type if each edge of Γ which is incident to v determines a splitting of G over an essential torus.

Recall that Lemma 2.1.9 shows that the $VPCn$ normal subgroup L of $G(v)$ with Fuchsian quotient is unique. Note that as G is torsion free, so is L , so that L is PDn . The following little result applied to any edge group K of v tells us that if v is of interior Seifert type, then L must be orientable. Note that L is normal in K with quotient which must be isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, as these are the only possible peripheral subgroups of a finitely generated Fuchsian group. Of course this question did not arise in the case of a 3-manifold as then L was the fundamental group of a closed 1-manifold, and the only such manifold is orientable.

Lemma 2.3.3. *Let K be an orientable $PD(n+1)$ group, and let L be a $VPCn$ normal subgroup of K with quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$. Then L is an orientable PDn group.*

Proof. As K is torsion free, so is L , and hence L is PDn . As $\mathbb{Z}_2 * \mathbb{Z}_2$ has an infinite cyclic subgroup of index 2, there is a subgroup K_0 of K , of index at most 2, such that K_0 contains L , and L is normal in K_0 with infinite cyclic quotient. As K is orientable, so is K_0 . Now Theorem 7.3 of [1] shows that L is orientable. \square

We will say that a Seifert fibre space component W of the characteristic submanifold of M is *adapted to ∂M* if there are no annuli in its frontier. In this case each boundary torus of W is either a component of ∂M or lies in the interior of M and so is a component of the frontier of W in M . Our next definition is the algebraic analogue of such a component of the characteristic submanifold of a 3-manifold.

Definition 2.3.4. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let Γ be a minimal graph of groups decomposition of G . Let v be a vertex of Γ such that*

$G(v)$ is a $VPCn$ -by-Fuchsian group, where the Fuchsian group is not finite nor two-ended.

Then v is of Seifert type adapted to ∂G if the following conditions hold:

1. If K is a peripheral subgroup of $G(v)$, then either K is a conjugate of a group in ∂G , or K is carried by an edge of Γ which is incident to v .
2. For each peripheral subgroup K of $G(v)$, there is at most one edge which is incident to v and carries K .
3. Each edge of Γ which is incident to v carries a peripheral subgroup of $G(v)$ and determines a splitting of G over an essential torus in G .

Remark 2.3.5. If v is of Seifert type adapted to ∂G , the two possibilities in 1) for each peripheral subgroup K of $G(v)$ are mutually exclusive. For an essential torus in G cannot be a conjugate of a group in ∂G .

Note that if v is of interior Seifert type, it is automatically of Seifert type adapted to ∂G .

If W is not adapted to ∂M , we can push into the interior of W those components of ∂W which meet ∂M in annuli to obtain a Seifert fibre space W' which is homeomorphic to W and adapted to ∂M . Note that each component of the closure of $W - W'$ is homeomorphic to $T \times I$. Recall that the annuli in which W meets ∂M must be vertical in W . Our next definition is the algebraic analogue of such a component of the characteristic submanifold of a 3-manifold M .

Definition 2.3.6. Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let Γ be a minimal graph of groups decomposition of G . Let v be a vertex of Γ such that $G(v)$ is a $VPCn$ -by-Fuchsian group, where the Fuchsian group is not finite nor two-ended. Let L denote the $VPCn$ normal subgroup of $G(v)$ with Fuchsian quotient.

Then v is of Seifert type if Γ can be refined by splitting at v to a graph of groups structure Γ' of G with the following properties:

1. There is a vertex v' of Γ' with $G(v') = G(v)$ such that v' is of Seifert type adapted to ∂G . Thus each edge of Γ' which is incident to v' carries a peripheral subgroup of $G(v')$ and determines a splitting of G over an essential torus in G .
2. The projection map $\Gamma' \rightarrow \Gamma$ sends v' to v and is an isomorphism apart from the fact that certain edges incident to v' are collapsed to v .

3. Let e denote an edge of Γ' which is incident to v' and collapsed to v . Thus $G(e)$ is a peripheral subgroup K of $G(v')$. Let w denote the other vertex of e . Then $G(w) = K$, and there is at least one other edge incident to w . Further for each such edge the associated edge splitting is dual to an essential annulus, and the boundary of each such annulus carries L .

Remark 2.3.7. When comparing this definition with the topological situation, think of Γ as being dual to the frontier of W , and the refinement Γ' as being dual to the union of the frontiers of W and of W' . Note that if v is of Seifert type adapted to ∂G , then v is trivially of Seifert type. One simply takes Γ' equal to Γ in the above definition.

Part 3) of the definition corresponds to the facts that each component of the closure of $W - W'$ is homeomorphic to $T \times I$, and that the frontier annuli of W must be vertical in W . The reason for the formulation involving the boundary of each annulus is that some of the annuli involved may be twisted, a phenomenon with no analogue in 3-manifold theory.

There are some special cases which are not covered by the above definitions. These occur when the Fuchsian quotient group of a vertex group is finite or two-ended. In the case of the characteristic submanifold of an orientable Haken 3-manifold, such a vertex corresponds to a component which is homeomorphic to one of $S^1 \times D^2$, $T \times I$, or a twisted I -bundle over the Klein bottle.

Definition 2.3.8. Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair, and let Γ be a minimal graph of groups decomposition of G . Let v be a vertex of Γ such that $G(v)$ is a $VPCn$ group.

Then v is of solid torus type if it is not isolated, and for each edge of Γ which is incident to v the associated edge splitting is dual to an essential annulus, and there is a $VPCn$ subgroup H of finite index in $G(v)$ such that the boundary of each such annulus carries H .

A vertex of solid torus type is of special solid torus type if either v has valence 3 and $H = G(v)$, or if v has valence 1 and H has index 2 or 3 in $G(v)$.

Remark 2.3.9. As mentioned after the previous definition, the annuli involved in this definition may be twisted. However, in the special case when v is of special solid torus type, the conditions imply that each of the annuli must be untwisted. This is proved during the proof of Theorem [2.3.14](#) in section [2.7](#).

Definition 2.3.10. Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair, and let Γ be a minimal graph of groups decomposition of G .

A vertex v of Γ is of special Seifert type if v has only one incident edge e , the splitting of G associated to e is dual to an essential torus, and $G(e)$ is a subgroup of index 2 in $G(v)$.

Remark 2.3.11. As $G(v)$ is a finite, torsion free extension of $G(e)$, it follows that $G(v)$ is also $PD(n+1)$. Also part 3) of Corollary [2.2.8](#) implies that $G(v)$ must be non-orientable.

Our last definition is the algebraic analogue of a component W of the characteristic submanifold of a 3-manifold which is homeomorphic to $T \times I$ or to $K \tilde{\times} I$, but has some annuli in its frontier. There are subcases here, depending on whether or not W has a component of its frontier which is a torus, and whether or not W contains a torus component of ∂M . It is not possible to have both. Note that in all cases, there is a Seifert fibration of W for which all the annuli in its frontier are vertical. As W is $T \times I$ or $K \tilde{\times} I$, this is equivalent to the condition that all the frontier annuli carry the same subgroup of $\pi_1(W)$ and that $\pi_1(W)$ splits over this subgroup. This is what we generalise in the definition below. Recall that as G is torsion free, a $VPC(n+1)$ subgroup is automatically $PD(n+1)$.

Definition 2.3.12. Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let Γ be a minimal graph of groups decomposition of G . A vertex v of Γ is of torus type if $G(v)$ is $VPC(n+1)$ and one of the following cases hold:

1. $G(v)$ is orientable and is one of the groups in ∂G , and for each edge of Γ which is incident to v the associated edge splitting is dual to an essential annulus. Further there is a $VPCn$ subgroup H of $G(v)$ such that the boundary of each such annulus carries H , and $G(v)$ splits over H .
2. $G(v)$ is orientable, one of the edges of Γ incident to v carries $G(v)$, and the associated edge splitting is dual to an essential torus in G . For each of the remaining edges of Γ incident to v , the associated edge splitting is dual to an essential annulus. Further there is a $VPCn$ subgroup H of $G(v)$ such that the boundary of each such annulus carries H , and $G(v)$ splits over H .
3. $G(v)$ is orientable, and for each edge of Γ which is incident to v the associated edge splitting is dual to an essential annulus. Further there is a $VPCn$ subgroup H of $G(v)$ such that the boundary of each such annulus carries H , and $G(v)$ splits over H . In addition Γ can be refined by splitting at v to a graph of groups structure Γ' of G such that the projection map $\Gamma' \rightarrow \Gamma$ sends

an edge e to v and otherwise induces a bijection of edges and vertices. The group $G(e)$ associated to e is equal to $G(v)$, and the edge splitting associated to e is dual to an essential torus in G .

4. $G(v)$ is non-orientable, and we denote the orientable subgroup of index 2 by $G(v)_0$. For each edge of Γ which is incident to v the associated edge splitting is dual to an essential annulus. Further there is a $VPCn$ subgroup H of $G(v)_0$ such that the boundary of each such annulus carries H , and $G(v)$ splits over H . In addition Γ can be refined by splitting at v to a graph of groups structure Γ' of G such that the projection map $\Gamma' \rightarrow \Gamma$ sends an edge e to v and otherwise induces a bijection of edges and vertices. The group associated to e is equal to $G(v)_0$, and the associated edge splitting is dual to an essential torus in G . Finally one vertex of e has valence 1, and associated group $G(v)$.

Remark 2.3.13. In part 3) of this definition, the comparable 3-manifold situation occurs when W is $T \times I$, and the frontier of W consists of annuli. Think of Γ as being dual to the frontier of W , and the refinement Γ' as being dual to the union of the frontier of W with the torus $T \times \{\frac{1}{2}\}$.

In part 4) of this definition, the comparable 3-manifold situation occurs when W is $K \tilde{\times} I$, and the frontier of W consists of annuli. Think of Γ as being dual to the frontier of W , and the refinement Γ' as being dual to the union of the frontier of W with a torus in the interior of W which is parallel to ∂W .

In all cases, the reason for the formulation involving the boundary of each annulus is that some of the annuli involved may be twisted, a phenomenon with no analogue in 3-manifold theory.

Note that Lemma 2.1.10 tells us that if a $VPC(n+1)$ group G splits over a subgroup H , then H is $VPCn$, and is normal in G with quotient which is isomorphic to \mathbb{Z} or $\mathbb{Z}_2 * \mathbb{Z}_2$.

Now we are ready to state the main result of this paper. Lemma 2.2.10 implies that if $(G, \partial G)$ is an orientable $PD(n+2)$ pair, then the decomposition $\Gamma_{n,n+1}(G)$ of Theorem 2.1.16 exists. That theorem tells us that for any group G , the V_0 -vertices of $\Gamma_{n,n+1}(G)$ are of four types, namely they are isolated, of $VPCk$ -by-Fuchsian type, where k is $n-1$ or n , or of commensuriser type. Our main result is the following theorem which asserts that $\Gamma_{n,n+1}(G)$ and its completion $\Gamma_{n,n+1}^c(G)$ (see Definition 2.1.17) have properties analogous to the topological picture in dimension 3. If $n=1$, and G is the fundamental group of an orientable Haken 3-manifold M , then $\Gamma_{1,2}(G)$ is dual to the frontier of $AT(M)$,

and $\Gamma_{1,2}^c(G)$ is dual to the frontier of $JSJ(M)$, where $AT(M)$ and $JSJ(M)$ are the submanifolds of M discussed at the start of section [2.1](#).

Theorem 2.3.14. (Main Result) *Let $n \geq 1$, and let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . Let $\mathcal{F}_{n,n+1}$ denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup, together with the equivalence classes of all n -canonical almost invariant subsets of G which are over a $VPC(n+1)$ subgroup. Finally let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G , and let $\Gamma_{n,n+1}^c$ denote the completion of $\Gamma_{n,n+1}$. Thus $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ are bipartite graphs of groups structures for G , with vertices of V_0 -type and of V_1 -type.*

Then $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ have the following properties:

1. Each V_0 -vertex v of $\Gamma_{n,n+1}$ satisfies one of the following conditions:
 - (a) v is isolated, and $G(v)$ is VPC of length n or $n+1$, and the edge splittings associated to the two edges incident to v are dual to essential annuli or tori in G .
 - (b) v is of $VPC(n-1)$ -by-Fuchsian type, and is of I -bundle type. (See Definition [2.3.1](#).)
 - (c) v is of $VPCn$ -by-Fuchsian type, and is of interior Seifert type. (See Definition [2.3.2](#).)
 - (d) v is of commensuriser type. Further v is of Seifert type (see Definition [2.3.6](#)), or of torus type (see Definition [2.3.12](#)) or of solid torus type (see Definition [2.3.8](#)).
2. The V_0 -vertices of $\Gamma_{n,n+1}^c$ obtained by the completion process are of special Seifert type (see Definition [2.3.10](#)) or of special solid torus type (see Definition [2.3.8](#)).
3. Each edge splitting of $\Gamma_{n,n+1}$ and of $\Gamma_{n,n+1}^c$ is dual to an essential annulus or torus in G .
4. Any nontrivial almost invariant subset of G over a $VPC(n+1)$ group and adapted to ∂G is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}$, and also by some V_0 -vertex of $\Gamma_{n,n+1}^c$.
5. If H is a $VPC(n+1)$ subgroup of G which is not conjugate into ∂G , then H is conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}^c$.

Remark 2.3.15. *Part 3) follows immediately from parts 1) and 2), as the definitions of the various types of V_0 -vertex in the statements of parts 1) and 2) all contain the requirement that the edge splittings be dual to an essential annulus or torus.*

Part 4) does not follow from the properties of an algebraic regular neighbourhood as an almost invariant subset of G over a $VPC(n+1)$ group which is adapted to ∂G need not be n -canonical, and so need not lie in the family $\mathcal{F}_{n,n+1}$. Note that, from [24], we know that there may be almost invariant subsets of G over $VPC(n+1)$ subgroups which are not adapted to ∂G .

Part 5) also does not follow from the properties of an algebraic regular neighbourhood as a $VPC(n+1)$ subgroup H of G may be non-orientable.

2.4 Torus Decompositions for $PD(n+2)$ Groups and Pairs

Before embarking on the proof of Theorem 2.3.14, we will consider a simpler graph of groups structure analogous to the torus decomposition $T(M)$ of a 3-manifold M , discussed in section 2.1

We will start this section by considering an orientable $PD(n+2)$ group G , where $n \geq 1$, and the graph of groups structure $\Gamma_{n+1}(G) = \Gamma(\mathcal{F}_{n+1} : G)$ of Theorem 2.1.14. Recall from Lemma 2.2.10 that such a group cannot admit any nontrivial almost invariant subset over a $VPC(\leq n)$ subgroup, so that $\Gamma_{n+1}(G)$ does exist. It is a reduced algebraic regular neighbourhood of \mathcal{F}_{n+1} in G , where \mathcal{F}_{n+1} is the collection of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPC(n+1)$ subgroups. Since G is orientable, the nontrivial almost invariant sets in \mathcal{F}_{n+1} are automatically over orientable $VPC(n+1)$ groups. Recall that if H is an orientable $VPC(n+1)$ subgroup of G , we call H an essential torus in G , and G possesses a unique nontrivial H -almost invariant subset, up to equivalence and complementation. Thus \mathcal{F}_{n+1} can be thought of as the collection of all essential tori in G . It will be convenient to denote $\Gamma_{n+1}(G)$ by $T_{n+1}(G)$, and we will call $T_{n+1}(G)$ the *torus decomposition* of G . We will also use $T_{n+1}^c(G)$ to denote the completion $\Gamma_{n+1}^c(G)$ of $\Gamma_{n+1}(G)$. See Definition 2.1.17

In chapter 12 of [22], we briefly discussed the connection between $T_{n+1}(G)$ and Kropholler's decomposition in [11] in the case of orientable $PD(n+2)$ groups. Using the present notation, it follows from Theorem 2.4.1 below that $T_{n+1}^c(G)$ is the same as Kropholler's decomposition. We will show shortly that

the analogous statement holds for the case of a $PD(n + 2)$ pair. In dimension 3, it is clear that $T_2(G)$ is the same as Castel's decomposition in [4] as his decomposition is defined to be $\Gamma_2(G)$. As discussed in section 2.1, if M is a closed orientable Haken 3-manifold and G equals $\pi_1(M)$, then the topological torus decomposition $T(M)$ determines the decomposition $T_2^c(G)$ of G . As in the statement of Theorem 2.1.14, it will be convenient to state Theorem 2.4.1 excluding the case when G is VPC . For if G is a $VPC(n + 2)$ group, then $\Gamma_{n+1}(G)$ consists of a single V_0 -vertex.

Theorem 2.4.1. *Let $n \geq 1$. If G is an orientable $PD(n + 2)$ group which is not VPC , then $T_{n+1}(G)$ and $T_{n+1}^c(G)$ have the following properties:*

1. Each V_0 -vertex v of $T_{n+1}(G)$ satisfies one of the following conditions:
 - (a) v is isolated, and $G(v)$ is a torus in G .
 - (b) v is of interior Seifert type. (See Definition 2.3.2.)
2. The V_0 -vertices of $T_{n+1}^c(G)$ obtained by the completion process are of special Seifert type. (See Definition 2.3.10.)
3. Each edge splitting of $T_{n+1}(G)$ and of $T_{n+1}^c(G)$ is dual to an essential torus in G .
4. If H is a $VPC(n + 1)$ subgroup of G , then H is conjugate into a V_0 -vertex group of $T_{n+1}^c(G)$.

Remark 2.4.2. *If G has no nontrivial almost invariant subsets which are over $VPC(n + 1)$ subgroups, then \mathcal{F}_{n+1} is empty and $T_{n+1}(G)$, and hence $T_{n+1}^c(G)$, consists of a single V_1 -vertex.*

Proof. 1) Theorem 2.1.14 tells us that $T_{n+1}(G)$ is a minimal, reduced bipartite, graph of groups decomposition of G , and that each V_0 -vertex of $T_{n+1}(G)$ is isolated, of $VPCn$ -by-Fuchsian type, or of commensuriser type. If v is a V_0 -vertex of commensuriser type, then v encloses an element X of \mathcal{F}_{n+1} which is over some $VPC(n + 1)$ group H , and $G(v)$ is of the form $Comm_G(H)$. Further X crosses weakly some of its translates by $Comm_G(H)$. In the present situation, G is $PD(n + 2)$ and H is $VPC(n + 1)$, so that the number of coends of H in G is 2. Now Proposition 7.4 of [22] implies that no almost invariant set can cross X or any of its translates weakly, so that V_0 -vertices of commensuriser type cannot

occur. It follows that each V_0 -vertex of $T_{n+1}(G)$ is isolated or of $VPCn$ -by-Fuchsian type. As each edge splitting is over a $VPC(n+1)$ subgroup of G , it follows that each edge splitting of $T_{n+1}(G)$ is dual to an essential torus in G . Hence any V_0 -vertex of $T_{n+1}(G)$ which is of $VPCn$ -by-Fuchsian type must be of interior Seifert type. This completes the proof of part 1) of the theorem.

2) The construction of $T_{n+1}^c(G)$ from $T_{n+1}(G)$ described in section 2.1 can only introduce V_0 -vertices of special Seifert type, so part 2) of the theorem holds.

3) Part 3) of the theorem follows immediately from parts 1) and 2).

4) First note that as G is torsion free, so is H . Thus H must be $PD(n+1)$.

If H is orientable, the pair (G, H) has two ends, so there is a nontrivial H -almost invariant subset X of G . As X is enclosed by some V_0 -vertex v of $T_{n+1}(G)$, it follows that H is conjugate into $G(v)$. Hence H is also conjugate into a V_0 -vertex group of $T_{n+1}^c(G)$, as required.

If H is non-orientable, let H_0 denote its orientable subgroup of index 2. As the pair (G, H_0) has two ends, there is a nontrivial H_0 -almost invariant subset of G . As $T_{n+1}(G)$ has no V_0 -vertices of commensuriser type, it follows from Theorem 2.1.14 and Remark 2.1.15 that H_0 must have small commensuriser, which we denote by K . This means that K contains H_0 with finite index, so that K is itself $VPC(n+1)$ and $PD(n+1)$. Note that K must contain H , so that K is non-orientable. We let K_0 denote its orientable subgroup of index 2. Note that K_0 is a maximal torus subgroup of G . The preceding paragraph shows that there is a V_0 -vertex v of $T_{n+1}(G)$ such that K_0 is conjugate into $G(v)$. As K contains K_0 with finite index, it follows that there is a vertex w of $T_{n+1}(G)$ such that K is conjugate into $G(w)$. If w is a V_0 -vertex, then K , and hence H , is conjugate into a V_0 -vertex group of $T_{n+1}^c(G)$, as required. So we now consider the case when w is a V_1 -vertex. In particular, v and w must be distinct. Thus there is an edge e of $T_{n+1}(G)$ which is incident to w such that, after a conjugation, $G(e)$ contains K_0 . As all the edge groups of $T_{n+1}(G)$ are torus groups, the group $G(e)$ must equal K_0 . If $E(w)$ denotes the family of subgroups of $G(w)$ which are edge groups for the edges incident to w , then Theorem 8.1 of [1] tells us that the pair $(G(w), E(w))$ is $PD(n+2)$. The commensuriser in $G(w)$ of $G(e) = K_0$ contains a conjugate of K and so is not equal to K_0 . Thus Lemma 2.2.7 shows that K_0 is the only element of the family $E(w)$, and $G(w)$ contains K_0 with index 2. Thus w has valence 1, and $G(w)$ equals a conjugate of K . Now it follows that w becomes a V_0 -vertex in the completion $T_{n+1}^c(G)$, so that K , and hence H , is conjugate into a V_0 -vertex group of $T_{n+1}^c(G)$, as required. \square

Next we discuss the torus decomposition of an orientable $PD(n+2)$ pair

$(G, \partial G)$ with non-empty boundary. Recall from section [2.2](#) the discussion of an essential torus in G . In particular, we let DG denote the orientable $PD(n + 2)$ group obtained by doubling G along its boundary. Then given an orientable $PD(n + 1)$ subgroup H of G , there is a H -almost invariant subset X_H of DG associated to H , and the intersection $Y_H = X_H \cap G$ is a H -almost invariant subset of G which is nontrivial unless H is conjugate into ∂G . Let \mathcal{T}_{n+1} denote the family of equivalence classes of all such nontrivial subsets Y_H of G , where H is $VPC(n + 1)$. Then the torus decomposition $T_{n+1}(G, \partial G)$ of $(G, \partial G)$ will be the reduced algebraic regular neighbourhood in G of \mathcal{T}_{n+1} . This is the natural definition, but it is not obvious that this algebraic regular neighbourhood exists. One immediate problem is that G may have nontrivial almost invariant subsets over $VPCn$ subgroups, so this decomposition is different from any of those proved to exist in [\[22\]](#). In order to show that $T_{n+1}(G, \partial G)$ exists, we will use the fact that DG does not have nontrivial almost invariant subsets over $VPCn$ subgroups, so we can apply results from [\[22\]](#).

Theorem 2.4.3. *Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair, such that ∂G is non-empty, and let \mathcal{T}_{n+1} denote the family of equivalence classes of almost invariant subsets Y_H of G described above. Then \mathcal{T}_{n+1} has a reduced algebraic regular neighbourhood $T_{n+1}(G, \partial G)$ in G . Further $T_{n+1}(G, \partial G)$ is adapted to ∂G .*

Proof. If G is VPC , part 2) of Corollary [2.2.9](#) tells us that either $(G, \partial G)$ is the trivial pair $(G, \{G, G\})$, or that ∂G is a single group S , and G contains S with index 2. In either case, the pair $(G, \partial G)$ admits no essential tori, so that \mathcal{T}_{n+1} is empty. Thus \mathcal{T}_{n+1} has a reduced algebraic regular neighbourhood which consists of a single V_1 -vertex, and this is trivially adapted to ∂G . For the rest of this proof we will assume that G is not VPC .

Recall our discussion at the start of section [2.2](#). The natural graph of groups structure Δ for the orientable $PD(n+2)$ group DG has two vertices w and \bar{w} and edges joining them which correspond to the groups of ∂G . Given an orientable $PD(n + 1)$ subgroup H of G , there is a H -almost invariant subset X_H of DG associated to H , and the intersection $Y_H = X_H \cap G$ is a H -almost invariant subset of G . An important point about X_H is that it is enclosed by the vertex w of Δ , where we identify G with $G(w)$, so that Y_H is adapted to ∂G .

Now consider $\mathcal{F}_{n+1}(DG)$, which is the collection of equivalence classes of all nontrivial almost invariant subsets of DG which are over $VPC(n + 1)$ subgroups. Recall that as DG is $PD(n + 2)$, it is torsion free so that a $VPC(n + 1)$ subgroup K of DG must be $PD(n + 1)$. Further, as DG is orientable, if there is a nontrivial almost invariant subset of DG which is over K , then K must

be orientable, and all such almost invariant subsets of DG are equivalent up to complementation. Also recall from Theorem 2.4.1 that the reduced algebraic regular neighbourhood $\Gamma(\mathcal{F}_{n+1}(DG) : DG)$ exists, and is denoted $T_{n+1}(DG)$, and its V_0 -vertices are either isolated or of $VPCn$ -by-Fuchsian type. As we will need to use our construction of unreduced algebraic regular neighbourhoods in [22], as corrected in [23], we note that Theorem 2.1.14 tells us that the unreduced algebraic regular neighbourhood of $\mathcal{F}_{n+1}(DG)$ in DG also exists. We will use the notation $\bar{\Gamma}(\mathcal{F}_{n+1}(DG) : DG)$ for this unreduced algebraic regular neighbourhood. As in the case of the reduced algebraic regular neighbourhood, its V_0 -vertices are either isolated or of $VPCn$ -by-Fuchsian type.

Let $\mathcal{E}_{n+1}(DG)$ denote the subfamily of $\mathcal{F}_{n+1}(DG)$ which consists of non-trivial almost invariant subsets of DG which are over subgroups of G which are non-peripheral in G . Note that $\mathcal{E}_{n+1}(DG)$ is G -invariant but is not DG -invariant. We claim that $\mathcal{E}_{n+1}(DG)$ possesses an unreduced algebraic regular neighbourhood $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$ in DG , and hence a reduced algebraic regular neighbourhood $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$. We recall from [22] that any finite subset of $\mathcal{F}_{n+1}(DG)$ possesses an unreduced algebraic regular neighbourhood in DG . As $\bar{\Gamma}(\mathcal{F}_{n+1}(DG) : DG)$ has no V_0 -vertices of commensuriser type, the same holds for an algebraic regular neighbourhood of any subset of $\mathcal{F}_{n+1}(DG)$. Thus the proof of Theorem 2.1.14 which we gave in chapter 12 of [22] shows that the V_0 -vertices of the unreduced algebraic regular neighbourhood of a finite subset of $\mathcal{F}_{n+1}(DG)$ must be either isolated or of $VPCn$ -by-Fuchsian type. Now we consider the construction in the proof of Theorem 9.2 of [22]. This constructs $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$ by expressing $\mathcal{E}_{n+1}(DG)$ as an ascending union of finite subsets \mathcal{E}_{n+1}^k . We obtain a sequence of bipartite graphs of groups decompositions $\bar{\Gamma}^k$ of DG for each of which, every V_0 -vertex must be either isolated or of $VPCn$ -by-Fuchsian type. The accessibility results of section 13 of [6] imply that the sequence must stabilise so that $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$ exists as required. Further each V_0 -vertex of $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$ is either isolated or of $VPCn$ -by-Fuchsian type.

Let T_{DG} denote the universal covering DG -tree of $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$, and recall that the V_0 -vertices of T_{DG} are the CCC's of all the translates by DG of elements of $\mathcal{E}_{n+1}(DG)$. Now we consider the construction of the unreduced algebraic regular neighbourhood of \mathcal{T}_{n+1} described in chapter 6 of [22]. One starts by choosing one element of \mathcal{T}_{n+1} from each equivalence class, and then considers the CCC's of these elements. There is a natural map φ from the equivalence classes of \mathcal{T}_{n+1} to the equivalence classes of $\mathcal{E}_{n+1}(DG)$, given by sending the

class of Y_H to the class of X_H . Recall that $Y_H = X_H \cap G$. Clearly if two elements of \mathcal{T}_{n+1} cross, then the corresponding elements of $\mathcal{E}_{n+1}(DG)$ also cross. Now recall that elements of $\mathcal{E}_{n+1}(DG)$ which cross must do so strongly. It follows that if two elements of $\mathcal{E}_{n+1}(DG)$ cross, then the corresponding elements of \mathcal{T}_{n+1} also cross. Hence φ induces a G -equivariant bijection between the collection P of all CCC's of \mathcal{T}_{n+1} and a subset Q of the V_0 -vertices of T_{DG} . It also follows that the pretree structures on P and Q are the same. In particular, the pretree structure on the collection P of all CCC's of \mathcal{T}_{n+1} is discrete. Now the proof of Theorem 3.8 of [22] shows that there is a bipartite G -tree T_G whose quotient by G is the unreduced algebraic regular neighbourhood of \mathcal{T}_{n+1} in G . Thus \mathcal{T}_{n+1} also has a reduced algebraic regular neighbourhood in G , so the torus decomposition $T_{n+1}(G, \partial G)$ of G exists.

Recall that each of the edge splittings of Δ crosses no element of $\mathcal{E}_{n+1}(DG)$, and so must be enclosed by some V_1 -vertex of the unreduced algebraic regular neighbourhood $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$. Thus we can refine $\bar{\Gamma}(\mathcal{E}_{n+1}(DG) : DG)$ by splitting at V_1 -vertices to obtain a graph of groups decomposition $\bar{\Gamma}\Delta$ of DG which also refines Δ . Recall that each element of $\mathcal{E}_{n+1}(DG)$ is enclosed by the vertex w of Δ . Thus if we remove the interiors of the edges of $\bar{\Gamma}\Delta$ which correspond to the edges of Δ , we will be left with a connected graph of groups $\bar{\Gamma}_w$, and the single vertex \bar{w} . The fundamental group of $\bar{\Gamma}_w$ is $G(w)$, which we continue to identify with G .

Claim: $\bar{\Gamma}_w$ is isomorphic to the unreduced algebraic regular neighbourhood of \mathcal{T}_{n+1} in G .

This claim immediately implies that the graph of groups Γ_w obtained by reducing $\bar{\Gamma}_w$ is isomorphic to the torus decomposition $T_{n+1}(G, \partial G)$. It also implies that each group in ∂G is conjugate into some vertex group of $T_{n+1}(G, \partial G)$, so that $T_{n+1}(G, \partial G)$ is adapted to ∂G .

To prove our claim, recall from two paragraphs previously that the map φ induces a G -equivariant injection from the V_0 -vertices of T_G to the V_0 -vertices of T_{DG} . It follows that φ induces a G -equivariant injection from the V_0 -vertices of T_G to the V_0 -vertices of $T_{\bar{\Gamma}\Delta}$. Let T denote the subtree of $T_{\bar{\Gamma}\Delta}$ spanned by the V_0 -vertices in the image of φ . As T_G is a minimal G -tree, T must be the minimal G -invariant subtree of $T_{\bar{\Gamma}\Delta}$. It follows that T is the universal covering G -tree of $\bar{\Gamma}_w$, proving that $\bar{\Gamma}_w$ is isomorphic to the unreduced algebraic regular neighbourhood of \mathcal{T}_{n+1} in G , as claimed. Note that it also follows that $T_{n+1}(G, \partial G)$ is the decomposition of G induced from $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$. \square

The properties of $T_{n+1}(G, \partial G)$ when ∂G is non-empty are similar to those in the case when ∂G is empty. The fact that $T_{n+1}(G, \partial G)$ is adapted to ∂G plays an important role. Before listing these properties, it will be convenient to introduce the completion $T_{n+1}^c(G, \partial G)$ of $T_{n+1}(G, \partial G)$ which is defined in the same way as we defined the completions of $\Gamma_{n+1}(G)$ and $\Gamma_{n,n+1}(G)$ in Definition 2.1.17.

The following result lists the properties of $T_{n+1}(G, \partial G)$ and its completion $T_{n+1}^c(G, \partial G)$ when ∂G is non-empty. As usual, it will be convenient to exclude the case when G is *VPC*. For in that case $T_{n+1}(G, \partial G)$ and $T_{n+1}^c(G, \partial G)$ are equal and consist of a single V_1 -vertex.

Theorem 2.4.4. *Let $n \geq 1$, and let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, such that ∂G is non-empty. If G is not *VPC*, then $T_{n+1}(G, \partial G)$ and $T_{n+1}^c(G, \partial G)$ have the following properties:*

1. *Each edge splitting of $T_{n+1}(G, \partial G)$ and of $T_{n+1}^c(G, \partial G)$ is dual to an essential torus in $(G, \partial G)$.*
2. *Each V_0 -vertex v of $T_{n+1}(G, \partial G)$ satisfies one of the following conditions:*
 - (a) *v is isolated, and $G(v)$ is an essential torus in G .*
 - (b) *v is of Seifert type adapted to ∂G . (See Definition 2.3.4. Note that this includes the possibility that v is of interior Seifert type.)*
3. *The V_0 -vertices of $T_{n+1}^c(G, \partial G)$ obtained by the completion process are of special Seifert type. (See Definition 2.3.10.)*
4. *If H is a $VPC(n+1)$ subgroup of G which is not conjugate into ∂G , then H is conjugate into a V_0 -vertex group of $T_{n+1}^c(G, \partial G)$.*

Remark 2.4.5. *It follows from property 4) that $T_{n+1}^c(G, \partial G)$ is the same as Kropholler's decomposition in [11]. In dimension 3, it is also easy to see that $T_2(G, \partial G)$ is the same as Castel's decomposition in [4]. Finally if M is an orientable Haken 3-manifold and $(G, \partial G)$ is the corresponding Poincaré duality pair, then the topological torus decomposition $T(M)$ determines the decomposition $T_2^c(G, \partial G)$ of G .*

Proof. It follows from the proof of Theorem 2.4.3 that the reduced graphs of groups decompositions $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$ and Δ have a common refinement $\Gamma\Delta$, obtained from $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$ by splitting at V_1 -vertices, which consists of $T_{n+1}(G, \partial G)$ and a single extra vertex \bar{w} which is joined to $T_{n+1}(G, \partial G)$

by edges e_1, \dots, e_m whose associated splittings are those of Δ . Note that the V_0 -vertices of $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$ and their incident edges are unaffected by the refinement process. Thus we will refer to the vertices of $\Gamma\Delta$ which are obtained from V_0 -vertices of $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$ as V_0 -vertices of $\Gamma\Delta$. As the V_0 -vertices of $\Gamma(\mathcal{E}_{n+1}(DG) : DG)$ are isolated or of $VPCn$ -by-Fuchsian type, the same holds for the V_0 -vertices of $\Gamma\Delta$.

1) Each edge of $T_{n+1}(G, \partial G)$, regarded as an edge of $\Gamma\Delta$, determines a splitting of the orientable $PD(n + 2)$ group DG over a $VPC(n + 1)$ subgroup. It follows that this splitting is dual to an essential torus in DG . Hence each edge splitting of $T_{n+1}(G, \partial G)$ is dual to an essential torus in $(G, \partial G)$, proving part 1) of the theorem.

2) Let v be an isolated V_0 -vertex of $\Gamma\Delta$. If one of the e_i 's is incident to v , then v has only one incident edge e in $T_{n+1}(G, \partial G)$ and the inclusion of $G(e)$ into $G(v)$ is an isomorphism. But this contradicts the minimality of $T_{n+1}(G, \partial G)$. Thus no e_i is incident to v , and v must be an isolated V_0 -vertex of $T_{n+1}(G, \partial G)$.

Now let v be a V_0 -vertex of $\Gamma\Delta$ of $VPCn$ -by-Fuchsian type. If no e_i is incident to v , then v is a V_0 -vertex of $T_{n+1}(G, \partial G)$ of $VPCn$ -by-Fuchsian type. As each edge splitting of $T_{n+1}(G, \partial G)$ is dual to an essential torus in $(G, \partial G)$, it follows that v is of interior Seifert type. If some e_i is incident to v , the associated edge group is a group in ∂G . It follows that v is a V_0 -vertex of $T_{n+1}(G, \partial G)$ which is of Seifert type adapted to ∂G .

Thus the V_0 -vertices of $T_{n+1}(G, \partial G)$ are isolated, or of Seifert type adapted to ∂G , which completes the proof of part 2) of the theorem.

3) The construction of $T_{n+1}^c(G, \partial G)$ from $T_{n+1}(G, \partial G)$ described in section 2.1 can only introduce V_0 -vertices of special Seifert type, so part 3) of the theorem holds.

4) First note that as G is torsion free, so is H . Thus H must be $PD(n + 1)$.

Suppose that H is orientable. The hypothesis that H is not conjugate into ∂G implies that H is an essential torus in $(G, \partial G)$, so that there is a nontrivial H -almost invariant subset X of G dual to H . As X is enclosed by some V_0 -vertex v of $T_{n+1}(G, \partial G)$, it follows that H is conjugate into $G(v)$. Hence H is also conjugate into a V_0 -vertex group of $T_{n+1}^c(G)$, as required.

Now suppose that H is non-orientable, and let H_0 denote its orientable subgroup of index 2. If H_0 is conjugate into a group S in ∂G , this conjugate of H_0 will be a $PD(n + 1)$ subgroup of the $PD(n + 1)$ group S and so will be of finite index. Thus S must itself be $VPC(n + 1)$ and be conjugate commensurable with H_0 . As H is not conjugate into S , it follows that $Comm_G(S) \neq S$, so Lemma 2.2.7 implies that G contains S with index 2. But this implies that G is $VPC(n + 1)$

which contradicts the hypothesis that G is not VPC . This contradiction shows that H_0 is not conjugate into ∂G .

It follows from the discussion in section 2.2 that H_0 is an essential torus in $(G, \partial G)$, so that there is a nontrivial H_0 -almost invariant subset of G which is dual to H_0 . As $T_{n+1}(G, \partial G)$ has no V_0 -vertices of commensuriser type, it follows from Theorem 2.1.14 and Remark 2.1.15 that H_0 must have small commensuriser, which we denote by K . This means that K contains H_0 with finite index, so that K is itself $VPC(n+1)$ and $PD(n+1)$. Note that K must contain H , so that K is non-orientable. We let K_0 denote its orientable subgroup of index 2. Note that K_0 is a maximal torus subgroup of G , and is not conjugate into ∂G . Thus there is a nontrivial K_0 -almost invariant subset of G which must be enclosed by some V_0 -vertex of $T_{n+1}(G, \partial G)$. It follows that there is a V_0 -vertex v of $T_{n+1}(G, \partial G)$ such that K_0 is conjugate into $G(v)$. As K contains K_0 with finite index, it follows that there is a vertex w of $T_{n+1}(G, \partial G)$ such that K is conjugate into $G(w)$. If w is a V_0 -vertex, then K , and hence H , is conjugate into a V_0 -vertex group of $T_{n+1}^c(G, \partial G)$, as required. So we now consider the case when w is a V_1 -vertex. In particular, v and w must be distinct. Thus there is an edge e of $T_{n+1}(G, \partial G)$ which is incident to w such that $G(e)$ contains K_0 . As all the edge groups of $T_{n+1}(G, \partial G)$ are torus groups, the group $G(e)$ must equal K_0 . Let $E(w)$ denote the family of subgroups of $G(w)$ which are edge groups for the edges incident to w .

Recall that each edge splitting of $T_{n+1}(G, \partial G)$ is over a $PD(n+1)$ group. If ∂G were empty so that G was a $PD(n+2)$ group, then Theorem 8.1 of [1] would tell us that the pair $(G(w), E(w))$ is $PD(n+2)$. As ∂G is not empty this need not be the case, but instead we apply Theorem 8.1 of [1] to the graph of groups structure $\Gamma\Delta$ of DG . This shows that the pair $(G(w), E(w))$ becomes $PD(n+2)$ when $E(w)$ is augmented by suitable groups in ∂G . Let $\overline{E}(w)$ denote this augmented family of subgroups of $G(w)$, so that the pair $(G(w), \overline{E}(w))$ is $PD(n+2)$. As the commensuriser in $G(w)$ of the group $G(e) = K_0$ is not equal to K_0 , Lemma 2.2.7 shows that K_0 is the only element of the family $\overline{E}(w)$, and $G(w)$ contains K_0 with index 2. Thus w has valence 1, and $G(w)$ equals a conjugate of K . Now it follows that w becomes a V_0 -vertex in the completion $T_{n+1}^c(G, \partial G)$, so that K , and hence H , is conjugate into a V_0 -vertex group of $T_{n+1}^c(G, \partial G)$, as required. \square

2.5 Further properties of Torus Decompositions

In the previous section, we showed that any $PD(n+2)$ pair has a torus decomposition, and established the basic properties of this decomposition. In this section, we will establish more detailed information about the vertices of this decomposition.

Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let w be a V_1 -vertex of $T_{n+1}(G, \partial G)$. Let $E(w)$ denote the family of subgroups of $G(w)$ which are edge groups for the edges incident to w . Recall that each edge splitting of $T_{n+1}(G, \partial G)$ is over a $PD(n+1)$ group. If ∂G is empty so that G is a $PD(n+2)$ group, then Theorem 8.1 of [1] tells us that the pair $(G(w), E(w))$ is an orientable $PD(n+2)$ pair. In general, the pair $(G(w), E(w))$ need not be $PD(n+2)$, but instead we apply Theorem 8.1 of [1] to the graph of groups structure $\Gamma\Delta$ of DG described in the proof of Theorem 2.4.4. This shows that the pair $(G(w), E(w))$ becomes $PD(n+2)$ when $E(w)$ is augmented by suitable groups in ∂G . As any essential torus in $(G, \partial G)$ is enclosed by some V_0 -vertex of $T_{n+1}(G, \partial G)$, it follows that any orientable $VPC(n+1)$ subgroup of $G(w)$ is conjugate into one of the groups in $E(w)$. It will be convenient to give a name to this property of a $PD(n+2)$ pair.

Definition 2.5.1. *An orientable $PD(n+2)$ pair $(G, \partial G)$ is atoroidal if any orientable $VPC(n+1)$ subgroup of G is conjugate into one of the groups in ∂G .*

Remark 2.5.2. *As G is torsion free, a $VPC(n+1)$ subgroup of G is also torsion free and hence is $PD(n+1)$. Thus it makes sense to say that such a subgroup is orientable. Recall from the preceding paragraph that if w is a V_1 -vertex of $T_{n+1}(G, \partial G)$, then the pair $(G(w), E(w))$ becomes $PD(n+2)$ when $E(w)$ is augmented by suitable groups in ∂G . The resulting $PD(n+2)$ pair is atoroidal.*

This is precisely analogous to the definition of atoroidal for a 3-manifold. In the case when an orientable atoroidal 3-manifold M has incompressible boundary, it is easy to show that M admits no essential annulus, unless M is homeomorphic to $T \times I$ or to a twisted I -bundle over the Klein bottle. We will now prove the algebraic analogue of this fact.

Proposition 2.5.3. *Let $(G, \partial G)$ be an orientable atoroidal $PD(n+2)$ pair, where $n \geq 1$. Let A and B be $VPC(n+1)$ groups in ∂G , possibly $A = B$. Let S and T be $VPCn$ subgroups of A and B respectively, and let g be an element of G such that $gSg^{-1} = T$. Then one of the following holds:*

1. A and B are the same element of ∂G , and $g \in A$.
2. A and B are distinct elements of ∂G , are the only groups in ∂G , and $A = G = B$. Thus $(G, \partial G)$ is the trivial pair $(G, \{G, G\})$.
3. A and B are the same element of ∂G . Further A is the only group in ∂G , and has index 2 in G .

Remark 2.5.4. *The hypothesis that there is g in G such that $gSg^{-1} = T$ means that the pair $(G, \partial G)$ admits an annulus. The conclusion of the proposition is that either this annulus is inessential (case 1) or that we have the special cases in 2) or 3).*

Proof. As $(G, \partial G)$ is atoroidal, any orientable $VPC(n+1)$ subgroup of G is conjugate into one of the groups in ∂G . Suppose that G contains a non-orientable $VPC(n+1)$ subgroup K . Then K has a subgroup of index 2 which must be conjugate into a group H of ∂G . As H and K are orientable $PD(n+1)$ groups, it follows that H and K are conjugate commensurable. As H cannot contain a non-orientable $VPC(n+1)$ subgroup, this implies that $Comm_G(H) \neq H$. Now Lemma 2.2.7 shows that ∂G consists of a single group H which has index 2 in G , so that we have case 3) of this proposition.

Thus in what follows we will assume that every $VPC(n+1)$ subgroup of G is orientable, and show that we have case 1) or case 2) of the proposition. We will consider separately the cases when A and B are the same or distinct.

Case: A and B are distinct elements of ∂G .

In this case, we will show that A and B are conjugate commensurable. Then Lemma 2.2.7 shows that we must have case 2) of the proposition.

We suppose that A and B are not conjugate commensurable, and will obtain a contradiction. After a suitable conjugation, we can arrange that $A \cap B$ is $VPCn$. Thus A , B and $A \cap B$ contain respectively finite index subgroups A' , B' and L such that L is normal in each of A' and B' , and $L \setminus A'$ and $L \setminus B'$ are both infinite cyclic (see Lemma 13.2 of [22]). Thus A' and B' are orientable $PD(n+1)$ groups, and L is PDn . Now let K denote the amalgamated free product $A' *_L B'$, so that L is normal in K with quotient a free group F of rank 2. We identify F with the fundamental group of a surface M which is a disc with two holes, in such a way that two of the boundary components of M carry the groups $L \setminus A'$ and $L \setminus B'$. Thus F together with the subgroups $L \setminus A'$ and $L \setminus B'$ and a third infinite cyclic subgroup forms a $PD2$ pair. The pre-images in K of the three boundary

groups of this pair yield three $PD(n+1)$ subgroups of K . Theorem 7.3 of [1] implies that K together with these subgroups forms a $PD(n+2)$ pair. Two of these three boundary subgroups of K are equal to A' and B' . As A' and B' are orientable and together generate K , it follows that the pair $(K, \partial K)$ is orientable. The inclusions of A' and B' into G determine a homomorphism of K into G , and we consider the image H in G of the third boundary subgroup $\partial_3 K$ of K . As $\partial_3 K$ is an extension of L by an infinite cyclic group, H is an extension of the $VPCn$ group L by a cyclic group. Thus H is $VPC(n+1)$ or $VPCn$.

If H is $VPC(n+1)$, it must be orientable as we are assuming that every $VPC(n+1)$ subgroup of G is orientable. As $(G, \partial G)$ is atoroidal, this implies that H is conjugate into a group in ∂G . Thus the map from K to G can be regarded as a map of $PD(n+2)$ pairs. Recall that the maps from the boundary subgroups A' and B' of K to the boundary subgroups A and B of G each have non-zero degree. As A and B are distinct, and K has only one other boundary group, it follows that the map from K to G must also have non-zero degree. Hence the image of K in G is a subgroup G' of finite index. Further L must be normal in G' . Recall that we are considering the case where $(G, \partial G)$ admits an annulus with fundamental group L whose boundary lies in the groups A and B of ∂G . As we are also assuming that A and B are distinct elements of ∂G , this annulus is automatically essential. As in Definition 2.2.19, an essential annulus with fundamental group L determines a nontrivial L -almost invariant subset of G . In particular, it follows that $e(G, L) > 1$, so that $e(G', L) > 1$. As L is normal in G' , it follows that $e(L \backslash G') > 1$. Now we apply Stallings' structure theorem [27] [28] for groups with more than one end. If $e(L \backslash G') = 2$, then $L \backslash G'$ is virtually infinite cyclic, so that G' , and hence G , must be $VPC(n+1)$. But then Corollary 2.2.9 implies that G , A and B are all equal, which contradicts our assumption that A and B are not commensurable. If $e(L \backslash G') = \infty$, then either $L \backslash G'$ is of the form $P *_R Q$, where R is finite, $P \neq R \neq Q$ and one of P and Q contains R with index at least 3, or $L \backslash G'$ is of the form $P *_R$, where R is finite and at least one of the inclusions of R into P is not an isomorphism. In either case, it is easy to see that $L \backslash G'$ contains infinitely many conjugacy classes of maximal infinite cyclic subgroups. It follows that $L \backslash G'$ contains infinitely many conjugacy classes of maximal two-ended subgroups. As a group is two-ended if and only if it is $VPC1$, the pre-images of these subgroups in G' form an infinite collection of conjugacy classes of maximal $VPC(n+1)$ subgroups of G' . Recall that we are assuming that every $VPC(n+1)$ subgroup of G is orientable. As G' is of finite index in G , there is a finite family $\partial G'$ of $VPC(n+1)$ subgroups of G' such that $(G', \partial G')$ is an orientable atoroidal $PD(n+2)$ pair. Thus any

maximal $VPC(n+1)$ subgroup of G' must be conjugate to one of the groups in $\partial G'$. As $\partial G'$ is a finite family, this is a contradiction, which completes the proof that H cannot be $VPC(n+1)$.

Now consider the case when H is $VPCn$. Recall that L is normal in $\partial_3 K$ with infinite cyclic quotient. Thus L is normal in H with finite cyclic quotient of some order d . There is a d -fold regular cover M_d of the surface M in which the pre-image of the third boundary component of M is a single circle C . Now M_d determines a subgroup K_d of K of index d , and L is normal in K_d with quotient $\pi_1(M_d)$. The boundary component C of M_d determines a boundary subgroup ∂_C of K_d . By construction the image of ∂_C in G is equal to L . We let \overline{M} be obtained from M_d by gluing a disc onto C , and let \overline{K} denote the corresponding quotient of K_d . Thus L is normal in \overline{K} with quotient $\pi_1(\overline{M})$, and \overline{K} yields an orientable $PD(n+2)$ pair with one less boundary subgroup than K_d . Further the homomorphism from K_d to G factors through \overline{K} . Each boundary subgroup of \overline{K} maps to a conjugate of A or B , so that the map from \overline{K} to G is a map of $PD(n+2)$ pairs. Again we have a map of non-zero degree, as it is of non-zero degree on the boundary. Now we argue exactly as in the preceding paragraph to obtain a contradiction. This completes the proof that if A and B are not conjugate commensurable, we have a contradiction, thus completing the proof that if A and B are distinct elements of ∂G , then we have case 2) of the proposition.

Case: A and B are the same element of ∂G .

Recall that we are assuming that every $VPC(n+1)$ subgroup of G is orientable. Thus case 3) cannot occur. Also recall that, as $A = B$, there are $VPCn$ subgroups S and T of A , and an element g of G such that $gSg^{-1} = T$. If g lies in A , we have case 1) of the proposition. Thus for the rest of this proof, we will suppose that g does not lie in A , and will obtain a contradiction.

As g does not lie in A , it follows that $(G, \partial G)$ admits an essential annulus with both ends in A . Let K denote the double of G along A . As $(G, \partial G)$ is an orientable $PD(n+2)$ pair, and A is one of the groups in the family ∂G , there is a natural structure of an orientable $PD(n+2)$ pair on K . As discussed at the end of section [2.2](#), we can double this essential annulus in $(G, \partial G)$ to obtain an essential torus in $(K, \partial K)$ which clearly crosses the torus in K represented by the subgroup A . Now we consider the uncompleted torus decomposition $T_{n+1}(K, \partial K)$ of the orientable $PD(n+2)$ pair $(K, \partial K)$. As K admits an essential torus, either this decomposition consists of a single V_0 -vertex or it has at least one edge.

If $T_{n+1}(K, \partial K)$ has at least one edge, then the associated splitting σ of K along an essential torus cannot cross any torus in K . In particular, it cannot

cross the torus A , nor can it equal this torus. It follows that σ determines an essential torus in $(G, \partial G)$. But this contradicts the fact that $(G, \partial G)$ is atoroidal. It follows that $T_{n+1}(K, \partial K)$ must consist of a single V_0 -vertex, so that either K is VPC or the pair $(K, \partial K)$ is $VPCn$ -by-Fuchsian.

If K is VPC , then G must also be VPC , so Corollary 2.2.9 tells us that either G has two boundary groups each equal to G , or G has one boundary group which is a subgroup of G of index 2. The first case is not possible as we assumed g does not lie in A , and the second case is not possible, as part 1) of Corollary 2.2.8 shows that G would be $VPC(n+1)$ and non-orientable which contradicts our assumption that every $VPC(n+1)$ subgroup of G is orientable.

If the pair $(K, \partial K)$ is $VPCn$ -by-Fuchsian, we let L denote the $VPCn$ normal subgroup and let Φ denote the quotient Fuchsian group. We can assume that K is not VPC , so Lemma 2.1.9 tells us that L is unique. Also part 1) of Lemma 2.1.12 tells us that as A is $VPC(n+1)$ the intersection $A \cap L$ must be $VPCn$ and hence of finite index in L . As K is the double of G along A , it follows that L must be conjugate into a vertex group of this splitting. As L is normal in K , it now follows that A must contain L . Thus G is itself isomorphic to a $VPCn$ -by-Fuchsian group, where the normal $VPCn$ subgroup is L . We denote the quotient group by Θ . As A is a $VPC(n+1)$ subgroup of G , the group Θ must be infinite. If Θ is two-ended, then G is $VPC(n+1)$, and we have a contradiction by the preceding paragraph. Thus we can assume that Θ is not two-ended. This implies that there are elements δ and ε in Θ of infinite order such that δ and ε have non-zero geometric intersection number. The pre-images in G of the infinite cyclic subgroups of Θ generated by δ and ε are $VPC(n+1)$ subgroups D and E of G . Note that $D \cap E = L$. By replacing δ and ε by their squares if needed, we can ensure that they are orientable elements of Φ , so that D and E will be orientable. As δ and ε have non-zero geometric intersection number, it follows that D and E are tori in K which cross. As D and E are subgroups of G , it follows that D and E are tori in $(G, \partial G)$ which cross. But this contradicts the hypothesis that $(G, \partial G)$ is atoroidal. This contradiction completes the proof of the proposition. \square

We now apply Proposition 2.5.3 to get information about the V_1 -vertices of the torus decomposition of a Poincaré duality pair.

Proposition 2.5.5. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let w be a V_1 -vertex of the uncompleted torus decomposition $T_{n+1}(G, \partial G)$. Let s and t be edges of $T_{n+1}(G, \partial G)$ which are incident to w , where possibly $s = t$. Let S and*

T be VPC^n subgroups of $G(s)$ and $G(t)$ respectively, and let g be an element of $G(w)$ such that $gSg^{-1} = T$. Then one of the following holds:

1. $s = t$, and $g \in G(s)$.
2. s and t are distinct and v is isolated, so that s and t are the only edges incident to w , and $G(s) = G(w) = G(t)$.
3. $s = t$, the vertex w has valence 1, and $G(s)$ has index 2 in $G(w)$.

Remark 2.5.6. If $n = 1$ and M is an orientable Haken 3-manifold, and a component W of $M - T(M)$ admits a π_1 -injective annulus with boundary in $fr(W)$, then either this annulus can be homotoped into $fr(W)$, or W is homeomorphic to $T \times I$ or to a twisted I -bundle over the Klein bottle.

Proof. Let $E(w)$ denote the family of subgroups of $G(w)$ which are edge groups for the edges incident to w . If ∂G is empty, then Theorem 8.1 of [1] shows that the pair $(G(w), E(w))$ is $PD(n+2)$. In general, as discussed just before Definition 2.5.1, the pair becomes $PD(n+2)$ when $E(w)$ is augmented by some groups in ∂G , and the $PD(n+2)$ pair obtained this way is atoroidal. Applying Proposition 2.5.3 to this pair yields three cases, which yield the three cases of this proposition. \square

Remark 2.5.7. If we consider the completed torus decomposition $T_{n+1}^c(G, \partial G)$ of the $PD(n+2)$ pair, then the third case in Proposition 2.5.5 cannot occur. For such V_1 -vertices of $T_{n+1}(G, \partial G)$ become V_0 -vertices when $T_{n+1}(G, \partial G)$ is completed to $T_{n+1}^c(G, \partial G)$.

We will also need information about the V_0 -vertices of the torus decomposition of a Poincaré duality pair. Part 1) of Theorem 2.4.4 states that a V_0 -vertex of the uncompleted torus decomposition $T_{n+1}(G, \partial G)$ must be isolated or of Seifert type adapted to ∂G . In the next result, we consider the second type of V_0 -vertex.

Lemma 2.5.8. Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let v be a V_0 -vertex of the uncompleted torus decomposition $T_{n+1}(G, \partial G)$ which is of Seifert type adapted to ∂G . (See Definition 2.3.4.) Let L denote the VPC^n normal subgroup of $G(v)$ with Fuchsian quotient Φ . Let s and t be edges of $T_{n+1}(G, \partial G)$ which are incident to v , where possibly $s = t$. Let S and T be VPC^n subgroups of $G(s)$ and $G(t)$ respectively, and let g be an element of $G(v)$ such that $gSg^{-1} = T$. Then one of the following holds:

1. $s = t$, and $g \in G(s)$.
2. S is commensurable with L .

Remark 2.5.9. *If $n = 1$ and M is an orientable Haken 3-manifold, then any component of $T(M)$ is a Seifert fibre space. The corresponding result is a standard result about Seifert fibre spaces. Namely that if W is a Seifert fibre space with incompressible boundary which admits a π_1 -injective annulus with boundary in ∂W , then either this annulus can be homotoped into ∂W , or this annulus is vertical in W .*

Proof. Let $E(v)$ denote the family of subgroups of $G(v)$ which are edge groups for the edges incident to v . If ∂G is empty, then as discussed just before Definition 2.5.1, the pair $(G(v), E(v))$ is orientable $PD(n + 2)$. In general, the pair becomes $PD(n + 2)$ when $E(v)$ is augmented by some groups in ∂G . Let $\overline{E}(v)$ denote this augmented family of groups. As v is of Seifert type adapted to ∂G , the normal subgroup L of $G(v)$ is contained in each group in $\overline{E}(v)$. Let $\partial\Phi$ denote the family of subgroups of Φ obtained by taking the quotient by L of each group in $\overline{E}(v)$. Thus the pair $(\Phi, \partial\Phi)$ is the orbifold fundamental group of a compact 2-dimensional orbifold $(F, \partial F)$. Note that as v is of Seifert type adapted to ∂G , the group Φ is not finite nor two-ended.

Let S', T' and g' denote the images of S, T and g in Φ , so that we have the equation $g'S'g'^{-1} = T'$ in Φ . If S' is finite, then $S \cap L$ has finite index in S , so that S is commensurable with L , and we have case 2) of the lemma. Otherwise S' is an infinite subgroup of a group H in $\partial\Phi$. Let A denote an infinite cyclic subgroup of S' , and let F_A denote the orbifold cover of F with fundamental group A . Thus F_A has a boundary component which carries A . Also, as A is torsion free, F_A is a surface. If g' does not lie in H , then this surface admits an essential annulus, and so must be an annulus. In particular, F_A is compact and hence a finite cover of F . As Φ is not two-ended, this is impossible so that g' must lie in H , and hence so does T' . Thus S', T' and g' all lie in the same group H in $\partial\Phi$, which means that we have case 1) of the lemma. \square

An important consequence of Proposition 2.5.5 is the following.

Lemma 2.5.10. *Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair, let V be a non-isolated V_0 -vertex of the uncompleted torus decomposition $T_{n+1}(G, \partial G)$ which is of Seifert type adapted to ∂G , and let L denote the VPC_n normal subgroup of $G(V)$ with Fuchsian quotient. Then*

$$G(V) = N_G(L) = \text{Comm}_G(L),$$

where $N_G(L)$ denotes the normalizer of L in G .

Remark 2.5.11. *If $n = 1$ and M is an orientable Haken 3-manifold, then any component of $T(M)$ is a Seifert fibre space, and so its fundamental group has an infinite cyclic normal subgroup. Let W be a component of $T(M)$, and L this normal subgroup. Then the corresponding result is that $\pi_1(W)$ is equal to the commensuriser of L in $\pi_1(M)$.*

Proof. The inclusions $G(V) \subset N_G(L) \subset \text{Comm}_G(L)$ are all clear. Thus it remains to prove that $\text{Comm}_G(L) \subset G(V)$.

Let T denote the universal covering G -tree of $T_{n+1}(G, \partial G)$ and let v denote a vertex of T above V with stabiliser $G(v)$ equal to $G(V)$. Suppose that $\text{Comm}_G(L)$ does not equal $G(V)$. Then there is an element g of $\text{Comm}_G(L)$ which does not fix v , and we let L' denote the intersection $L \cap gLg^{-1}$. Thus L' is a $VPCn$ subgroup of G which fixes both v and gv , and hence fixes every edge on the path λ joining v to gv .

If w is a V_1 -vertex of λ , then L' fixes w and two distinct incident edges. Now Proposition 2.5.5 shows that w has valence 2 in T and that the two incident edges each have the same stabiliser. It also implies that $G(w)$ contains this stabiliser with index 1 or 2, but we will not need to distinguish these cases. Recall that any V_0 -vertex of $T_{n+1}(G, \partial G)$ is isolated or of Seifert type adapted to ∂G . If v_0 is an interior V_0 -vertex of λ which is of Seifert type adapted to ∂G , then Lemma 2.5.8 shows that L' must be commensurable with the $VPCn$ subgroup L_0 of $G(v_0)$ which is normal in $G(v_0)$ with Fuchsian quotient.

Now let μ denote the subinterval of λ between v and the first V_0 -vertex w of λ which is of Seifert type adapted to ∂G . Possibly μ equals λ . The preceding discussion shows that any pair of adjacent edges of μ have the same stabiliser. Thus the stabiliser of μ equals an edge group H of $G(v)$, which is also an edge group of $G(w)$. Hence the subgroup K of G generated by $G(v)$ and $G(w)$ equals $G(v) *_H G(w)$, and K has the natural structure of an orientable $PD(n+2)$ pair. The splitting of K over H is over an essential torus in K . Recall that L is the $VPCn$ normal subgroup of $G(v)$ with Fuchsian quotient, and let L'' denote the corresponding $VPCn$ normal subgroup of $G(w)$. If w equals gv , then $L'' = gLg^{-1}$, so that L' is a subgroup of L'' . If w is not equal to gv , then the discussion in the preceding paragraph shows that L' is commensurable with L'' . As L' is contained in L , it follows that in either case L and L'' are commensurable. As each is a normal subgroup of H with quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, part 2) of Lemma 2.1.11 shows that L and L'' must be equal. Thus the pair $(K, \partial K)$ is $VPCn$ -by-Fuchsian. It follows that there is an essential torus in K which

crosses the essential torus determined by H , and hence an essential torus in G which crosses the torus determined by H . But this contradicts the fact that the edge splittings of $T_{n+1}(G, \partial G)$ do not cross any essential torus in G .

We conclude that every element of $Comm_G(L)$ lies in $G(V)$. Thus $G(V) = N_G(L) = Comm_G(L)$ as required. \square

There is another more technical consequence of Proposition 2.5.5 which will only be needed in the case when G is a $PD(n+2)$ group, i.e. when ∂G is empty, but the general result is no more difficult. Before stating this result it will be convenient to have the following definition.

Definition 2.5.12. *Let Γ be a minimal graph of groups structure for a group G , let v be a vertex of Γ , and let X be a H -almost invariant subset of G which is enclosed by v .*

Then X is peripheral in v if X is equivalent to the almost invariant subset of G associated to some edge of Γ which is incident to v .

The word peripheral here is very natural, but the reader should note that when $(G, \partial G)$ is a Poincaré duality pair, this idea may have nothing to do with ∂G .

Lemma 2.5.13. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let H and H' be essential tori in $(G, \partial G)$ whose dual almost invariant sets X and X' are enclosed by V_0 -vertices V and V' of $T_{n+1}(G, \partial G)$. Suppose that $H \cap H'$ is a $VPCn$ subgroup K of G . Suppose further that X is not peripheral in V . Then V must be of Seifert type adapted to ∂G , each of X and X' is enclosed by V , and K is commensurable with the $VPCn$ normal subgroup L of $G(V)$ with Fuchsian quotient.*

Proof. As X is not peripheral in V , the vertex V of $T_{n+1}(G, \partial G)$ cannot be isolated. Thus Theorem 2.4.4 implies it must be of Seifert type adapted to ∂G . Now part 1) of Lemma 2.1.12 tells us that the intersection $H \cap L$ must be $VPCn$, and hence of finite index in L .

Suppose that V equals V' . Then part 1) of Lemma 2.1.12 tells us that the intersection $H' \cap L$ must also be $VPCn$, and hence of finite index in L . Thus the intersection $H \cap H' \cap L$ is of finite index in L , and so is also $VPCn$. As $K = H \cap H'$ is assumed to be $VPCn$, it follows that K and L must be commensurable, as required.

In what follows, we will suppose that V and V' are distinct. Let T denote the universal covering G -tree of $T_{n+1}(G, \partial G)$. Let v be a vertex above V which is fixed by H , and let v' be a vertex above V' which is fixed by H' . Then K fixes

the path λ joining v to v' . Now we are in much the same situation as in the proof of Lemma 2.5.10, with v' in place of gv . Thus if w is a V_1 -vertex of λ , then w has valence 2 in T and the two incident edges each have the same stabiliser, and if v_0 is an interior V_0 -vertex of λ which is of Seifert type adapted to ∂G , then K must be commensurable with the $VPCn$ normal subgroup L_0 of $G(v_0)$ with Fuchsian quotient.

Now we consider the edge e of λ which is incident to v . As K fixes e , it is a subgroup of the torus group $G(e)$. Our hypothesis that X is not peripheral in V implies that $H \cap G(e)$ is commensurable with the normal $VPCn$ subgroup L of $G(v)$. As K is a $VPCn$ subgroup of $G(e)$ and of H , it follows that K must be commensurable with L . It remains to prove that X' must be enclosed by V .

As in the proof of Lemma 2.5.10, we let μ denote the subinterval of λ between v and the first V_0 -vertex v'' of λ which is of Seifert type adapted to ∂G . As before any pair of adjacent edges of μ have the same stabiliser. Thus the edge group $G(e)$ is equal to an edge subgroup of $G(v'')$.

If v'' is not equal to v' , so that μ is properly contained in λ , the above discussion shows that K must be commensurable with the normal $VPCn$ subgroup L'' of $G(v'')$. As K is commensurable with L and L'' , it follows that L and L'' are commensurable. As each is a normal subgroup of $G(e)$ with quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, Lemma 2.1.11 shows that L and L'' must be equal. Now Lemma 2.5.10 yields a contradiction. We deduce that μ must equal λ .

Let e' denote the edge of λ which is incident to v' . As all consecutive edges of λ have the same stabiliser, it follows that $G(e) = G(e')$. If X' is not peripheral in v' , the preceding argument with the roles of X and X' reversed shows that K must be commensurable with the normal subgroup L' of $G(v')$. If X' is peripheral in v' but is not equivalent to the almost invariant set associated to the edge e' , then Lemma 2.5.8 shows that K must be commensurable with L' . Thus either K is commensurable with L' , or X' is equivalent to the almost invariant subset of G associated to e' . In the first case, we derive a contradiction as before. In the second case, we recall that each interior vertex of λ is isolated or of special Seifert type. Thus consecutive edges of λ not only have the same stabilisers but they have equivalent associated almost invariant subsets of G . Hence X' must be equivalent to the almost invariant subset of G associated to e , so that X' is enclosed by v , and hence by V , as required. This completes the proof of the lemma. \square

2.6 Enclosing properties of Annulus-Torus Decompositions

Let G denote any almost finitely presented group which has no nontrivial almost invariant subsets over $VPC(< n)$ subgroups. Recall that $\mathcal{F}_{n,n+1}$ denotes the family of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPCn$ subgroups and of equivalence classes of all n -canonical almost invariant subsets of G which are over $VPC(n+1)$ subgroups. The decomposition $\Gamma_{n,n+1}$ of G is the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$, and its V_0 -vertices correspond to the cross connected components of $\mathcal{F}_{n,n+1}$. Some of its properties are described in Theorem 2.1.16. For brevity, we will denote $\mathcal{F}_{n,n+1}$ by \mathcal{F} in the rest of this section. In the case when $(G, \partial G)$ is a $PD(n+2)$ pair, we will also consider the family \mathcal{F}' of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPCn$ subgroups together with equivalence classes of all nontrivial almost invariant subsets of G which are over $VPC(n+1)$ subgroups and are adapted to ∂G . Note that, by Lemma 2.2.13, nontrivial almost invariant subsets of G which are over $VPC(n+1)$ subgroups and are adapted to ∂G are dual to essential tori in $(G, \partial G)$. Thus from the topological point of view, it seems very natural to consider the family \mathcal{F}' . However, one obvious reason why our theory in [22] discusses algebraic regular neighbourhoods of $\mathcal{F} = \mathcal{F}_{n,n+1}$ rather than \mathcal{F}' is that, for general groups, there is no analogue of \mathcal{F}' . Another reason is that elements of $\mathcal{F}' - \mathcal{F}$ need not be n -canonical and our methods in [22] cannot handle this situation. In this section, we will show that \mathcal{F} is contained in \mathcal{F}' (Corollary 2.6.4) and that \mathcal{F} and \mathcal{F}' have the same reduced algebraic regular neighbourhoods (Theorem 2.6.17). This technical result will be used in section 2.7 of this paper in a crucial way. It is the algebraic analogue of the situation discussed in the second paragraph of section 2.1, with $\Gamma_{n,n+1}$ being the algebraic analogue of $AT(M)$, the family \mathcal{F}' being the algebraic analogue of the family of all essential annuli and tori in M , and the family \mathcal{F} being the algebraic analogue of the family of all essential annuli in M together with those essential tori in M which do not cross any essential annulus in M .

First we will show that \mathcal{F} is contained in \mathcal{F}' . To do this we need to consider almost invariant subsets of G which are not adapted to ∂G . We start with the following simple result.

Lemma 2.6.1. *Let $n \geq 1$, and let $(G, \partial G)$ be an orientable $PD(n+2)$ pair. Suppose that X is a nontrivial almost invariant subset of G over a $VPC(n+1)$ group H and that X is not adapted to ∂G . Then there is a subgroup S_i of G with a*

conjugate in ∂G such that $H \cap S_i$ is $VPCn$.

Proof. As X is not adapted to ∂G , there is S_i in ∂G , and $g \in G$, such that both $X \cap gS_i$ and $X^* \cap gS_i$ are not H -finite. By replacing S_i by a conjugate, we can arrange that both $X \cap S_i$ and $X^* \cap S_i$ are not H -finite. Thus both of them are not $(H \cap S_i)$ -finite. Let K denote $H \cap S_i$. Then $X \cap S_i$ is a nontrivial almost invariant subset of S_i which is over K , so that $e(S_i, K) > 1$. As K is a subgroup of the $VPC(n+1)$ group H , it must be $VPC(\leq n+1)$. As K must have infinite index in the $PD(n+1)$ group S_i , Strebel's result [29] shows that K has cohomological dimension $\leq n$. Thus K is $VPC(\leq n)$.

Recall the following long exact cohomology sequence from page [18]

$$H^0(G; \mathbb{Z}_2 E) \rightarrow H^0(G; PE) \rightarrow H^0(G; PE/\mathbb{Z}_2 E) \rightarrow H^1(G; \mathbb{Z}_2 E) \rightarrow H^1(G; PE) \rightarrow$$

Here H is a subgroup of a group G , and E denotes $H \backslash G$. Also $H^0(G; PE) \cong \mathbb{Z}_2$, and $e(G, H)$ equals the dimension over \mathbb{Z}_2 of $H^0(G; PE/\mathbb{Z}_2 E)$. Thus if $e(G, H) > 1$, then $H^1(G; \mathbb{Z}_2 E)$ is non-zero. In the setting of the present lemma, as $e(S_i, K) > 1$ we see that $H^1(S_i; \mathbb{Z}_2(K \backslash S_i))$ must be non-zero. As S_i is $PD(n+1)$, this last group is isomorphic to $H_n(S_i; \mathbb{Z}_2(K \backslash S_i))$ which is in turn isomorphic to $H_n(K, \mathbb{Z}_2)$. Thus $H_n(K, \mathbb{Z}_2)$ is non-zero. As K is torsion free and $VPC(\leq n)$, it follows that K must be $VPCn$, so that $H \cap S_i$ is $VPCn$ as required. \square

Now we can prove the following.

Proposition 2.6.2. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair. Suppose that X is a nontrivial almost invariant subset of G which is over a $VPC(n+1)$ subgroup H and is not adapted to ∂G . Then X crosses some nontrivial almost invariant subset of G which is over some $VPCn$ subgroup of G and is dual to an essential annulus.*

Remark 2.6.3. *In particular, if such X exists, then there are essential annuli in the pair $(G, \partial G)$ which are over $VPCn$ subgroups of H .*

Proof. The proof of this result is suggested by our arguments in [21]. Since X is not adapted to ∂G , the proof of Lemma [2.6.1] shows that, after conjugation, there is a group S_i in ∂G such that $X \cap S_i$ is a nontrivial almost invariant subset of S_i over $H \cap S_i$, and $H \cap S_i$ is $VPCn$. We denote $H \cap S_i$ by K . By replacing H by a subgroup of finite index if necessary, we can arrange that H is orientable, that K has a finite index subgroup L which is normal in H , and that the quotient $L \backslash H$ is infinite cyclic. Theorem 7.3 of [1] now implies that L is orientable.

As in section [2.2](#), we choose an aspherical space M with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . We denote the cover of M corresponding to L by M_L . As S_i contains L , there is a component Σ of ∂M_L with fundamental group L . Since L is normal in H with infinite cyclic quotient, the quotient $L \setminus H$ acts naturally on M_L , and we obtain infinitely many components of ∂M_L which are translates of Σ and have fundamental group L . Let Z denote the support of a 0-cochain on M_H with finite coboundary which represents the element of $H_e^0(M_H; \mathbb{Z}_2)$ determined by $H \setminus X$. Let $p : M_L \rightarrow M_H$ denote the covering projection. As $X \cap S_i$ is a nontrivial L -almost invariant subset of S_i , the vertices of $p(\Sigma)$ in M_H meet both Z and Z^* in infinite sets. As X is H -invariant, the vertices of the image of each translate of Σ also meet both Z and Z^* in infinite sets.

As the number of these translates of Σ is infinite, we can apply Lemma [2.2.22](#). As in that lemma, let $\Sigma_1, \Sigma_2, \Sigma_3$ and Σ_4 denote four distinct translates of Σ by elements of $L \setminus H$, and let A_{ij} denote the annulus in M_L with fundamental group L and joining Σ_i and Σ_j . Let Y_{ij} denote the L -almost invariant subset of G dual to A_{ij} . Let Z_{ij} denote the support of a 0-cochain on M_L with finite coboundary which represents the element of $H_e^0(M_L; \mathbb{Z}_2)$ determined by $L \setminus Y_{ij}$. The proof of Lemma [2.2.22](#) shows that there are distinct integers i, j, k and l such that Z_{ij} separates Σ_k and Σ_l , i.e. Σ_k is almost contained in Z_{ij} and Σ_l is almost contained in Z_{ij}^* , or vice versa. It follows that X must cross Y_{ij} because each of the four corners of the pair (Z, pZ_{ij}) is infinite, as it has infinite intersection with $p\Sigma_k$ or $p\Sigma_l$. As Y_{ij} is dual to an essential annulus, this completes the proof of the lemma. \square

The point of this proposition is the following corollary.

Corollary 2.6.4. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let \mathcal{F} and \mathcal{F}' be defined as above. If X is a n -canonical almost invariant subset of G over a $VPC(n+1)$ subgroup H , then X is automatically adapted to ∂G . Thus \mathcal{F} is contained in \mathcal{F}' .*

Remark 2.6.5. *As X is adapted to G , it follows from Lemma [2.2.13](#) that H must be orientable.*

Proof. If X is not adapted to ∂G , then Proposition [2.6.2](#) tells us that X crosses some nontrivial almost invariant set over some $VPCn$ subgroup of G , which contradicts the hypothesis that X is n -canonical. Hence the definitions of \mathcal{F} and \mathcal{F}' show that \mathcal{F} is contained in \mathcal{F}' , as claimed. \square

Next we give several results about how elements of $\mathcal{F}' - \mathcal{F}$ can cross elements of \mathcal{F} .

Lemma 2.6.6. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let \mathcal{F} and \mathcal{F}' be defined as above. Let X be an element of $\mathcal{F}' - \mathcal{F}$ which crosses a nontrivial almost invariant set Y over a $VPCn$ group K . Then the following hold:*

1. *If Y is dual to an essential annulus, then X crosses Y strongly.*
2. *There is an almost invariant set Y' over a group K' commensurable with K such that Y' is dual to an essential annulus and X crosses Y' strongly.*

Remark 2.6.7. *If $n = 1$ and M is an orientable Haken 3-manifold, the corresponding result holds. For X corresponds to a torus which must be homotopic into a component W of $T(M)$ which meets ∂M . Such W is a Seifert fibre space whose intersection with ∂M consists of tori and vertical annuli. Thus W is filled by vertical annuli, so X must cross one of them.*

Note that in this topological situation, an annulus can never cross a torus strongly, for the intersection of their fundamental groups must be of finite index in the fundamental group of the annulus.

Proof. As X lies in $\mathcal{F}' - \mathcal{F}$, it is a nontrivial almost invariant subset of G which is over an orientable $VPC(n+1)$ group H and is adapted to ∂G .

1) Suppose that Y is dual to an essential annulus A . As discussed immediately after the proof of Lemma 2.2.22, the double DG of G along ∂G contains a torus which is the double of the annulus A . We let DY denote the DK -almost invariant subset of DG associated to this torus. As X is adapted to ∂G , Lemma 2.2.5 yields a H -almost invariant subset \bar{X} of DG such that $\bar{X} \cap G$ equals X . As X and Y cross, it follows that \bar{X} and DY must also cross. As H and DK are both orientable $VPC(n+1)$ subgroups of the orientable $PD(n+2)$ group DG , each has two coends in DG . Now Proposition 7.4 of [22] shows that \bar{X} and DY must cross each other strongly. This implies that X crosses Y strongly, thus completing the proof of part 1).

2) As in section 2.2, we choose an aspherical space M with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Consider the cover M_K , the element of $H_f^1(M_K; \mathbb{Z})$ corresponding to $K \setminus Y$ and its dual class α in $H_{n+1}(M_K, \partial M_K; \mathbb{Z})$. The boundary of α has support in only finitely many components $\Sigma_1, \dots, \Sigma_k$ of ∂M_K . Note that Proposition 2.2.20 tells us that there must be at least one such component of ∂M_K , and each such

component must carry a subgroup of finite index in K . By replacing K by a suitable subgroup of finite index, we can arrange that K is orientable and that each of $\Sigma_1, \dots, \Sigma_k$ carries K itself. As K is orientable, we know that $k \geq 2$. For each pair of distinct integers i and j between 1 and k , there is an essential untwisted annulus A_{ij} in M_K whose boundary lies in $\Sigma_i \cup \Sigma_j$. Let Y_{ij} denote the dual K -almost invariant subset of G . Let Z_{ij} denote the support of a 0-cochain on M_K with finite coboundary which represents the element of $H_e^0(M_K; \mathbb{Z}_2)$ determined by $K \setminus Y_{ij}$. We will show that X must cross one of these Y_{ij} 's. This will be the required Y' . Now part 1) will imply that X crosses Y' strongly.

Let Z denote the support of a 0-cochain on M_H with finite coboundary which represents the element of $H_e^0(M_H; \mathbb{Z}_2)$ determined by $H \setminus X$. As X is adapted to ∂G , we know that for each component Σ of ∂M_H the vertices of Σ are almost all in Z or almost all in Z^* . Further the vertices of Σ are H -infinite, as X is nontrivial, so that the vertices of Σ cannot be almost all in Z and almost all in Z^* . Thus if X crosses none of the Y_{ij} 's, it follows that, after replacing X by X^* if needed, the vertices of the images in M_H of each Σ_i , $1 \leq i \leq k$, almost all lie in Z . As X and Y_{ij} do not cross, it now follows that $Y_{ij} \leq X$ or $Y_{ij}^* \leq X$. By replacing Y_{ij} by its complement if needed, we can arrange that $Y_{ij} \leq X$, for all i and j . Part 2) of Proposition 2.2.21 tells us that Y is equivalent to a sum of some of the Y_{ij} 's and their complements. If there are no complements in the sum, then we have $Y \leq X$, which contradicts the assumption that X crosses Y . The same inequality holds if the number of complements in the sum is even. If the number of complements in the sum is odd, then we have $X^* \leq Y$, which again contradicts the assumption that X crosses Y . This contradiction shows that X must cross some Y_{ij} and so completes the proof of the lemma. \square

Two easy consequences are the following results.

Lemma 2.6.8. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let \mathcal{F} and \mathcal{F}' be defined as above. Let X be an element of $\mathcal{F}' - \mathcal{F}$ which crosses nontrivial almost invariant sets Y and Y' over $VPCn$ groups L and L' respectively. Then L and L' are commensurable.*

Remark 2.6.9. *If $n = 1$ and M is an orientable Haken 3-manifold, the corresponding result holds. For X corresponds to a torus which crosses an annulus in M , and so must be homotopic into a component W of $T(M)$ which meets ∂M . Such W is a Seifert fibre space whose intersection with ∂M consists of tori and vertical annuli. If W has a unique Seifert fibration, up to isotopy, it follows that all essential annuli in M which are homotopic into W carry commensurable groups as this group must*

be commensurable with the fibre group of W . Otherwise, W is homeomorphic to $T \times I$ or to $K \tilde{\times} I$, and in either case $W \cap \partial M$ must consist of annuli alone. In this case, all essential annuli in M which are homotopic into W carry commensurable groups as their boundaries lie in $W \cap \partial M$.

Proof. As X lies in $\mathcal{F}' - \mathcal{F}$, it is a nontrivial almost invariant subset of G which is over an orientable $VPC(n+1)$ group H and is adapted to ∂G . By part 2) of Lemma 2.6.6, we may assume that X crosses Y and Y' strongly. As X crosses Y strongly, it follows that $e(H, H \cap L) \geq 2$. As H is $VPC(n+1)$, it follows that $H \cap L$ is $VPCn$ and hence of finite index in L . Thus by replacing L by a subgroup of finite index, we can assume that it is a subgroup of H . Similarly we can assume that L' is also a subgroup of H . If L and L' are not commensurable, it follows that some element of L' is hyperbolic with respect to Y which implies that Y' crosses Y strongly. Now Corollary 7.10 of [22] implies that L has small commensuriser which contradicts the fact that H commensurises L . This contradiction shows that L and L' must be commensurable as required. \square

Lemma 2.6.10. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let \mathcal{F} and \mathcal{F}' be defined as above. Let X be an element of $\mathcal{F}' - \mathcal{F}$ and let X' be an element of \mathcal{F}' dual to a torus in $(G, \partial G)$. Suppose that X crosses X' , and also crosses a nontrivial almost invariant set Y over a $VPCn$ group L . Then X' crosses a nontrivial almost invariant set Y' over a $VPCn$ group L' commensurable with L . In particular, X' lies in $\mathcal{F}' - \mathcal{F}$.*

Remark 2.6.11. *If $n = 1$ and M is an orientable Haken 3-manifold, the corresponding result is clear. For X corresponds to a torus which crosses an annulus in M , and so must be homotopic into a component W of $T(M)$ which meets ∂M . As X crosses X' , the corresponding tori cross, so both must be homotopic into W . In addition, each of these tori cannot be homotopic to a boundary component of W . Now any such torus in W must cross an annulus in W , so that X' lies in $\mathcal{F}' - \mathcal{F}$.*

Proof. The hypotheses imply that X and X' are nontrivial almost invariant subsets of G which are over orientable $VPC(n+1)$ subgroups H and H' respectively, and are adapted to ∂G . As they are adapted to ∂G , they can be extended to almost invariant sets in DG over the same groups H and H' . As DG is $PD(n+2)$, and H and H' are each $PD(n+1)$, it follows that H and H' each has two coends in DG , so that these extended almost invariant sets must cross strongly if at all. As X and X' cross, it follows that they must cross each other strongly. Let L' denote the intersection $H \cap H'$, which is $VPCn$. By part 2) of

Lemma 2.6.6, we may assume that X crosses Y strongly, and the proof of Lemma 2.6.8 shows that we can assume that L is a subgroup of H .

If L and L' are not commensurable subgroups of H , then some element of L' is hyperbolic with respect to Y . This implies that X' crosses Y strongly, thus proving the lemma in this case.

Now suppose that L and L' are commensurable. Lemma 13.1 of [22] implies that there are finite index subgroups H_1 of H and L_1 of L such that L_1 is normal in H_1 with infinite cyclic quotient. We let Γ denote the Cayley graph of G with respect to some finite generating set, and consider the action of $L_1 \backslash H_1$ on the graph $L_1 \backslash \Gamma$. Let h be an element of $H_1 - L_1$. As $L_1 \backslash \delta Y$ is finite, there is a finite connected subcomplex C of $L_1 \backslash \Gamma$ which contains $L_1 \backslash \delta Y$. Thus for all but finitely many elements g of $L_1 \backslash H_1$, we have $gC \cap C$ empty, so that $g(L_1 \backslash Y)$ and $L_1 \backslash Y$ are nested. As X crosses Y strongly, the intersection $Y \cap H_1$ is a nontrivial L_1 -almost invariant subset of H_1 . As $L_1 \backslash H_1$ is infinite cyclic, there is a power h^m of h such that $Y \cap H_1 \subset h^m(Y \cap H_1)$. Thus by replacing h by a suitable power if needed, we can arrange that $Y \subset hY$. Also Y cannot be equivalent to hY . For if this happens, then L_1 has infinite index in $\{g \in H_1 : gY \text{ is equivalent to } Y\}$, and we can apply Theorem 5.8 from [26] to the action of $L_1 \backslash H_1$ on $L_1 \backslash \Gamma$. The proof of this result implies that H_1 must have finite index in G , which is impossible as X is a nontrivial almost invariant subset of G over H .

As h normalises L_1 , each translate $h^n Y$ of Y is also L_1 -almost invariant. If X' crosses Y or any translate $h^n Y$, we will have proved the lemma. Thus we can suppose that X' does not cross $h^n Y$, for any integer n . By replacing X' and Y by their complements if needed, we can arrange that $X' \leq Y$. (If we replace Y by its complement, we simultaneously replace h by its inverse, in order to preserve the inclusion $Y \subset hY$.) Now this implies that $X' \leq h^k Y$ for all $k \geq 0$. We claim that the inequality $X' \leq h^k Y$ cannot hold for every integer k . To see this pick a finite generating set for G and let C denote the corresponding Cayley graph for G . Now recall that given a pair of nontrivial almost invariant subsets U and W of G , each over a finitely generated subgroup of G , there is an integer d such that if $U \leq gW$ then U is contained in the d -neighbourhood of gW , where distances are measured in C . As the intersection of all the $h^k Y$ is empty, the inequalities $X' \leq h^k Y$, for every k , would imply that X' is empty, which is a contradiction. This completes the proof of the claim. (Alternatively the claim follows immediately using the fact, proved in [19], that X' and Y are equivalent to almost invariant sets in very good position, which implies that the partial orders on E induced by inclusion and by \leq are the same.) Thus there must be a least integer k such that $X' \not\leq h^k Y$. Now by replacing Y by $h^k Y$ if

needed, we can suppose that $k = 0$. Thus we have $X' \leq Y$ and $X' \not\leq h^{-1}Y$.

Recall that L and $L' = H \cap H'$ are commensurable. This implies that h and its powers do not lie in L' . Let h_+ denote $\{h^k : k \geq 0\}$, let h_- denote $\{h^k : k \leq 0\}$, and let h_\pm denote $h_+ \cup h_-$. As X and X' cross strongly, it follows that $h_\pm \cap X'$ and $h_\pm \cap X'^*$ each contain points which are arbitrarily far from $\delta X'$. In particular, both sets are infinite. The fact that $X' \leq Y \subset hY \subset h^2Y \subset \dots$ implies that $h_+ \cap X'$ is finite, so that $h_- \cap X'$ must be infinite.

As we are assuming that X' does not cross $h^n Y$, for any integer n , we know that X' does not cross $h^{-1}Y$. Also recall that $X' \not\leq h^{-1}Y$, so that we must have one of the inequalities $X' \leq h^{-1}Y^*$, $h^{-1}Y^* \leq X'$ or $h^{-1}Y \leq X'$. If $X' \leq h^{-1}Y^*$, then $X' \leq h^{-k}Y^*$, for every $k \geq 0$, so that $h_- \cap X'$ is finite, which is a contradiction. If $h^{-1}Y^* \leq X'$, the inequality $X' \leq Y$ implies that $h^{-1}Y^* \leq Y$, which is impossible as $h^{-1}Y \subset Y$. Thus we must have $h^{-1}Y \leq X'$. As $X' \leq Y$, we know $Y^* \leq X'^*$. Hence X' crosses the nontrivial L_1 -almost invariant set $Y' = Y^* \cup h^{-1}Y$, completing the proof of the lemma. \square

Remark 2.6.12. Example 2.13 of [24] shows that Lemmas 2.6.8 and 2.6.10 are not valid if X is not adapted to ∂G . In that example, G is the fundamental group of an orientable Haken 3-manifold M constructed by gluing two Seifert fibre spaces M_1 and M_2 along a boundary torus T , so that the Seifert fibrations do not match. Thus the given decomposition of M is essentially its JSJ decomposition. Denote $\pi_1(T)$ by H , and let σ' denote the splitting of G over H determined by T . Note that σ' is adapted to ∂G . Let X' denote a H -almost invariant subset of G associated to σ' .

In this example, each M_i has at least one boundary component other than T , and so admits essential annuli disjoint from T . In particular M itself admits essential annuli, so that G does possess nontrivial almost invariant subsets over VPC1 subgroups. We described a splitting σ of G over H which crosses σ' , and is not adapted to ∂G . Let X denote a H -almost invariant subset of G associated to this splitting. Thus X crosses X' . We also showed that X crosses annuli in M_1 and in M_2 . As annuli in M_1 and M_2 carry incommensurable subgroups of G , Lemma 2.6.8 fails. It follows from results we proved in [24] that X' is 1-canonical, i.e. X' does not cross any nontrivial almost invariant subset of G over a VPC1 subgroup. Thus Lemma 2.6.10 also fails.

Now we can start on the proof that \mathcal{F} and \mathcal{F}' have the same algebraic regular neighbourhoods. We want to show that the reduced algebraic regular neighbourhood $\Gamma(\mathcal{F}')$ of \mathcal{F}' is isomorphic to the reduced algebraic regular neighbourhood of \mathcal{F} , which is $\Gamma_{n,n+1}(G)$. It seems natural to approach this by proving that each element of \mathcal{F}' is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}$, but this seems hard to do

directly. Instead we will consider our construction of unreduced algebraic regular neighbourhoods from [22]. We will show that \mathcal{F} and \mathcal{F}' have the same unreduced algebraic regular neighbourhoods which will immediately imply that they also have the same reduced algebraic regular neighbourhoods. Briefly our construction in [22] goes as follows. Let E denote a G -invariant family of nontrivial almost invariant subsets of G , and assume that the elements of E are in good position. Thus we have a G -invariant partial order \leq on E , essentially given by almost inclusion. A cross-connected component (CCC) of E is an equivalence class of the equivalence relation generated by crossing. We let P denote the collection of CCC's of E . We showed that there is a natural idea of betweenness on the elements of P which gives it a pretree structure. If this pretree is discrete, we can construct a bipartite G -tree T with P as its V_0 -vertices, and the graph of groups $\Gamma = G \setminus T$ is the unreduced algebraic regular neighbourhood of the family E .

Remark 2.6.13. *As G is finitely generated, it may be that we could simplify this approach using pretrees by one which uses cubings and very good position, as discussed in section 9 of [6], but we have not attempted to do this.*

Let $P(\mathcal{F})$ denote the collection of all CCC's of elements of \mathcal{F} , and let $P(\mathcal{F}')$ denote the collection of all CCC's of elements of \mathcal{F}' . Thus $P(\mathcal{F})$ and $P(\mathcal{F}')$ have natural pretree structures. As we know that \mathcal{F} has an unreduced algebraic regular neighbourhood, we know that $P(\mathcal{F})$ is a discrete pretree. Recall that Corollary 2.6.4 shows that \mathcal{F} is contained in \mathcal{F}' . This inclusion yields a natural map $\varphi : P(\mathcal{F}) \rightarrow P(\mathcal{F}')$, which is clearly G -equivariant. We will show that φ is a G -equivariant bijection such that φ and φ^{-1} preserve the pretree idea of betweenness. Thus φ induces a G -equivariant isomorphism of the pretrees. This implies that the pretree determined by \mathcal{F}' is discrete, so that \mathcal{F}' has an unreduced algebraic regular neighbourhood in G which is isomorphic to the unreduced algebraic regular neighbourhood of \mathcal{F} . Thus the reduced algebraic regular neighbourhood of \mathcal{F}' in G is isomorphic to $\Gamma_{n,n+1}$ as required.

Recall that $\mathcal{F}' - \mathcal{F}$ consists of all those nontrivial almost invariant subsets of G which are over orientable $VPC(n+1)$ subgroups and are adapted to ∂G , but are not n -canonical. In particular every element of $\mathcal{F}' - \mathcal{F}$ crosses some element of \mathcal{F} . It follows immediately that $\varphi : P(\mathcal{F}) \rightarrow P(\mathcal{F}')$ is surjective. We will show in the proof of Theorem 2.6.17 that it is injective. Note also that φ clearly preserves betweenness in the sense that if three CCC's A, B and C of $P(\mathcal{F})$ satisfy ABC , i.e. B is between A and C , and if the image vertices A', B' and C' of $P(\mathcal{F}')$ are distinct, then $A'B'C'$. But it is not automatic that φ^{-1} preserves betweenness,

and this needs proof in the special case under consideration. We give here a simple example which makes clear that this is a real and subtle problem.

Example 2.6.14. *Let M denote the compact surface obtained from the 2-disc by removing the interiors of two subdiscs. Thus $G = \pi_1(M)$ is free of rank 2. Let λ and μ denote two simple essential arcs properly embedded in M and intersecting transversely in a single point which cannot be removed by an isotopy of λ and μ . Let $N(\lambda)$ and $N(\lambda \cup \mu)$ denote regular neighbourhoods of λ and $\lambda \cup \mu$ respectively, and let $\Gamma(\lambda)$ and $\Gamma(\lambda \cup \mu)$ denote the bipartite graphs of groups decompositions of G which are determined by the frontiers in M of $N(\lambda)$ and $N(\lambda \cup \mu)$ respectively. Then $\Gamma(\lambda)$ and $\Gamma(\lambda \cup \mu)$ will be non-isomorphic graphs of groups. Now λ and μ determine almost invariant subsets $X(\lambda)$ and $X(\mu)$ of G obtained by lifting them to arcs $\bar{\lambda}$ and $\bar{\mu}$ in the universal cover of M and choosing one side of the lift. We can regard $\Gamma(\lambda)$ as the algebraic regular neighbourhood of $X(\lambda)$ and can regard $\Gamma(\lambda \cup \mu)$ as the algebraic regular neighbourhood of the family $\{X(\lambda), X(\mu)\}$. As λ is a simple arc on M , the translates of $\bar{\lambda}$ are disjoint, so that the family $E(\lambda)$ of translates of $X(\lambda)$ and its complement is nested. Thus each CCC of $E(\lambda)$ consists of a single translate of $X(\lambda)$. If we let $E(\lambda, \mu)$ denote the family of translates of $X(\lambda)$ and $X(\mu)$ and their complements, then each CCC of $E(\lambda, \mu)$ will consist of a single translate of $X(\lambda)$ and a single translate of $X(\mu)$. In each case, the stabiliser of each CCC is trivial. Thus if $P(\lambda)$ and $P(\lambda, \mu)$ denote the families of CCC's of $E(\lambda)$ and $E(\lambda, \mu)$, the natural equivariant map φ from $P(\lambda)$ to $P(\lambda, \mu)$ is a bijection. Further it is clear that φ preserves betweenness. But this cannot be true for the inverse map φ^{-1} . For the fact that $\Gamma(\lambda)$ and $\Gamma(\lambda \cup \mu)$ are not isomorphic implies that the pretree structures on $P(\lambda)$ and $P(\lambda, \mu)$ must be different.*

The following technical result is the key to handling this difficulty.

Lemma 2.6.15. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, let \mathcal{F} and \mathcal{F}' be defined as above, and let $\varphi : P(\mathcal{F}) \rightarrow P(\mathcal{F}')$ be the natural map. Let X be an element of $\mathcal{F}' - \mathcal{F}$ which crosses an almost invariant set Y over a VPC^n group K , and let B denote the CCC of $P(\mathcal{F})$ which contains Y so that the CCC $\varphi(B)$ of $P(\mathcal{F}')$ contains X and Y . Then*

1. Y cannot be isolated in \mathcal{F} .
2. If U and V belong to CCC's A and C of $P(\mathcal{F})$ such that $A \neq B \neq C$, and if $U \leq X \leq V$, then there is an element Z of B such that $U \leq Z \leq V$.

Remark 2.6.16. *In the 3-manifold setting, so that $n = 1$, part 1) is easy to see. For let M be a Haken manifold with incompressible boundary, and suppose that an essential torus T in M crosses an essential annulus A . Then both must be homotopic into a component W of $JSJ(M)$ which is a Seifert fibre space which meets ∂M . As such a component W is filled by essential annuli, either A crosses some essential annulus or it is homotopic into a frontier component of W . The second case would imply that A crosses no essential torus of M , which is a contradiction, so that A must cross some essential annulus in M , as required.*

Proof. As X lies in \mathcal{F}' , it is a nontrivial almost invariant subset of G which is over an orientable $VPC(n+1)$ group H and is adapted to ∂G .

1) If Y is isolated in \mathcal{F} , then part 4) of Proposition 2.2.21 implies that Y is dual to an annulus. Now part 1) of Lemma 2.6.6 implies that X crosses Y strongly. Thus $e(H, H \cap K) \geq 2$, so that $H \cap K$ must be $VPCn$, and so of finite index in K . Now Lemma 13.1 of [22] shows that, by replacing H and K by subgroups of finite index, we can suppose that K is a subgroup of H and is normal in H with infinite cyclic quotient. Let h be an element of $H - K$. As K is normal in H , the translate hY of Y is also K -almost invariant. As Y is assumed to be isolated in \mathcal{F} , no translate $h^n Y$ of Y can cross Y . As in the proof of Lemma 2.6.10, by replacing h by a suitable power if needed, we can suppose that $Y \subset hY$. Also as in that proof, Y cannot be equivalent to hY . Now it follows that Y crosses the nontrivial K -almost invariant set $hY^* \cup h^{-1}Y$, so that Y cannot be isolated. This contradiction completes the proof of part 1).

2) If X crosses Y weakly, part 2) of Lemma 2.6.6 shows that there is a K' -almost invariant set Y' such that K' is commensurable with K , and Y' is dual to an annulus. It also shows that X crosses Y' strongly and hence that H commensurises K' . In particular K' , and hence K , has large commensuriser. Now Proposition 8.6 of [22] shows that all nontrivial almost invariant subsets of G over subgroups commensurable with K belong to a single CCC apart from a finite number which are isolated. (Note that this proposition as stated only applies to the case when K is $VPC1$, but essentially the same proof works in general. This is discussed in section 14 of [6].) As X crosses Y and Y' , part 1) tells us that Y and Y' cannot be isolated, so it follows that Y and Y' belong to the same CCC B .

It follows from the preceding paragraph that, by replacing Y by Y' if needed, we can assume that Y is dual to an annulus, so that X crosses Y strongly.

As in part 1), we can further assume that K is a subgroup of H and is normal in H with infinite cyclic quotient. In particular, as H commensurises K , it follows

that H must preserve the CCC B , so that for every $g \in H$, we have gY belongs to B . As no element of the CCC A can cross any element of B , it follows that, for every $g \in H$, we have $U \leq gY$ or $U \leq gY^*$. We will write $U \leq gY^{(*)}$ to indicate that one of these two inequalities holds. Similarly, for every $g \in H$, we have $gY^{(*)} \leq V$. By replacing Y by its complement if needed, we can suppose that $U \leq Y$.

Now we consider the action of $K \setminus H$ on $K \setminus \Gamma$, and let h be an element of $H - K$. As in the proof of Lemma 2.6.10, by replacing h by a suitable power, we can suppose that $Y \subset hY$. Also as in that proof, Y cannot be equivalent to $h^n Y$, for any non-zero value of n . As $U \leq Y$, the inclusion $Y \subset hY$ implies that $U \leq h^k Y$ for all $k \geq 0$. As in the proof of Lemma 2.6.10, the inequality $U \leq h^k Y$ cannot hold for every integer k , and by replacing Y by a suitable translate $h^k Y$ if needed, we can suppose that $U \leq Y$ and $U \not\leq h^{-1} Y$. Thus we must have $U \leq h^{-1} Y^* \subset h^{-2} Y^*$, and hence $U \leq Y \cap h^{-2} Y^*$. Let R denote the intersection $Y \cap h^{-2} Y^*$. As $h^{-2} Y^*$ is K -almost invariant, it follows that R is also K -almost invariant. Now let Z denote the intersection $R \cap X$. We claim that Z is a K -almost invariant subset of G . Certainly $KZ = Z$ as each of R and X is invariant under the left action of K . The coboundary δZ of Z is the union of subsets of δR and δX . As δR is K -finite, it remains to show that $\delta Z \cap \delta X$ is K -finite. Now δX is H -finite, and contains the union of the translates $h^n(\delta Z \cap \delta X)$, for every integer n . As the translates of R by powers of h^2 are all disjoint, it follows that the translates of $\delta Z \cap \delta X$ by powers of h^2 are all disjoint. Now it follows that $\delta Z \cap \delta X$ must be K -finite, so that δZ itself is K -finite as required. Recall that we have the inequalities $U \leq R$ and $U \leq X$, so that we also have $U \leq Z$. Finally we claim that as $Z = Y \cap h^{-2} Y^* \cap X$, it crosses $hZ = hY \cap h^{-1} Y^* \cap X$. This is because none of the four corners of the pair (Z, hZ) is small, which follows from the inclusions $U \subset Z \cap hZ$, $hU \subset Z^* \cap hZ$, $h^{-1} U \subset Z \cap hZ^*$ and $X^* \subset Z^* \cap hZ^*$. Thus Z is a nontrivial and non-isolated almost invariant subset of G over K and hence must belong to the CCC B . Now the inequalities $U \leq Z \leq X \leq V$ complete the proof of part 2) of the lemma. \square

Theorem 2.6.17. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair. Let \mathcal{F} denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPCn$ subgroups and of equivalence classes of all n -canonical almost invariant subsets of G which are over $VPC(n+1)$ subgroups. Thus the reduced algebraic regular neighbourhood of \mathcal{F} exists and is $\Gamma_{n,n+1}$. Let \mathcal{F}' denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over $VPCn$ subgroups together with all nontrivial almost invariant subsets of G which*

are over $VPC(n+1)$ subgroups and are adapted to ∂G . Then \mathcal{F}' has a reduced algebraic regular neighbourhood in G , and this is naturally isomorphic to $\Gamma_{n,n+1}$.

Proof. Recall our discussion immediately before Example 2.6.14. There is a natural G -equivariant surjection $\varphi : P(\mathcal{F}) \rightarrow P(\mathcal{F}')$. We need to show that φ is an injection and that $P(\mathcal{F})$ and $P(\mathcal{F}')$ have the same pretree idea of betweenness.

Suppose that φ is not injective. Then there must be elements Y_0 and Y_1 of \mathcal{F} , belonging to distinct CCC's v_0 and v_1 of $P(\mathcal{F})$, and a sequence X_1, \dots, X_n of elements of $\mathcal{F}' - \mathcal{F}$ such that X_1 crosses Y_0 , and X_n crosses Y_1 , and X_i crosses both X_{i-1} and X_{i+1} , for each i such that $2 \leq i \leq n-1$. As $Y_0 \in \mathcal{F}$, and $X_1 \in \mathcal{F}' - \mathcal{F}$, Lemma 2.6.10 shows that Y_0 cannot be dual to a torus. Similarly Y_1 cannot be dual to a torus. Thus Y_0 and Y_1 are over $VPCn$ subgroups of G .

Let X_i be almost invariant over H_i , and Y_j over K_j , and note that each H_i is orientable. We claim that $H_1 \cap K_0$ has finite index in K_0 . If X_1 crosses Y_0 strongly, then $e(H_1, H_1 \cap K_0) \geq 2$. As H_1 is $VPC(n+1)$, it follows that $H_1 \cap K_0$ must be $VPCn$ and so the claim follows in this case. If X_1 crosses Y_0 weakly, then part 2) of Lemma 2.6.6 tells us that there is a nontrivial almost invariant set Y' over a group K' commensurable with K_0 such that X_1 crosses Y' strongly. As before this implies that $H_1 \cap K'$ must be $VPCn$, so it follows that $H_1 \cap K_0$ must also be $VPCn$, and the claim follows in this case also.

Now Lemma 13.1 of [22] implies that a subgroup of finite index in H_1 commensurises $H_1 \cap K_0$ and hence commensurises K_0 itself, so that K_0 has large commensuriser in G . Proposition 8.6 of [22] shows that the collection of equivalence classes of all nontrivial almost invariant subsets of G which are over a subgroup commensurable with K_0 form a single CCC apart from finitely many isolated elements. Part 1) of Lemma 2.6.15 shows that Y_0 cannot be isolated. Thus the CCC v_0 of $P(\mathcal{F})$ which contains Y_0 contains all the nontrivial and non-isolated almost invariant subsets of G which are over a subgroup commensurable with K_0 . As X_1 crosses an element of v_0 , and also crosses X_2 , Lemma 2.6.10 shows that X_2 must cross some nontrivial almost invariant subset Y_2 of G which is over a subgroup commensurable with K_0 . Again part 1) of Lemma 2.6.15 shows that Y_2 cannot be isolated. It follows that Y_2 lies in the CCC v_0 . Now, by induction, it follows that each X_i must cross some element of v_0 . Thus X_n crosses an element Z_0 of v_0 and the element Y_1 of v_1 . Lemma 2.6.8 shows that Z_0 and Y_1 must have commensurable stabilisers, and Lemma 2.6.15 shows that Y_1 cannot be isolated. But this implies that Y_1 also lies in the CCC v_0 , which contradicts our assumption. This completes the proof that φ is injective and hence is a bijection.

Finally Lemma [2.6.15](#) shows that the betweenness relations on the two pre-trees $P(\mathcal{F}')$ and $P(\mathcal{F}'')$ are the same, which completes the proof of the theorem. \square

2.7 Proof of the Main Theorem

In this section we use our work from previous sections to show how our main result, Theorem [2.3.14](#), follows from Theorem [2.1.16](#). One of the things we need to prove when $(G, \partial G)$ is an orientable $PD(n+2)$ pair is that all the edge splittings of $\Gamma_{n,n+1}(G)$ are dual to essential annuli and tori. If an edge of $\Gamma_{n,n+1}(G)$ is incident to a V_0 -vertex v which is isolated or of $VPCk$ -by-Fuchsian type, with k equal to $n-1$ or n , it is trivial that the edge group is $VPCn$ or $VPC(n+1)$. But if v is of commensuriser type, then we do not even know that the edge group is finitely generated. However we will show in this section that when $(G, \partial G)$ is an orientable $PD(n+2)$ pair the edge groups of $\Gamma_{n,n+1}(G)$ are all $VPCn$ or $VPC(n+1)$. Assuming this, the following result shows that the edge splittings of $\Gamma_{n,n+1}(G)$ are all dual to essential annuli and tori.

Lemma 2.7.1. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . Let $\mathcal{F}_{n,n+1}$ denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup, together with equivalence classes of all n -canonical almost invariant subsets of G which are over a $VPC(n+1)$ subgroup. Let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G . (See Theorem [2.1.16](#).) Finally let e be an edge of $\Gamma_{n,n+1}$ with associated edge splitting σ .*

1. *If $G(e)$ is $VPCn$, then σ is dual to an essential annulus in G .*
2. *If $G(e)$ is $VPC(n+1)$, then σ is dual to an essential torus in G .*

Proof. Let X denote the $G(e)$ -almost invariant subset of G associated to the edge splitting σ . As σ is an edge splitting of $\Gamma_{n,n+1}$, it follows that X crosses no element of $\mathcal{F}_{n,n+1}$.

1) If $G(e)$ is $VPCn$, then X is automatically an element of $\mathcal{F}_{n,n+1}$, and hence is an isolated element. Now part 5) of Proposition [2.2.21](#) implies that X is dual to an essential annulus in G , as required.

2) If $G(e)$ is $VPC(n+1)$, the fact that X crosses no nontrivial almost invariant subset of G over a $VPCn$ subgroup implies that X is n -canonical and so

is again an element of $\mathcal{F}_{n,n+1}$. Now Corollary 2.6.4 and Remark 2.6.5 show that X is adapted to ∂G and that $G(e)$ is orientable, so that X is dual to an essential torus in G , as required. \square

Recall that Theorem 2.1.16 states that each V_0 -vertex v of $\Gamma_{n,n+1}$ satisfies one of the following conditions:

- 1) v is isolated, and $G(v)$ is VPC of length n or $n + 1$.
- 2) v is of $VPCk$ -by-Fuchsian type, where k equals $n - 1$ or n .
- 3) v is of commensuriser type, so that $G(v)$ is the full commensuriser $Comm_G(H)$ for some VPC subgroup H of length n or $n + 1$, such that $e(G, H) \geq 2$.

We will consider cases 2) and 3) in the lemmas which follow. We start with case 2), where v is of $VPCk$ -by-Fuchsian type.

Lemma 2.7.2. *Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair such that G is not VPC , and let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G . Let v be a V_0 -vertex of $\Gamma_{n,n+1}$ which is of $VPCk$ -by-Fuchsian type, where k equals $n - 1$ or n .*

1. *If $k = n$, then v is of interior Seifert type (see Definition 2.3.2).*
2. *If $k = n - 1$, then v is of I -bundle type (see Definition 2.3.1).*

Proof. 1) As v is of $VPCn$ -by-Fuchsian type, the groups associated to the edges of $\Gamma_{n,n+1}$ incident to v are all $VPC(n + 1)$. Now Lemma 2.7.1 implies that the edge splittings of G associated to these edges are dual to essential tori. It follows that v is of interior Seifert type, as claimed.

2) In this case $G(v)$ is $VPC(n - 1)$ -by-Fuchsian where the Fuchsian quotient group Θ is not finite nor two-ended, and there is exactly one edge of $\Gamma_{n,n+1}$ which is incident to v for each peripheral subgroup K of $G(v)$ and this edge carries K . Let $E(v)$ denote the collection of these edge groups. Note that each group in $E(v)$ is $VPCn$. Thus Lemma 2.7.1 implies that the edge splittings of G associated to the edges of $\Gamma_{n,n+1}$ which are incident to v are dual to essential annuli. Note also that it is possible that the family $E(v)$ is empty. In this case, $\Gamma_{n,n+1}$ consists of a single V_0 -vertex v , and $G = G(v)$ is $VPC(n - 1)$ -by-Fuchsian.

As in section 2.2, we choose an aspherical space M with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Recall that in order to prove that v is of I -bundle type, we need to show that there are two distinct components Σ and T of $\partial \widetilde{M}$ such that the induced action of $G(v)$ on \widetilde{M} preserves the union of Σ and T , and for each peripheral subgroup

K of $G(v)$, if e_K denotes the edge of Γ which is incident to v and carries K , then the edge splitting associated to e_K is given by the essential annulus $K_{\Sigma, T}$.

Finding Σ and T

We claim that the pair $(G(v), E(v))$ is $PD(n+1)$. Let L denote the $VPC(n-1)$ normal subgroup of $G(v)$ with Fuchsian quotient Θ . Let $\partial\Theta$ denote the family of subgroups of Θ which are the images of the groups in $E(v)$, so that the pair $(\Theta, \partial\Theta)$ is a Fuchsian pair. If Θ is torsion free, then the pair $(\Theta, \partial\Theta)$ is $PD2$. In this case, as L is $PD(n-1)$, Theorem 7.3 of [1] implies that the pair $(G(v), E(v))$ is $PD(n+1)$. In general Θ has a torsion free subgroup of finite index, and the pre-image in $G(v)$ of this subgroup yields a $PD(n+1)$ pair of finite index in $G(v)$. As $G(v)$ is torsion free, it follows that the pair $(G(v), E(v))$ itself is $PD(n+1)$, as claimed. Note that this pair need not be orientable.

Consider any orientation preserving non-peripheral element α in Θ of infinite order. As Θ is not finite nor two-ended, there is another orientation preserving non-peripheral element β in Θ of infinite order such that α and β have non-zero geometric intersection number. Thus α and β determine nontrivial almost invariant subsets of Θ such that each is over an infinite cyclic subgroup, each is adapted to $\partial\Theta$, and they cross strongly. The pre-images in $G(v)$ of these subsets of Θ are almost invariant subsets of $G(v)$ such that each is over a $VPCn$ subgroup, each is adapted to $E(v)$, and they cross strongly. We denote these $VPCn$ subgroups of $G(v)$ by C and D . Now Lemma 2.2.5 shows us that we can extend these sets to almost invariant subsets P and Q of G over C and D respectively such that P and Q are enclosed by v . As the almost invariant subsets of $G(v)$ cross strongly, P and Q must also cross strongly. Now Proposition 7.2 of [22] shows that C and D must each have two coends in G . Note that $C \cap D$ equals the $VPC(n-1)$ normal subgroup L of $G(v)$. Now we consider the covers M_L, M_C and M_D of M with fundamental groups L, C and D respectively.

Case: C and D are orientable.

As G possesses a nontrivial C -almost invariant subset, Proposition 2.2.20 tells us that M_C must have at least two boundary components which carry a subgroup of C of finite index. As C has two coends in G , part 3) of Proposition 2.2.21 shows that the number of such boundary components of M_C and of each finite cover of M_C is exactly 2. Hence each of these two boundary components of M_C must carry C itself. We denote these two boundary components by $\partial_1 M_C$ and $\partial_2 M_C$. Note that it follows that C preserves precisely two boundary components of the universal cover \widetilde{M} of M , and that all other C -orbits of boundary

components of \widetilde{M} are infinite. Now as in section [2.2](#), there is an untwisted annulus A and a map $\theta : (A, \partial A) \rightarrow (M_C, \partial M_C)$ which is an isomorphism on fundamental groups and sends $\partial_1 A$ into $\partial_1 M_C$ and $\partial_2 A$ into $\partial_2 M_C$. Similarly D preserves precisely two boundary components of the universal cover \widetilde{M} of M , and all other D -orbits of boundary components of \widetilde{M} are infinite. We denote by $\partial_1 M_D$ and $\partial_2 M_D$ the images in M_D of the two boundary components of \widetilde{M} which are preserved by D . Also there is an untwisted annulus B and a map $\phi : (B, \partial B) \rightarrow (M_D, \partial M_D)$ which is an isomorphism on fundamental groups and sends $\partial_1 B$ into $\partial_1 M_D$ and $\partial_2 B$ into $\partial_2 M_D$.

Now we consider the pre-images of these annuli in the common cover M_L of M_C and M_D . Above θ we have a map θ_L into M_L of a two-ended cover A_L of A , and θ_L maps ∂A_L into the two components of ∂M_L which lie above $\partial_1 M_C$ and $\partial_2 M_C$. Similarly above ϕ we have a map ϕ_L into M_L of a two-ended cover B_L of B , and ϕ_L maps ∂B_L into the two components of ∂M_L which lie above $\partial_1 M_D$ and $\partial_2 M_D$. The fact that P crosses Q strongly implies that $\theta_L(\partial_1 A_L)$ must cross the image of ϕ_L strongly, in the natural sense that $\theta_L(\partial_1 A_L)$ contains points arbitrarily far from the image of ϕ_L and on each side. In particular $\theta_L(\partial_1 A_L)$ must meet the image of ϕ_L , and hence must meet $\phi_L(\partial_1 B_L)$ or $\phi_L(\partial_2 B_L)$. As the same argument applies to $\theta_L(\partial_2 A_L)$, it follows that the two components of ∂M_L which contain $\theta_L(\partial_1 A_L)$ and $\theta_L(\partial_2 A_L)$ are the same as the two components of ∂M_L which contain $\phi_L(\partial_1 B_L)$ and $\phi_L(\partial_2 B_L)$. Hence C and D preserve the same two boundary components Σ and T of the universal cover \widetilde{M} of M .

Case: *one or both of C and D is not orientable.*

We apply the above arguments to their orientable subgroups of index 2. We will see that the action of the group generated by C and D on the boundary components of \widetilde{M} has one orbit with the two elements Σ and T , and all other orbits are infinite.

Let Θ_0 denote the subgroup of Θ generated by all orientation preserving non-peripheral elements, and let $G(v)_0$ denote the pre-image of Θ_0 in $G(v)$. Thus Θ_0 has index at most 2 in Θ , so that $G(v)_0$ has index at most 2 in $G(v)$. Consider the action of $G(v)_0$ on the boundary components of \widetilde{M} . As any two orientation preserving non-peripheral elements of Θ of infinite order belong to a finite sequence of such elements each crossing the next, and as such elements generate Θ_0 , it follows that Σ and T form one or two orbits under the action of $G(v)_0$ and that all other orbits are infinite. It follows that the same statement holds for the action of $G(v)$ on the boundary components of \widetilde{M} . Thus Σ and T are the

required components of $\partial\widetilde{M}$.

If the family $E(v)$ of edge splittings associated to edges of $\Gamma_{n,n+1}$ incident to v is empty, we have shown that v is of I -bundle type. In this case, either ∂G consists of two copies of G , or ∂G consists of a single group which has index 2 in G . It remains to deal with the case when $E(v)$ is non-empty.

The edge splittings of v

Let K denote a peripheral subgroup of $G(v)$, let e_K denote the edge of $\Gamma_{n,n+1}$ which is incident to v and carries K , and let $\theta_K : (A_K, \partial A_K) \rightarrow (M, \partial M)$ be an essential annulus which is dual to the edge splitting associated to e_K . We need to show that the annulus θ_K equals $K_{\Sigma, T}$ which was defined prior to Lemma [2.2.15](#). In general there can be many distinct annuli all carrying the same group, so there is something to be proved.

Let S denote the stabiliser of Σ , so that S is one of the groups in ∂G . We first consider the case when no element of $G(v)$ interchanges Σ and T , so that $G(v)$ is contained in S . As v is a vertex of $\Gamma_{n,n+1}$ with associated group $G(v)$, there is a natural induced decomposition of S as the fundamental group of a graph of groups $\Gamma(S)$, with a vertex V with associated group $S \cap G(v) = G(v)$. Further the stars of v in $\Gamma_{n,n+1}$ and of V in $\Gamma(S)$ are isomorphic, i.e. there is a bijection between the edges of $\Gamma_{n,n+1}$ incident to v and the edges of $\Gamma(S)$ incident to V , and corresponding edges have the same associated groups. Note that $\Gamma(S)$ is probably not minimal, and may well be infinite. However, as S is finitely generated, there is a finite minimal subgraph $\Gamma_\mu(S)$ of $\Gamma(S)$ which carries S .

Recall from the start of this proof that the pair $(G(v), E(v))$ is $PD(n+1)$. In particular, as we are now assuming that $E(v)$ is non-empty, $G(v)$ itself is not $PD(n+1)$. Also each group in $E(v)$ is $VPCn$. As S is $PD(n+1)$, and $G(v)$ is not $PD(n+1)$ nor $VPCn$, it follows that the minimal graph $\Gamma_\mu(S)$ must contain V , and at least one edge incident to V . As S is $PD(n+1)$, Theorem 8.1 of [\[1\]](#) tells us that the pair $(G(V), E_\mu(V))$ is $PD(n+1)$, where $E_\mu(V)$ denotes the family of subgroups of $G(V)$ associated to edges of $\Gamma_\mu(S)$ incident to V . As the pair $(G(v), E(v))$ is $PD(n+1)$, and $G(V) = G(v)$ and $E_\mu(V)$ is a subfamily of $E(v)$, it follows that the families $E_\mu(V)$ and $E(v)$ must be equal. In particular every edge of $\Gamma(S)$ which is incident to V must also be an edge of $\Gamma_\mu(S)$. Recall that K is a group in $E(v)$, that e_K denotes the edge of $\Gamma_{n,n+1}$ which is incident to v and carries K , and that $\theta_K : (A_K, \partial A_K) \rightarrow (M, \partial M)$ denotes an essential annulus which is dual to the edge splitting associated to e_K . Let φ_K denote the lift of θ_K into $M_{G(v)}$. As the edge of $\Gamma(S)$ corresponding to e_K is an edge of $\Gamma_\mu(S)$, it follows that the splitting of G over K associated to e_K induces a splitting of S

over K . This implies that φ_K must have a boundary component on the image of Σ in $M_{G(v)}$. The same argument applied to the stabiliser of T shows that φ_K must also have a boundary component on the image of T in $M_{G(v)}$. Hence the annulus θ_K must be $K_{\Sigma, T}$, which completes the proof of the lemma, on the assumption that no element of $G(v)$ interchanges Σ and T .

If there are elements of $G(v)$ which interchange Σ and T , we let S denote the stabiliser of Σ , and let $G(v)_0$ denote $S \cap G(v)$. Note that in this case the images of Σ and T in $M_{G(v)}$ are the same. Then we make essentially the same argument as in the preceding paragraph. This time the vertex V of $\Gamma(S)$ has associated group $G(v)_0$. Given the edge e_K of the star of v in $\Gamma_{n, n+1}$ which has associated group K , the star of V in $\Gamma(S)$ has either one corresponding edge with associated group K or it has two corresponding edges each with associated group K_0 , where $K_0 = K \cap G(v)_0$. As the pair $(G(v), E(v))$ is $PD(n+1)$, it follows that the pair $(G(V), E(V))$ is also $PD(n+1)$. Hence as above, the minimal subgraph $\Gamma_\mu(S)$ of $\Gamma(S)$ which carries S must contain V and each edge of $\Gamma(S)$ which is incident to V . Thus, as before, the lift φ_K of the annulus θ_K into $M_{G(v)}$ must have all of its boundary on the image of Σ . Note that there are two cases here, depending on whether the annulus A_K is twisted or untwisted. In either case it follows that the annulus θ_K must be $K_{\Sigma, T}$, which completes the proof of the lemma. \square

Now we consider the case when v is a V_0 -vertex of $\Gamma_{n, n+1}$ which is of commensuriser type. This is the most difficult case, and we will need most of our previous work in this paper. Note that in Theorem [2.1.16](#), the vertex group $G(v) = \text{Comm}_G(H)$ need not even be finitely generated. Before we start, here is a preliminary result.

Lemma 2.7.3. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let $\Gamma_{n, n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n, n+1}$ in G . Let v be a V_0 -vertex of $\Gamma_{n, n+1}$ of commensuriser type such that $G(v)$ is the full commensuriser $\text{Comm}_G(H)$ for some $VPCn$ subgroup H of G with $e(G, H) \geq 2$.*

Then v encloses two almost invariant subsets X and X' of G , each over a subgroup of H of finite index, and each dual to an annulus, such that X and X' cross.

Proof. As $\Gamma_{n, n+1}$ is the reduced algebraic regular neighbourhood of $\mathcal{F}_{n, n+1}$ in G , any V_0 -vertex arises from a cross-connected component (CCC) of $\mathcal{F}_{n, n+1}$. As v is of commensuriser type, all the crossings in this CCC must be weak and all the almost invariant sets in this CCC are over groups commensurable with H . As in section [2.2](#), we consider an aspherical space M with fundamental group

G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . As v is of commensuriser type, the number of coends of H in G must be at least 4. Now part 4) of Proposition [2.2.21](#) tells us H has a subgroup K of finite index such that ∂M_K has 4 (or more) components each of which carries K . Now Lemma [2.2.22](#) shows that we can find almost invariant subsets X and X' of G , each over K , and each dual to an annulus, such that X and X' cross. When $n = 1$, Proposition 8.6 of [\[22\]](#) shows that X and X' must be enclosed by the V_0 -vertex v of $\Gamma_{n,n+1}$, and essentially the same argument applies for all values of n . \square

Recall that we want to consider a V_0 -vertex v of $\Gamma_{n,n+1}$ which is of commensuriser type, and that we do not even know that $G(v) = \text{Comm}_G(H)$ is finitely generated. We first deal with two special cases when $\text{Comm}_G(H)$ is certainly finitely generated.

Lemma 2.7.4. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G . Let v be a V_0 -vertex of $\Gamma_{n,n+1}$ which is of commensuriser type, such that $G(v)$ is the full commensuriser $\text{Comm}_G(H)$ for some $VPCn$ subgroup H of G with $e(G, H) \geq 2$.*

If H has finite index in $G(v)$, then v is of solid torus type (see Definition [2.3.8](#)).

Proof. Let e_1, \dots, e_m denote the edges of $\Gamma_{n,n+1}$ which are incident to v , and denote the associated subgroups of $G(v)$ by H_1, \dots, H_m . The hypothesis implies that $G(v) = \text{Comm}_G(H)$ is itself $VPCn$. Hence each H_i is also $VPC(\leq n)$. Now Lemma [2.2.10](#) implies that G cannot split over a $VPC(< n)$ subgroup. It follows that each H_i is $VPCn$, and now Lemma [2.7.1](#) tells us that the edge splitting of G associated to e_i is dual to an essential annulus A_i . It remains to show that the boundaries of these annuli all carry the same subgroup of $G(v)$. We write ∂H_i for the group carried by the boundary of A_i . Note that as each H_i is a $VPCn$ subgroup of the $VPCn$ group $G(v)$, it must be of finite index in $G(v)$. Thus each H_i is commensurable with H .

Lemma [2.7.3](#) shows that v encloses two almost invariant subsets X and X' of G , each over a subgroup of H of finite index, and each dual to an annulus, such that X and X' cross. By replacing H by a subgroup of finite index, we can assume that X and X' are both H -almost invariant. As discussed in section [2.2](#) the doubles of these annuli are tori T and T' in DG . These tori must cross and are therefore enclosed by the same V_0 -vertex V of $T_{n+1}(DG)$. Hence V is not an isolated vertex, so that V must be of $VPCn$ -by-Fuchsian type. We

let L denote the $VPCn$ normal subgroup of $G(V)$ with Fuchsian quotient. As T and T' are non-peripheral tori enclosed by V , Lemma 2.5.13 shows that the intersection group $T \cap T'$ must be commensurable with L . As $T \cap T'$ contains the $VPCn$ group H , it follows that H is commensurable with L , and hence that each H_i is also commensurable with L . Let T_i denote the torus in DG obtained by doubling the annulus A_i . The tori T_i and T have the $VPCn$ subgroup $H \cap H_i$ in common. As T is non-peripheral in V , Lemma 2.5.13 shows that T_i must also be enclosed by V . Now Lemma 2.7.5 below shows that ∂H_i must equal L . As this holds for all i , it shows that the groups ∂H_i are all equal, as required. This completes the proof that v is of solid torus type. \square

Lemma 2.7.5. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and suppose that G has a splitting over a $VPCn$ subgroup H_i which is dual to an essential annulus A_i in $(G, \partial G)$. Let ∂H_i denote the group carried by the boundary of A_i , and let T_i denote the torus in DG obtained by doubling A_i . Let V denote a vertex of $T_{n+1}(DG)$ of $VPCn$ -by-Fuchsian type, and let L denote the normal $VPCn$ subgroup of $G(V)$ with Fuchsian quotient. Suppose that T_i is enclosed by V , and that H_i is commensurable with L . Then ∂H_i equals L .*

Proof. Observe that ∂H_i is a normal subgroup of T_i with quotient Q isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, depending on whether the annulus A_i is untwisted or twisted. As ∂H_i is assumed to be commensurable with L , part 2) of Lemma 2.1.12 implies that ∂H_i equals $L \cap T_i$, so that ∂H_i must be contained in L .

As in section 2.2, it will again be convenient to consider an aspherical space M with fundamental group G and with aspherical subspaces which correspond to ∂G whose union is denoted by ∂M . Let Σ be a component of ∂M such that one end of A_i is in Σ . The splitting of G over H_i determined by A_i induces a splitting of $\pi_1(\Sigma)$ over ∂H_i . Thus part 3) of Corollary 2.2.8 shows that ∂H_i is a maximal orientable $VPCn$ subgroup of $\pi_1(\Sigma)$. Now DG splits over $\pi_1(\Sigma)$ and this induces the splitting of T_i over ∂H_i . As ∂H_i equals $L \cap T_i$, we can identify the quotient $Q = T_i / \partial H_i$ with a subgroup of the Fuchsian quotient group of $G(V)$ by L . Now we consider the full pre-image \overline{T}_i of Q in $G(V)$. Thus L is normal in \overline{T}_i with quotient Q . In particular \overline{T}_i is a $VPC(n+1)$ subgroup of $G(V)$ which contains T_i with finite index. As the splitting of DG over $\pi_1(\Sigma)$ induces a splitting of T_i over ∂H_i , it follows that it induces a splitting of \overline{T}_i over some subgroup L' of $\pi_1(\Sigma)$ which contains ∂H_i with finite index. Thus L and L' must be commensurable. As \overline{T}_i splits over L' , Lemma 2.1.10 shows that L' must be a normal $VPCn$ subgroup of \overline{T}_i with quotient which is isomorphic to \mathbb{Z} or $\mathbb{Z}_2 * \mathbb{Z}_2$. Now part 2) of Lemma 2.1.11 shows that L' must equal L . In particular,

it follows that $\partial H_i \subset L \subset \pi_1(\Sigma)$. Recall from Lemma 2.3.3 that L must be orientable. Now the maximality of ∂H_i among orientable $VPCn$ subgroups of $\pi_1(\Sigma)$ implies that ∂H_i must equal L , as required. \square

Next we consider a case where $Comm_G(H)$ contains H with infinite index, but still must be finitely generated.

Lemma 2.7.6. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G . Let v be a V_0 -vertex of $\Gamma_{n,n+1}$ which is of commensuriser type, such that $G(v)$ is the full commensuriser $Comm_G(H)$ for some $VPCn$ subgroup H of G with $e(G, H) \geq 2$.*

If $G(v)$ is $VPC(n+1)$, then v is of torus type (see Definition 2.3.12).

Proof. As $G(v)$ is $VPC(n+1)$, and G cannot split over a $VPC(<n)$ subgroup, it follows that the group associated to each edge of $\Gamma_{n,n+1}$ which is incident to v must be $VPCn$ or $VPC(n+1)$. Now Lemma 2.7.1 implies that the edge splittings of G associated to these edges are dual to essential annuli or tori. Let $\Sigma(v)$ denote the collection of all these splittings. Recall that Definition 2.3.12 includes four cases, but in each case, at most one of the splittings in $\Sigma(v)$ can be dual to an essential torus.

We start by showing that this condition holds.

The splittings in $\Sigma(v)$

Note that any splitting of G dual to an essential torus must be over a maximal orientable $VPC(n+1)$ subgroup of G , by part 3) of Corollary 2.2.8. Thus if $G(v)$ is orientable, any such edge splitting must be over $G(v)$. If $G(v)$ is non-orientable, then any such edge splitting must be over $G(v)_0$, the orientable subgroup of $G(v)$ of index 2. Let $T_{n,n+1}$ denote the universal covering G -tree of $\Gamma_{n,n+1}$, and let w denote a vertex of $T_{n,n+1}$ above v and with stabiliser $G(v)$. If $G(v)$ is orientable, suppose that two of the splittings in $\Sigma(v)$ are dual to an essential torus, and if $G(v)$ is non-orientable, suppose that one of the splittings in $\Sigma(v)$ is dual to an essential torus. In either case, there are two distinct edges of $T_{n,n+1}$ which are incident to w and have the same stabiliser, which is $G(v)$ or $G(v)_0$. Let X and Y denote the almost invariant subsets of G associated to these edges. As each is dual to the same essential torus in $(G, \partial G)$ they must be equivalent. Now Corollary 4.16 of [22] implies that w must have valence 2, and hence is isolated. This implies that either v is isolated, or that v has valence 1 with edge group of index 2 in $G(v)$. Either case contradicts our assumption that

v is of commensuriser type. We conclude that if $G(v)$ is orientable, then at most one of the splittings in $\Sigma(v)$ can be dual to a torus, and if $G(v)$ is non-orientable, then none of the splittings in $\Sigma(v)$ can be dual to a torus. Further if $G(v)$ is orientable and one of the splittings in $\Sigma(v)$ is dual to a torus, the minimality of $\Gamma_{n,n+1}$ shows that there must also be a splitting in $\Sigma(v)$ which is dual to an essential annulus. Thus in all cases, $\Sigma(v)$ must contain at least one splitting which is dual to an essential annulus.

Notation

Let e_1, \dots, e_m denote all those edges of $\Gamma_{n,n+1}$ which are incident to v and have associated edge splitting dual to an essential annulus. Denote the associated $VPCn$ subgroups of $G(v)$ by H_1, \dots, H_m . Let T_i denote the torus in DG obtained by doubling the annulus A_i associated to the edge splitting of G over H_i .

Lemma 2.7.3 implies that v encloses two almost invariant subsets X and X' of G , each dual to an annulus, such that X and X' cross. Further we can assume that X and X' are both H -almost invariant. The doubles of these annuli are tori in DG which must cross, and are therefore both enclosed by a V_0 -vertex V of $T_{n+1}(DG)$. Thus neither torus is peripheral in V , and V is of $VPCn$ -by-Fuchsian type. If we let L denote the $VPCn$ normal subgroup of $G(V)$ with Fuchsian quotient, then H is commensurable with L . Now Lemma 2.5.10 shows that $G(V) = Comm_{DG}(L)$ which equals $Comm_{DG}(H)$. As $G(v) = Comm_G(H)$, it follows that $G(v)$ is a subgroup of $G(V)$.

Case 1: $G(v)$ is orientable.

Thus $G(v)$ is an essential torus in DG , and we let W denote the almost invariant subset of DG which is over $G(v)$. As $G(v)$ is a subgroup of $G(V)$, it follows that W is enclosed by V .

We will suppose that W is not peripheral in V . We will not need to consider here the case when W is peripheral in V , although this can occur.

As H_i is a subgroup of $G(v)$, the tori T_i and $G(v)$ have the $VPCn$ subgroup H_i in common. Then Lemma 2.5.13 shows that the almost invariant subset of DG associated to the torus T_i must be enclosed by V , and that H_i is commensurable with L . Now Lemma 2.7.5 shows that ∂H_i must equal L . As ∂H_i is a subgroup of $G(v)$ which in turn is a subgroup of $G(V)$, and as L is normal in $G(V)$, it follows that ∂H_i is a normal $VPCn$ subgroup of $G(v)$ with quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, so that $G(v)$ splits over ∂H_i . This proves part of cases 1), 2) and 3) of the definition of torus type (Definition 2.3.12).

We next consider how the $VPC(n+1)$ group $G(v)$ sits in G . There are three subcases.

Case 1a): $G(v)$ is conjugate to a subgroup of some group in ∂G .

Note that this implies that no edge of $\Gamma_{n,n+1}$ incident to v has associated splitting dual to an essential torus. As in section [2.2](#) it will again be convenient to consider an aspherical space M with fundamental group G and with aspherical subspaces which correspond to ∂G whose union is denoted by ∂M . Let Σ be a component of ∂M such that $G(v)$ is a subgroup of $\pi_1(\Sigma)$. As $\pi_1(\Sigma)$ must contain $G(v)$ with finite index, it is conjugate into some vertex group of $\Gamma_{n,n+1}$. Thus either $\pi_1(\Sigma)$ equals $G(v)$, or some edge incident to v has associated group equal to $G(v)$. As the second case is impossible, $G(v)$ must equal $\pi_1(\Sigma)$. Consider the cover M_H of M with fundamental group equal to H . There will be a component of ∂M_H which covers Σ and has fundamental group H . As we already know that M_H admits essential annuli which carry H , it follows that there is an essential annulus in M which carries H and has one end on Σ . Doubling such an annulus yields a torus in DG which crosses the torus Σ . It follows that the $G(v)$ -almost invariant subset W of DG associated to $G(v)$ is not peripheral in the V_0 -vertex V of $T_{n+1}(DG)$ which encloses W , so the argument in the preceding paragraph applies and shows that each ∂H_i equals L . As above, it also follows that $G(v)$ splits over ∂H_i . Thus we have case 1) of Definition [2.3.12](#).

Case 1b): $G(v)$ is not conjugate to a subgroup of some group in ∂G .

In this case, there is a nontrivial $G(v)$ -almost invariant subset of G which is adapted to ∂G , and we denote this set by Y . The $G(v)$ -almost invariant subset W of DG is enclosed by G , and Y is equal to $W \cap G$, and is adapted to ∂G . Now Theorem [2.6.17](#) tells us that Y must be enclosed by some V_0 -vertex w of $\Gamma_{n,n+1}$. If w is not equal to v , then there must be an edge of $\Gamma_{n,n+1}$ which is incident to v , and carries $G(v)$. Thus in any case, Y is enclosed by v . Now we have two subcases depending on whether or not Y is peripheral in v .

If Y is not peripheral in v , then Y must cross some almost invariant subset Z of G which is over a subgroup commensurable with H and belongs to the CCC of $\mathcal{F}_{n,n+1}$ which gives rise to v . Lemma [2.6.6](#) shows that Y must cross some almost invariant subset Z' of G also over a subgroup commensurable with H and dual to an annulus in $(G, \partial G)$. Now part 1) of Lemma [2.6.15](#) shows that Z' cannot be isolated in $\mathcal{F}_{n,n+1}$ and so must also belong to the CCC of $\mathcal{F}_{n,n+1}$ which gives rise to v . We conclude that Y crosses some annulus in $(G, \partial G)$ enclosed by v . It follows that W crosses the torus in DG which is the double of this annulus.

As above this implies that W is not peripheral in V , and hence that each ∂H_i equals L , and that $G(v)$ splits over ∂H_i . As v is a V_0 -vertex of $\Gamma_{n,n+1}$, and Y is enclosed by v and is over $G(v)$, it follows that Y crosses no torus in G so that Y determines a splitting of G over $G(v)$. As Y is enclosed by v and is not peripheral in v , this shows that we have case 3) of Definition [2.3.12](#).

If Y is peripheral in v , we let e be the edge of $\Gamma_{n,n+1}$ to which Y is associated. We also let X_i denote the H_i -almost invariant subset of G associated to the edge e_i . Now the definition of betweenness in our construction of an algebraic regular neighbourhood in [\[22\]](#) implies that there is an element Z of the CCC which gives rise to v such that Z lies between X_i and Y . More precisely, there is a nontrivial almost invariant subset Z of G which is over a subgroup commensurable with H and which is enclosed by v such that $Y^{(*)} \leq Z \leq X_i^{(*)}$, where $Y^{(*)}$ denotes one of Y or Y^* . As H_i stabilises both X_i and Y , it follows that H_i lies within some bounded neighbourhood of δZ . In turn this implies that a subgroup of finite index in H_i must stabilise Z . Thus H_i and H must be commensurable. As H is commensurable with L , it follows that H_i is commensurable with L , and now Lemma [2.7.5](#) shows that ∂H_i must equal L . As above, it also follows that $G(v)$ splits over ∂H_i . Thus we have case 2) of Definition [2.3.12](#).

This completes the proof that v is of torus type, in the case when $G(v)$ is orientable.

Case 2: $G(v)$ is non-orientable.

Let W denote the almost invariant subset of DG which is over $G(v)_0$. As before W is enclosed by V . Recall that V is the V_0 -vertex of $T_{n+1}(DG)$ with $G(V) = \text{Comm}_{DG}(L) = \text{Comm}_{DG}(H)$, and that $G(v)$ is a subgroup of $G(V)$.

If $G(v)_0$ is conjugate to a subgroup of some group in ∂G , then the non-orientable $PD(n+1)$ group $G(v)$ is commensurable with that group in ∂G . Now part 1) of Corollary [2.2.8](#) shows that G itself must be a non-orientable $PD(n+1)$ group, and that G must contain $G(v)_0$ with finite index. In particular it would imply that G is $VPC(n+1)$, which is excluded in our hypotheses. Thus $G(v)_0$ cannot be conjugate to a subgroup of any group in ∂G , so we let $Y = W \cap G$, a nontrivial $G(v)_0$ -almost invariant subset of G which is adapted to ∂G . As before Y must be enclosed by v . As we showed earlier that no edge incident to v can have associated splitting dual to an essential torus, it follows that Y cannot be peripheral in v . As in the case when $G(v)$ is orientable, this implies that W is not peripheral in V , and hence that each ∂H_i equals L , the $VPCn$ normal subgroup of $G(V)$ with Fuchsian quotient. Thus L is also a normal subgroup of $G(v)$, and the quotient $G(v)/L$ is a $VPC1$ subgroup of the Fuchsian quotient

$G(V)/L$. Hence $G(v)/L$ is isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, so that $G(v)$ splits over L . Thus $G(v)$ splits over ∂H_i , for each i , as required by part 4) of Definition [2.3.12](#).

As v is a V_0 -vertex of $\Gamma_{n,n+1}$, and Y is enclosed by v and is over a subgroup of index 2 in $G(v)$, it follows that Y determines a splitting σ of G over $G(v)_0$. As $G(v)$ contains $G(v)_0$ with finite index, it must be conjugate into one of the vertex groups K of this splitting. Note that the splitting σ is dual to an essential torus in $(G, \partial G)$. Thus, if ∂G is empty, Theorem 8.1 of [\[1\]](#) shows that the pair formed by K and one or two copies of $G(v)_0$ is $PD(n+2)$, where there will be two copies of $G(v)_0$ if σ is a HNN extension and only one copy otherwise. In general, as discussed just before Definition [2.5.1](#), the pair becomes $PD(n+2)$ when some family of groups in ∂G is added to the copies of $G(v)_0$. As K contains a conjugate of $G(v)$, one of the copies of $G(v)_0$ in ∂K is not equal to its own commensuriser in K . Thus Lemma [2.2.7](#) implies that K contains this copy of $G(v)_0$ with index 2, so that K must be a conjugate of $G(v)$, and ∂K consists only of $G(v)_0$. In particular it follows that σ must be an amalgamated free product and not a HNN extension.

As Y is enclosed by v , we can refine $\Gamma_{n,n+1}$ by splitting at v to obtain a graph of groups structure Γ' of G such that the projection map $\Gamma' \rightarrow \Gamma_{n,n+1}$ sends an edge e to v and otherwise induces a bijection of edges and vertices. The group associated to e is equal to $G(v)_0$, and the associated edge splitting is σ . As $G(v)$ contains $G(v)_0$ with finite index, one vertex of e must carry $G(v)$ and the other must carry $G(v)_0$. Let w denote the vertex of e with $G(w) = G(v)$. Recall from the preceding paragraph that σ is an amalgamated free product and that one vertex group of σ is conjugate to $G(v)$. It follows that the edge e of Γ' is separating, and that if we remove the interior of e from Γ' then the component of the resulting subgraph which contains w must carry the group $G(v)$. As $G(w)$ equals $G(v)$, the minimality of $\Gamma_{n,n+1}$ implies that this subgraph must consist solely of w , so that w has valence 1 in Γ' . This shows that we have case 4) of Definition [2.3.12](#), and so completes the proof that v is of torus type in all cases. \square

Now we consider the general situation of a V_0 -vertex v of $\Gamma_{n,n+1}$ which is of commensuriser type. Note that a priori, the group $G(v)$ need not be finitely generated, but we show not only that $G(v)$ is finitely generated, but also describe its structure.

Lemma 2.7.7. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$*

in G . Let v be a V_0 -vertex of $\Gamma_{n,n+1}$ which is of commensuriser type, such that $G(v)$ is the full commensuriser $Comm_G(H)$ for some $VPCn$ subgroup H of G with $e(G, H) \geq 2$.

If H has infinite index in $G(v)$, and $G(v)$ is not $VPC(n+1)$, then v is of Seifert type (see Definition [2.3.6](#)).

Proof. Lemma [2.7.3](#) implies that v must enclose two almost invariant subsets of G , each dual to an annulus and crossing each other. Further we can assume that each is over H . The doubles of these annuli are tori in DG which must cross, and are therefore both enclosed by a V_0 -vertex V of $T_{n+1}(DG)$. Thus neither torus is peripheral in V , and V is of $VPCn$ -by-Fuchsian type. If we let L denote the $VPCn$ normal subgroup of $G(V)$ with Fuchsian quotient Φ , then H is commensurable with L . Now Lemma [2.5.10](#) tells us that $G(V) = Comm_{DG}(L)$. It follows that $G(V)$ is also equal to $Comm_{DG}(H)$, and so $G(V)$ contains $Comm_G(H) = G(v)$. Thus $G(v)$ is itself a $VPCn$ -by-Fuchsian group, where the normal $VPCn$ subgroup H' is commensurable with H . (But note that at this stage it is still possible that $G(v)$ is not finitely generated!) As we are assuming H has large commensuriser, it follows that the Fuchsian quotient group $\Theta = G(v)/H'$ must be infinite. Further as $G(v)$ is not $VPC(n+1)$, this quotient cannot be two-ended. This implies that there are elements α and β in Θ of infinite order such that α and β have non-zero geometric intersection number. The pre-images in $G(v)$ of the infinite cyclic subgroups of Θ generated by α and β are $VPC(n+1)$ subgroups A' and B' of $G(v)$. Note that $A' \cap B' = H'$.

If we regard α and β as elements of Φ , the pre-images in $G(V)$ of the same infinite cyclic subgroups are $VPC(n+1)$ subgroups A and B of $G(V)$. By replacing α and β by their squares if needed, we can ensure that they are orientable elements of Φ , so that A and B will be orientable. As α and β have non-zero geometric intersection number, it follows that A and B are tori in DG which cross. As A' and B' are subgroups of finite index in A and B respectively, they also are tori in DG which cross. As A' and B' are subgroups of $G(v)$ which is a subgroup of G , it follows that A' and B' are tori in $(G, \partial G)$ which are enclosed by v and which cross.

Now we consider the torus decomposition $T_{n+1}(G, \partial G)$. Each of A' and B' must be enclosed by some V_0 -vertex of $T_{n+1}(G, \partial G)$. As A' and B' cross, they must both be enclosed by a single V_0 -vertex u of $T_{n+1}(G, \partial G)$, and neither is peripheral in u . Thus u is not isolated, and so Theorem [2.4.4](#) shows that u must be of Seifert type adapted to ∂G . Let L' denote the normal subgroup of $G(u)$ with Fuchsian quotient. Recall that A' and B' are tori in $(G, \partial G)$ which cross

and that $A' \cap B' = H'$. Lemma 2.5.13 implies that H' is commensurable with L' . Now Lemma 2.5.10 implies that $G(u)$ equals $Comm_G(L')$. It follows that $G(u) = Comm_G(L') = Comm_G(H') = Comm_G(H) = G(v)$. Note that as $G(u)$ is finitely generated, it follows that $G(v)$ must also be finitely generated. As each of H' and L' is a normal $VPCn$ subgroup of $G(v)$ with Fuchsian quotient, Lemma 2.1.9 shows that H' and L' must be equal.

Now Theorem 2.6.17 shows that each essential torus in $(G, \partial G)$ which is enclosed by the vertex u of $T_{n+1}(G, \partial G)$ is also enclosed by some V_0 -vertex v of $\Gamma_{n,n+1}$. As the tori which are enclosed by u and are not peripheral in u form a single CCC, they must all be enclosed by v . It follows that the vertex u of $T_{n+1}(G, \partial G)$ is enclosed by the vertex v of $\Gamma_{n,n+1}$. Thus there is a refinement Γ' of $\Gamma_{n,n+1}$ and a vertex v' of Γ' , such that the projection map $p : \Gamma' \rightarrow \Gamma_{n,n+1}$ sends v' to v and is an isomorphism apart from the fact that certain edges incident to v' are mapped to v . Further the vertex v' , like u , is of Seifert type adapted to ∂G , and $G(v') = G(u)$ maps isomorphically to $G(v)$. This last fact implies that if e is an edge of Γ' which is incident to v' and mapped to v , then the other vertex w of e has associated group equal to $G(e)$. As Γ' is minimal there is at least one other edge incident to w , and each such edge must carry a subgroup of the $VPC(n+1)$ group $G(w) = G(e)$. Thus each such edge carries a $VPC(\leq n+1)$ subgroup of G . As usual, Lemma 2.7.1 tells us that the associated splitting of G must be dual to an essential annulus or torus in $(G, \partial G)$. If one of these other edges incident to w has associated splitting dual to an essential torus, the edge group must equal $G(e)$, so that the vertex w of Γ' has two incident edges with associated splittings over the same essential torus. In this case, as in the proof of Lemma 2.7.6, Corollary 4.16 of [22] implies that w must have valence 2, and we modify Γ' by collapsing the edge e . By repeating this process we will arrange that if e is an edge of Γ' which is incident to v' and mapped to v , with the other vertex of e being w , then each edge e_i , other than e , which is incident to w must carry a $VPCn$ subgroup H_i of $G(w) = G(e)$, so that the associated edge splitting is dual to an annulus A_i in $(G, \partial G)$. As usual we write ∂H_i for the group carried by ∂A_i . At this point we have proved that most of Definition 2.3.6 holds. It remains to show that ∂H_i equals L' , for each i , where L' is the normal subgroup of $G(u) = G(v)$ with Fuchsian quotient.

The image of e_i in $\Gamma_{n,n+1}$ is an edge incident to v , so its associated edge splitting is enclosed by v . The edge splitting of G associated to the edge e of Γ' is dual to a torus T which is also enclosed by v . We claim that the torus T cannot be peripheral in v . To see this we need to consider the universal covering G -trees T' and $T_{n,n+1}$ of Γ' and $\Gamma_{n,n+1}$ respectively. Let V' be a vertex of T' above v'

with stabiliser equal to $G(v')$. Let E be an edge of T' above e which is incident to V' and has stabiliser equal to $G(e)$, let W denote the other vertex of E , and let E_i denote an edge of T' above e_i which is incident to W and has stabiliser equal to H_i . Orient E and E_i towards V' , and let Z_E and Z_i denote the associated almost invariant subsets of G . The orientations imply that $Z_i^* \leq Z_E^*$. If T were peripheral in v , the fact that Z_i is enclosed by v would imply that $Z_E^* \leq Z_i$ or $Z_E^* \leq Z_i^*$. The first inequality would imply that $Z_i^* \leq Z_i$, which is obviously impossible. The second inequality would imply that Z_i and Z_E were equivalent. This is impossible as their stabilisers are not commensurable, as H_i is $VPCn$ and $G(e)$ is $VPC(n+1)$. This contradiction shows that the torus T cannot be peripheral in v , as claimed.

Now let T_i denote the torus in DG obtained by doubling the annulus A_i with fundamental group H_i . As T is not peripheral in v , it must cross some almost invariant subset of G which is over a subgroup commensurable with H and belongs to the CCC of $\mathcal{F}_{n,n+1}$ which gives rise to v . As in the proof of Lemma 2.7.6, Lemmas 2.6.6 and 2.6.15 imply that T crosses some annulus enclosed by v , and so crosses the torus obtained by doubling this annulus. Lemma 2.5.13 then implies that T and this torus are enclosed by the vertex V of $T_{n+1}(DG)$ and are not peripheral in V . As H_i is contained in $G(e)$, Lemma 2.5.13 now shows that T_i is enclosed by V , and that H_i is commensurable with L . (Recall that L is the $VPCn$ normal subgroup of $G(V)$ with Fuchsian quotient Φ .) Lemma 2.7.5 then shows that ∂H_i must equal L . In particular, $G(e)$ contains L . As $G(e)$ is an edge torus of the V_0 -vertex u of $T_{n+1}(G, \partial G)$, it must contain L' . As L and L' are commensurable normal $VPCn$ subgroups of $G(e)$ each of which has quotient isomorphic to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$, Lemma 2.1.11 implies that L' equals L . We conclude that ∂H_i equals L' , for each i . We have now established all the requirements in Definition 2.3.6, so that v must be of Seifert type in $\Gamma_{n,n+1}$, as required. This completes the proof of Lemma 2.7.7. \square

We are finally ready to prove our main result. For the convenience of the reader we restate it.

Theorem 3.4.1. (Main Result) *Let $n \geq 1$, and let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . Let $\mathcal{F}_{n,n+1}$ denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup, together with the equivalence classes of all n -canonical almost invariant subsets of G which are over a $VPC(n+1)$ subgroup. Finally let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G , and let $\Gamma_{n,n+1}^c$ denote the comple-*

tion of $\Gamma_{n,n+1}$. Thus $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ are bipartite graphs of groups structures for G , with vertices of V_0 -type and of V_1 -type.

Then $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ have the following properties:

1. Each V_0 -vertex v of $\Gamma_{n,n+1}$ satisfies one of the following conditions:
 - (a) v is isolated, and $G(v)$ is VPC of length n or $n + 1$, and the edge splittings associated to the two edges incident to v are dual to essential annuli or tori in G .
 - (b) v is of $VPC(n - 1)$ -by-Fuchsian type, and is of I -bundle type. (See Definition [2.3.1](#))
 - (c) v is of $VPCn$ -by-Fuchsian type, and is of interior Seifert type. (See Definition [2.3.2](#))
 - (d) v is of commensuriser type. Further v is of Seifert type (see Definition [2.3.6](#)), or of torus type (see Definition [2.3.12](#)) or of solid torus type (see Definition [2.3.8](#)).
2. The V_0 -vertices of $\Gamma_{n,n+1}^c$ obtained by the completion process are of special Seifert type (see Definition [2.3.10](#)) or of special solid torus type (see Definition [2.3.8](#)).
3. Each edge splitting of $\Gamma_{n,n+1}$ and of $\Gamma_{n,n+1}^c$ is dual to an essential annulus or torus in G .
4. Any nontrivial almost invariant subset of G over a $VPC(n + 1)$ group and adapted to ∂G is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}$, and also by some V_0 -vertex of $\Gamma_{n,n+1}^c$.
5. If H is a $VPC(n + 1)$ subgroup of G which is not conjugate into ∂G , then H is conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}^c$.

Remark 2.7.8. Part 3) follows immediately from parts 1) and 2), as the definitions of the various types of V_0 -vertex in the statements of parts 1) and 2) all contain the requirement that the edge splittings be dual to an essential annulus or torus.

Part 4) does not follow from the properties of an algebraic regular neighbourhood as an almost invariant subset of G over a $VPC(n + 1)$ group which is adapted to ∂G need not be n -canonical, and so need not lie in the family $\mathcal{F}_{n,n+1}$. Note that, from [\[24\]](#), we know that there may be almost invariant subsets of G over $VPC(n + 1)$ subgroups which are not adapted to ∂G .

Part 5) also does not follow from the properties of an algebraic regular neighbourhood as a $VPC(n+1)$ subgroup H of G may be non-orientable.

Proof. 1) Recall that Theorem 2.1.16 states that each V_0 -vertex v of $\Gamma_{n,n+1}$ satisfies one of the following conditions:

- a) v is isolated, and $G(v)$ is VPC of length n or $n+1$.
- b) v is of $VPCk$ -by-Fuchsian type, where k equals $n-1$ or n .
- c) v is of commensuriser type, and $G(v)$ is the full commensuriser $Comm_G(H)$ for some VPC subgroup H of length n or $n+1$, such that $e(G, H) \geq 2$.

We will consider each type of V_0 -vertex of $\Gamma_{n,n+1}$ in turn.

a) Suppose that a V_0 -vertex v of $\Gamma_{n,n+1}$ is isolated, and $G(v)$ is VPC of length n or $n+1$. Let e denote either of the two edges of $\Gamma_{n,n+1}$ incident to v . Then $G(e)$ equals $G(v)$ and so is VPC of length n or $n+1$. Now Lemma 2.7.1 implies that the edge splitting of G associated to e is dual to an essential annulus or torus. It follows that we have case 1a) of Theorem 2.3.14.

b) Suppose that a V_0 -vertex v of $\Gamma_{n,n+1}$ is of $VPCk$ -by-Fuchsian type, where k equals $n-1$ or n . If $k = n$, part 1) of Lemma 2.7.2 shows that v is of interior Seifert type, so that we have case 1c) of Theorem 2.3.14. If $k = n-1$, part 2) of Lemma 2.7.2 shows that v is of I -bundle type, so that we have case 1b) of Theorem 2.3.14.

c) Suppose that v is a V_0 -vertex of $\Gamma_{n,n+1}$ which is of commensuriser type, and that $G(v)$ is the full commensuriser $Comm_G(H)$ for some VPC subgroup H of length n or $n+1$, such that $e(G, H) \geq 2$.

If H has length $n+1$, we recall from Theorem 2.1.16 and Remark 2.1.15 that v encloses elements X and Y of $\mathcal{F}_{n,n+1}$ which are over a subgroup H' of finite index in H , and which cross weakly. Now Corollary 2.6.4 and Remark 2.6.5 imply that as X and Y are n -canonical, they must be adapted to ∂G , and H' must be orientable. Thus X and Y are dual to essential tori in $(G, \partial G)$. Now Lemma 2.2.5 tells us that there are H' -almost invariant subsets \bar{X} and \bar{Y} of DG such that $\bar{X} \cap G$ equals X and $\bar{Y} \cap G$ equals Y . As X and Y cross weakly, it follows that \bar{X} and \bar{Y} also cross weakly. But Proposition 7.4 of [22] implies that no almost invariant subset of DG can cross \bar{X} weakly, as H' has only 2 coends in DG . This contradiction shows that v cannot be of commensuriser type.

If H has length n , there are three cases depending on the index of H in $Comm_G(H)$. In all three cases, we have case 1d) of Theorem 2.3.14.

If this index is finite, Lemma 2.7.4 shows that v is of solid torus type.

If this index is infinite and $Comm_G(H)$ is $VPC(n+1)$, Lemma 2.7.6 shows that v is of torus type.

In the remaining case, Lemma 2.7.7 shows that v is of Seifert type.

We have now shown that each V_0 -vertex of the uncompleted graph of groups $\Gamma_{n,n+1}$ satisfies part 1) of the theorem.

2) A V_0 -vertex of $\Gamma_{n,n+1}^c$ which is obtained by the completion process arises from a V_1 -vertex w of $\Gamma_{n,n+1}$. It suffices to show that w is of special Seifert type or of special solid torus type. Recall that either $G(w)$ is $VPC(n+1)$ and w has a single incident edge e with $G(e)$ of index 2 in $G(w)$, or $G(w)$ is $VPCn$ and w has a single incident edge e with $G(e)$ of index 2 or 3 in $G(w)$, or $G(w)$ is $VPCn$ and w has three incident edges each carrying $G(w)$. We consider each case separately.

Case: $G(w)$ is $VPC(n+1)$, and w has a single incident edge e with $G(e)$ of index 2 in $G(w)$.

Lemma 2.7.1 shows that the edge splitting associated to e is dual to an essential torus. It follows that w is of special Seifert type.

In the remaining cases, each edge incident to w carries a $VPCn$ group, so that Lemma 2.7.1 shows that the associated edge splitting is dual to an essential annulus.

Case: $G(w)$ is $VPCn$, and w has a single incident edge e with $G(e)$ of index 2 or 3 in $G(w)$.

Let H denote $G(e)$, let A denote the annulus associated to e , and let ∂H denote the subgroup of H carried by ∂A . We claim that A is untwisted. Assuming this claim, it follows that ∂H equals H and so has index 2 or 3 in $G(w)$, which implies that w is of special solid torus type. It remains to prove the claim.

As usual we choose an aspherical space M with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Let M_H denote the cover of M with fundamental group equal to H . If A is twisted, then its lift into M_H has boundary in a single component Σ of ∂M_H whose fundamental group must equal ∂H . First suppose that H is normal in $G(w)$, so that the quotient $G(w)/H$ acts on M_H . Thus ∂M_H has two or three distinct components whose fundamental group equals ∂H , and each of these components contains the boundary of an essential twisted annulus. The double cover $M_{\partial H}$ must have two boundary components above each such component of ∂M_H , each with fundamental group equal to ∂H , giving a total of four or six such boundary components of $M_{\partial H}$. Part 3) of Proposition 2.2.21 now shows that H has at least 4 coends in G . This implies that $\Gamma_{n,n+1}$ has a V_0 -vertex v of commensuriser type with $G(v) = Comm_G(H)$. But w is a V_1 -vertex of $\Gamma_{n,n+1}$

and $G(w)$ commensurises H , so that $G(w) \subset G(v)$. As w has a single incident edge e and $G(e) \neq G(w)$, this is impossible. This contradiction shows that if H is normal in $G(w)$, then the annulus A must be untwisted, as claimed. Now suppose that H is not normal in $G(w)$, so that H has index 3 in $G(w)$. Let H_1 denote the intersection of the conjugates of H in $G(w)$. Thus H_1 is a normal subgroup of $G(w)$ of index some power of 3. If A_1 denotes the cover of A with fundamental group H_1 , the fact that this cover has odd degree implies that A_1 is itself a twisted annulus in M_1 . Now the preceding argument yields a contradiction, showing that the annulus A must be untwisted as claimed. This completes the proof that w must be of special solid torus type when $G(w)$ is $VPCn$ and w has a single incident edge e with $G(e)$ of index 2 or 3 in $G(w)$.

Case: $G(w)$ is $VPCn$, and w has three incident edges e_1, e_2 and e_3 , each carrying $G(w)$.

Let K denote $G(w)$, and let A_i denote the annulus associated to the edge e_i . As in the preceding case, we claim that each A_i is untwisted. Assuming this claim, it follows that the boundary of each A_i carries K , which implies that w is of special solid torus type. It remains to prove the claim.

As K is a torsion free $VPCn$ group it is also PDn . Thus an annulus with fundamental group K is untwisted if and only if K is orientable.

Now suppose that K is non-orientable, so that each A_i is twisted. As usual we choose an aspherical space M with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Let M_K denote the cover of M with fundamental group equal to K , and let M_0 denote the cover of M with fundamental group equal to K_0 , the orientable subgroup of K of index 2. The lift of each annulus A_i into M_K has boundary in a single component Σ_i of ∂M_K whose fundamental group must equal K_0 . As the A_i 's determine non-conjugate splittings of G , the Σ_i 's must be distinct. As above, the double cover M_0 of M_K has two boundary components above each Σ_i each with fundamental group equal to K_0 . It follows that K has at least 6 coends in G , which implies that $\Gamma_{n,n+1}$ has a V_0 -vertex v of commensuriser type with $G(v) = Comm_G(K)$. Further any almost invariant subset of G over a subgroup commensurable with K must be enclosed by v . In particular, for each i , the K -almost invariant subset of G associated to e_i is enclosed by v . This implies that there is a path λ_i in $\Gamma_{n,n+1}$ with v at one end and with the edge e_i at the other end, and each interior vertex of λ_i is isolated. As $\Gamma_{n,n+1}$ is reduced bipartite, each λ_i contains at most two edges. As v is a V_0 -vertex and w is a V_1 -vertex, it follows that each λ_i consists of the single edge e_i . Thus each e_i has v and w as its endpoints. Let Γ_K denote

the subgraph of $\Gamma_{n,n+1}$ which consists of the union of the e_i 's. As each e_i and w has associated group K , and $G(v)$ commensurises K , it follows that the group carried by Γ_K also commensurises K . But as $G(v)$ is the full commensuriser of K , this is impossible. This contradiction shows that K must be orientable, so that each A_i is untwisted, as claimed. This completes the proof that w must be of special solid torus type when $G(w)$ is $VPCn$ and w has three incident edges each carrying $G(w)$, and so completes the proof of part 2) of the theorem.

3) This follows immediately from parts 1) and 2), as the definitions of the various types of V_0 -vertex in the statements of parts 1) and 2) all contain the requirement that the edge splittings be dual to an essential annulus or torus.

4) Consider a nontrivial almost invariant subset of G over a $VPC(n+1)$ group and adapted to ∂G . We need to show that this set is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}$, and by some V_0 -vertex of $\Gamma_{n,n+1}^c$. By definition, any such set lies in the family \mathcal{F}' . Now Theorem 2.6.17 shows that $\Gamma_{n,n+1}$ equals the algebraic regular neighbourhood of the family \mathcal{F}' . Thus any element of \mathcal{F}' is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}$. The construction of the completion $\Gamma_{n,n+1}^c$ of $\Gamma_{n,n+1}$ shows that any element of \mathcal{F}' is also enclosed by some V_0 -vertex of $\Gamma_{n,n+1}^c$. This completes the proof of part 4).

5) Let H be a $VPC(n+1)$ subgroup of G which is not conjugate into ∂G . We need to show that H is conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}^c$. First note that as G is torsion free, so is H . Thus H must be $PD(n+1)$.

Suppose that H is orientable. The hypothesis that H is not conjugate into ∂G implies that H is an essential torus in $(G, \partial G)$, so that there is a nontrivial H -almost invariant subset X of G dual to H . Now Theorem 2.6.17 shows that X is enclosed by a V_0 -vertex of $\Gamma_{n,n+1}$, so that H is conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}$, and hence is also conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}^c$ as required.

Now we will suppose that H is non-orientable and is not conjugate into any V_0 -vertex group of $\Gamma_{n,n+1}$. The proof of part 4) of Theorem 2.4.4 shows that the commensuriser K of H in G is itself a non-orientable $PD(n+1)$ subgroup of G . Let K_0 denote the orientation subgroup of K . Then K_0 is a maximal torus subgroup of G , and the proof of part 4) of Theorem 2.4.4 shows that it is not conjugate into ∂G . As in the preceding paragraph, it follows that there is a V_0 -vertex v of $\Gamma_{n,n+1}$, so that K_0 is conjugate into $G(v)$. Our assumption that H is not conjugate into any V_0 -vertex group of $\Gamma_{n,n+1}$ implies that the same is true for K . As K contains K_0 with finite index, there is a vertex w of $T_{n+1}(G, \partial G)$ such that K is conjugate into $G(w)$, and w must be a V_1 -vertex. Hence there is an edge e of $\Gamma_{n,n+1}$ which is incident to w such that $G(e)$ contains a conjugate

of K_0 . As all the edge groups of $\Gamma_{n,n+1}$ are annulus or torus groups, the group $G(e)$ must equal this conjugate of K_0 . Let σ denote the splitting of G over K_0 determined by the edge e , so that σ is dual to an essential torus in G , and let L denote the vertex group of σ which contains $G(w)$. If ∂G is empty, Theorem 8.1 of [1] shows that the pair formed by L and one or two copies of K_0 is $PD(n+2)$, where there will be two copies of K_0 if σ is a HNN extension and only one copy otherwise. In general, as discussed just before Definition 2.5.1, the pair becomes $PD(n+2)$ when some family of groups in ∂G is added to the copies of K_0 . As L contains a conjugate of K , one of the copies of K_0 in ∂L is not equal to its own commensuriser in L . Thus Lemma 2.2.7 implies that L contains this copy of K_0 with index 2, and ∂L consists only of K_0 . In particular it follows that σ must be an amalgamated free product and not a HNN extension. As L contains a conjugate of K , and both K and L contain K_0 with index 2, it follows that $G(w)$ is equal to L and must be a conjugate of K . Hence the edge e of $\Gamma_{n,n+1}$ is separating, and if we remove the interior of e from $\Gamma_{n,n+1}$, then the component Γ_w of the resulting subgraph which contains w must carry the group $G(w)$. Now the minimality of $\Gamma_{n,n+1}$ implies that Γ_w consists solely of w , so that w has valence 1 in $\Gamma_{n,n+1}$. It follows that w becomes a V_0 -vertex in the completion $\Gamma_{n,n+1}^c$, so that K , and hence H , is conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}^c$, as required. \square

We observe the following result which again shows the similarity between the algebra in this paper and the topology of 3-manifolds.

Lemma 2.7.9. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC. Suppose that X is a n -canonical almost invariant subset of G which is over a $VPC(n+1)$ group H , and that H intersects some group in ∂G in a $VPCn$ group L . Then X is isolated in $\mathcal{F}_{n,n+1}$.*

Remark 2.7.10. *If $n = 1$ and M is an orientable Haken 3-manifold, the corresponding result holds. For X corresponds to an essential torus T in M which crosses no essential annulus in M , such that $\pi_1(T)$ intersects the fundamental group of some boundary component of M in an infinite cyclic subgroup. This second condition implies that T must be homotopic into a component W of $T(M)$ which meets ∂M , and the fact that T crosses no essential annulus in M now implies that T must be homotopic into a torus component of the frontier of W in M . In particular, T crosses no essential annulus or torus in M .*

Proof. As usual, Lemma 13.1 of [22] tells us that H has a subgroup H' of finite index which normalises a subgroup L' of finite index in L . Also as usual, we

choose an aspherical space M with fundamental group G and with aspherical subspaces corresponding to ∂G whose union is denoted ∂M . Let M' denote the covering space of M with fundamental group L' . The hypothesis that H intersects some group in ∂G in the group L implies that there is a component of $\partial M'$ with fundamental group L' . The action of H'/L' on M' yields an infinite family of distinct such boundary components. Now the proof of part 3) of Lemma 2.2.21 shows that $e(G, L')$ is infinite.

As H' normalises L' , it also follows that L' has large commensuriser. Thus there is a V_0 -vertex v of $\Gamma_{n,n+1}$ of commensuriser type such that $G(v) = \text{Comm}_G(L')$, so that $G(v)$ must contain H' . Also $\Gamma_{n,n+1}$ has a V_0 -vertex w which encloses X .

First we suppose that v and w are distinct. It follows that there is a path joining v and w such that H' lies in the edge group of each edge on the path. Let e be an edge on this path. Recall that all edge groups of $\Gamma_{n,n+1}$ are $VPCn$ or $VPC(n+1)$. Thus $G(e)$ must be $VPC(n+1)$ and must contain H' with finite index. Let Y denote the almost invariant subset of G associated to the edge splitting of G given by e . Lemma 2.7.1 tells us that Y is adapted to ∂G . Now Corollary 2.6.4 tells us that X is also adapted to ∂G . But a H' -almost invariant subset of G which is adapted to ∂G is unique up to equivalence and complementation, so it follows that X must be equivalent to Y or Y^* . As Y is isolated in $\mathcal{F}_{n,n+1}$, it follows that X is also isolated in $\mathcal{F}_{n,n+1}$.

Now suppose that $v = w$. Recall from Theorem 2.3.14 that any V_0 -vertex of $\Gamma_{n,n+1}$ of commensuriser type must be of Seifert type, of torus type, or of solid torus type. The last case cannot occur here, as $G(v)$ contains the $VPC(n+1)$ group H' . Thus v is of Seifert type, or of torus type. In the second case, $G(v)$ is $VPC(n+1)$ and so must contain H' with finite index. Thus any almost invariant set enclosed by v and corresponding to a torus must be equivalent to X or to X^* . Now the hypothesis that X crosses no essential annulus in M implies that X crosses no element of the CCC of $\mathcal{F}_{n,n+1}$ which gives rise to v . As X is enclosed by v , it must be associated to an edge splitting of $\Gamma_{n,n+1}$ associated to an edge incident to v , and so X must be isolated in $\mathcal{F}_{n,n+1}$. Finally if v is of Seifert type, we use the facts that v is of $VPCn$ -by-Fuchsian type, and that tori enclosed by v correspond to loops in the base orbifold X_v , and annuli enclosed by v correspond to arcs in X_v . Now a loop in X_v which crosses no arc must be peripheral in X_v . It follows again that X must be associated to an edge splitting of $\Gamma_{n,n+1}$ associated to an edge incident to v , and so X must be isolated in $\mathcal{F}_{n,n+1}$. This completes the proof of the lemma. \square

2.8 Comparing the decompositions of a $PD(n+2)$ pair and its double

In this section, we will apply Theorem [2.3.14](#) to understand the effect of doubling on our decompositions $\Gamma_{n,n+1}(G)$ and $\Gamma_{n,n+1}^c(G)$ of a $PD(n+2)$ pair $(G, \partial G)$.

First we need to improve our description of these decompositions. The same description suffices for both decompositions, so we will only work with $\Gamma_{n,n+1}^c(G)$. In section [2.2](#) we used aspherical spaces to clarify the concept of an essential annulus in a $PD(n+2)$ pair $(G, \partial G)$. Now we need to greatly refine those ideas in order to clarify "how vertices of $\Gamma_{n,n+1}^c(G)$ meet ∂G ". Let $\partial G = \{S_1, \dots, S_m\}$, and recall that in section [2.2](#), we used a mapping cylinder construction to make a $K(G, 1)$ with $K(S_i, 1)$'s as disjoint subspaces.

Given any group G and a graph of groups decomposition $\Gamma(G)$ of G , there is a general construction of an aspherical space X with $\pi_1(X) = G$ whose structure mimics the graph of groups $\Gamma(G)$. This is called a graph of spaces. For each vertex group $G(v)$ of $\Gamma(G)$, we choose a corresponding aspherical space $K(G(v), 1)$, and for each edge group $G(e)$ we choose a $K(G(e), 1)$ and then take its product with the unit interval I . We construct X from the disjoint union of all the $K(G(v), 1)$ and $K(G(e), 1) \times I$ by gluing each end of each $K(G(e), 1) \times I$ to the appropriate $K(G(v), 1)$, by a map inducing the appropriate inclusion of fundamental groups. Thus there is a natural map from X to $\Gamma(G)$ defined by collapsing each $K(G(v), 1)$ and each $K(G(e), 1)$ to a point. We will apply this construction to the decomposition $\Gamma_{n,n+1}^c(G)$. Further, for each edge e of $\Gamma_{n,n+1}^c(G)$ whose associated splitting is dual to an annulus Λ with fundamental group H , we will choose our $K(G(e), 1)$ to be Λ , which is an I -bundle over a $K(H, 1)$, as discussed in section [2.2](#). Thus $\partial\Lambda$, the boundary of the annulus Λ , is the corresponding ∂I -bundle over $K(H, 1)$.

Next we will carry out a similar construction of an aspherical space Y which represents ∂G . Each boundary component of each annulus associated to an edge splitting of $\Gamma_{n,n+1}^c(G)$ induces a splitting of one of the S_i 's. Thus each S_i can be decomposed as a graph of groups structure with edges corresponding to boundary components of these annuli. For each S_i , we make a corresponding graph of spaces construction of a space Y_i , with $\pi_1(Y_i) = S_i$. Further, if an edge e of this graph of groups decomposition of S_i has associated splitting over a boundary component of the annulus Λ , we will choose the corresponding $K(G(e), 1)$ to be homeomorphic to that component of $\partial\Lambda$. Let Y denote the disjoint union of the Y_i 's. Note that as S_i is a $PD(n+1)$ group and these splittings are over PDn

groups, it follows from [1] that each vertex space of Y is naturally a $PD(n+1)$ pair.

Finally we combine the above constructions using a mapping cylinder construction as follows. We take the disjoint union of the space X constructed above from $\Gamma_{n,n+1}^c(G)$ with the product $Y \times I$, and glue $Y \times \{0\}$ to X so that each edge space glues by a homeomorphism to a boundary component of the appropriate annulus Λ , and each vertex space is glued to the appropriate vertex space of X . Denote the resulting space by M , and denote $Y \times \{1\}$ by ∂M . We now have a picture which mimics the topology of the JSJ decomposition of a 3-manifold, as each edge splitting of $\Gamma_{n,n+1}^c(G)$ which is dual to an annulus Λ is represented by an embedding of Λ in M with $\partial\Lambda$ embedded in ∂M . Note that M has a natural projection p to $\Gamma_{n,n+1}^c(G)$ obtained by collapsing the constituent edge and vertex spaces to a point.

In order to complete the analogy with 3-manifold topology, we proceed as follows. We subdivide $\Gamma_{n,n+1}^c(G)$ by adding a new vertex at the middle of each edge e , and then for each vertex v of $\Gamma_{n,n+1}^c(G)$, we let M_v denote the pre-image under p of the star of v in this subdivision. Let $\partial_1 M_v$ denote the intersection of M_v with the pre-image under p of the new vertices, let $\partial_0 M_v$ denote $M_v \cap \partial M$, and let ∂M_v denote the union $\partial_0 M_v \cup \partial_1 M_v$. Each component of $\partial_1 M_v$ is an annulus or torus in M . Each component of $\partial_0 M_v$ is either a component of ∂M , or is naturally a $PD(n+1)$ pair with boundary equal to the boundary of some annuli in M . Thus we have finally assigned meaning to the "intersection with ∂G of a vertex of $\Gamma_{n,n+1}^c(G)$ ".

If v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of interior Seifert type, of special Seifert type, or is isolated with torus group, then $\partial_1 M_v$ consists entirely of essential tori in $(G, \partial G)$, and $\partial_0 M_v$ is empty. Thus the "intersection of v with ∂G " is empty.

If v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of I -bundle type, then $\partial_1 M_v$ consists entirely of essential annuli in $(G, \partial G)$, and $\partial_0 M_v$ consists of one or two $PD(n+1)$ pairs the union of whose boundary components is equal to the boundary of the annuli forming $\partial_1 M_v$. If $\partial_0 M_v$ consists of two $PD(n+1)$ pairs, each includes into M_v by an isomorphism of fundamental groups, so that the I -bundle is trivial. Otherwise, $\partial_0 M_v$ consists of a single $PD(n+1)$ pair such that the image of $\pi_1(\partial_0 M_v)$ in $\pi_1(M_v)$ has index 2, so that the I -bundle is twisted. Thus the "intersection of v with ∂G " is the ∂I -bundle associated to the I -bundle.

We note that if Σ is a torus in ∂G , then it must be conjugate into some vertex group of $\Gamma_{n,n+1}(G)$ or of $\Gamma_{n,n+1}^c(G)$. Otherwise there is an essential annulus in $(G, \partial G)$ with a boundary component in Σ , but then the V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ which encloses that annulus must be of commensuriser type so that $G(v)$ con-

tains Σ .

Now we can describe exactly what we mean by doubling $\Gamma_{n,n+1}(G)$ or $\Gamma_{n,n+1}^c(G)$. Again the same description suffices for both. Let DM denote the space obtained by doubling M along ∂M , so that DM is a $K(DG, 1)$. We denote the two copies of M in DM by M and \overline{M} , and let τ denote the involution of DM which interchanges M and \overline{M} . We will describe a family of disjoint tori in DM which determines the decomposition of DG which we want. The annuli and tori in M which correspond to the edges of $\Gamma_{n,n+1}^c(G)$ determine tori in DM , as follows. A torus T in M yields two tori T and τT in DM , and an annulus Λ in M yields a torus $D\Lambda = \Lambda \cup \tau\Lambda$ in DM . In addition, for each torus Σ in ∂G which is enclosed by a V_1 -vertex of $\Gamma_{n,n+1}^c(G)$, we add two parallel copies of the corresponding component of ∂M , one copy in M and the other in \overline{M} . Clearly the tori in this family are all disjoint. Now this family of disjoint tori in DM determines a graph of groups structure of DG , which we denote by $D\Gamma_{n,n+1}^c$. Thus all the edges of $D\Gamma_{n,n+1}^c$ have associated splittings dual to tori in DG .

There is a natural map from DM to $D\Gamma_{n,n+1}^c$, and it is easy to describe the vertex spaces of DM . If Σ is a torus in ∂G which is conjugate into a V_1 -vertex group of $\Gamma_{n,n+1}^c(G)$, the two parallel copies of Σ in DM together bound a copy of $\Sigma \times I$, which corresponds to an isolated vertex of $D\Gamma_{n,n+1}^c$. We label such a vertex as a V_0 -vertex of $D\Gamma_{n,n+1}^c$. The other vertex spaces of DM arise from vertex spaces of M . If M_v is disjoint from ∂M , so that $\partial_0 M_v$ is empty, there are two corresponding vertex spaces M_v and τM_v of DM each homeomorphic to M_v . If v is a V_1 -vertex of $\Gamma_{n,n+1}^c(G)$ such that $\partial_0 M_v$ consists only of torus components of ∂M , there are again two corresponding vertex spaces, each homeomorphic to M_v . If v is a V_1 -vertex of $\Gamma_{n,n+1}^c(G)$ such that $\partial_0 M_v$ is not empty and does not consist only of torus components of ∂M , there is one corresponding vertex space obtained by doubling M_v along the non-torus components of $\partial_0 M_v$. Finally, if v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ such that $\partial_0 M_v$ is non-empty, there is one corresponding vertex space obtained by doubling M_v along $\partial_0 M_v$. We define each of the corresponding vertices of $D\Gamma_{n,n+1}^c$ to be of type V_0 or V_1 so as to be of the same type as v . With this labelling, $D\Gamma_{n,n+1}^c$ is bipartite. Note that an isolated vertex of $D\Gamma_{n,n+1}^c$ either arises from a torus in ∂G which is conjugate into a V_1 -vertex group of $\Gamma_{n,n+1}^c(G)$, or it arises from an isolated annulus or torus vertex of $\Gamma_{n,n+1}^c(G)$. As $\Gamma_{n,n+1}^c(G)$ is reduced, it follows that $D\Gamma_{n,n+1}^c$ is also reduced.

Next we need to consider more detail about the structure of the V_0 -vertex groups of $\Gamma_{n,n+1}^c(G)$. Recall that if v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of interior Seifert type, then v is of $VPCn$ -by-Fuchsian type. Let L denote the $VPCn$ normal

subgroup of $G(v)$. Then the quotient group $G(v)/L$ is not virtually cyclic and is the orbifold fundamental group of a compact 2-orbifold X_v . Further there is exactly one edge of $\Gamma_{n,n+1}^c(G)$ which is incident to v for each peripheral subgroup K of $G(v)$, and this edge carries K . Thus there is a natural projection of M_v to X_v , in which $\partial M_v = \partial_1 M_v$ maps onto ∂X_v . This precisely mirrors the picture in a 3-manifold of an interior Seifert fibre space component of the characteristic submanifold. We will show that a similar picture occurs for any V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ of commensuriser type, again mirroring the situation for 3-manifolds. Thus, in all these cases, there is a compact 2-orbifold X_v and a natural projection of M_v to X_v in which ∂M_v maps onto ∂X_v . Note that any V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ of commensuriser type is of peripheral type, so that $\partial_0 M_v$ is non-empty, which introduces some new aspects to the discussion. Note also that if v is of peripheral Seifert type, then $G(v)$ is $VPCn$ -by-Fuchsian and so v has a natural base orbifold X_v , but even this is not clear if v is of torus type or of solid torus type.

Consider a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of commensuriser type. For a given such vertex v , there is a $VPCn$ subgroup L of $G(v)$ such that for each edge of $\Gamma_{n,n+1}^c(G)$ which is incident to v and associated to a splitting over an annulus Λ , the group carried by each component of $\partial\Lambda$ is L . It follows that $\partial_0 M_v$ consists of a disjoint union of torus components of ∂M and of annuli whose boundary components carry L . Thus each component of ∂M_v is either a torus component of $\partial_0 M_v$ or of $\partial_1 M_v$, or is a union of annuli in $\partial_0 M_v$ and $\partial_1 M_v$.

Let T be a component of ∂M_v which is a union of annuli in $\partial_0 M_v$ and $\partial_1 M_v$. Either all the annuli in T are untwisted and glued in a circular pattern, or there are precisely two twisted annuli in T separated by a chain of untwisted annuli. Note that all these annuli carry subgroups of G , and so have torsion free $VPCn$ fundamental group. It follows that $\pi_1(T)$ is a torsion free $VPC(n+1)$ group and so is $PD(n+1)$. We claim that $\pi_1(T)$ is an orientable $PD(n+1)$ group. By our construction of M , we know that T has a neighbourhood homeomorphic to an I -bundle over T . Further this I -bundle must be trivial as T is a boundary component of M_v . Now it follows that there is an excision isomorphism between $H_{n+2}(T \times I, T \times \partial I)$ and $H_{n+2}(M, \partial M) \cong \mathbb{Z}$. As $H_{n+2}(T \times I) = 0$, and $H_{n+1}(T \times \partial I) \cong H_{n+1}(T) \oplus H_{n+1}(T)$, it follows that $H_{n+1}(T)$ is non-zero, which implies that $\pi_1(T)$ is an orientable $PD(n+1)$ group, as required. We will abuse terminology and say that T is a torus.

Note that an untwisted annulus has a natural projection to the unit interval I , and a twisted annulus has a natural projection to the 1-orbifold Q which is the quotient of I by a reflection involution. Hence if the torus T is a union of

untwisted annuli, it has a natural map to the circle S^1 such that the restriction to each annulus is projection to a unit interval. And if T contains two twisted annuli, it has a natural map to the 1-orbifold C which is the quotient of S^1 by a reflection involution such that the restriction to each untwisted annulus is projection to a unit interval, and the restriction to each twisted annulus is projection to a copy of Q . Now recall that our aim is to show that if v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of commensuriser type, then there is a compact 2-orbifold X_v and a natural projection of M_v to X_v in which ∂M_v maps onto ∂X_v . We will show further that this projection can be chosen so that its restriction to each annulus component of $\partial_0 M_v$ and of $\partial_1 M_v$ is the natural projection to a 1-suborbifold of ∂X_v which is isomorphic to I or Q , as appropriate. π

We start by considering the case when v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of torus type, so that $G(v)$ is $VPC(n+1)$ and hence $PD(n+1)$.

Lemma 2.8.1. *Let v be a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of torus type. Then the following hold:*

1. *If $G(v)$ is an orientable $PD(n+1)$ group, then ∂M_v consists of two tori such that each includes into M_v inducing an isomorphism of fundamental groups.*
2. *If $G(v)$ is a non-orientable $PD(n+1)$ group, and if $G(v)_0$ denote its orientation subgroup, then ∂M_v consists of a single torus whose inclusion into M_v induces an injection of fundamental groups with image $G(v)_0$.*
3. *In either case, there is a compact 2-orbifold X_v such that $G(v)$ is L -by- $\pi_1^{orb}(X_v)$, and a natural map from M_v to X_v such that ∂M_v maps onto ∂X_v . Further, each annulus in $\partial_0 M_v$ and $\partial_1 M_v$ maps to a 1-suborbifold of ∂X_v by the natural map. If the annulus is untwisted, its image in ∂X_v is isomorphic to the unit interval I , and if the annulus is twisted, its image in ∂X_v is isomorphic to Q , the quotient of I by a reflection involution.*

Proof. 1) The above discussion shows that ∂M_v consists of tori. There is an excision isomorphism between $H_{n+2}(M_v, \partial M_v)$ and $H_{n+2}(M, \partial M) \cong \mathbb{Z}$, and the fundamental class of ∂M_v maps to zero in $H_{n+1}(M_v)$. Now consider the long exact homology sequence of the pair $(M_v, \partial M_v)$. As $G(v)$ is $VPC(n+1)$ and orientable, we know that $H_{n+2}(M_v) = 0$, and $H_{n+1}(M_v) \cong \mathbb{Z}$. As $H_{n+1}(T) \cong \mathbb{Z}$ for any torus T , it follows that ∂M_v consists of at most two tori.

In case 1) of Definition [2.3.12](#), there is a torus Σ in $\partial_0 M_v$ whose inclusion into M_v induces an isomorphism of fundamental groups.

In case 2) of Definition [2.3.12](#) there is a torus Σ in $\partial_1 M_v$ whose inclusion into M_v induces an isomorphism of fundamental groups.

It follows that in either of these cases, ∂M_v consists of two tori and that the inclusion into M_v of the second torus T induces an isomorphism $H_{n+1}(T) \rightarrow H_{n+1}(M_v)$. Thus the map from $\pi_1(T)$ to $\pi_1(M_v)$ is onto. As $\pi_1(T)$ is $VPC(n+1)$, the kernel of this map must be finite. Now the fact that $\pi_1(T)$ is torsion free implies that this map is an isomorphism, as required.

In case 3) of Definition [2.3.12](#) there is a torus contained in M_v whose fundamental group equals $G(v)$. We could further refine our above construction of $(M, \partial M)$ to arrange that M_v contains such a torus Σ . This must separate M_v into two pieces each with fundamental group $G(v)$. By considering each of these pieces separately as in the preceding paragraph, we can again conclude that each component of ∂M_v includes into M_v by an isomorphism of fundamental groups.

2) If $G(v)$ is a non-orientable $PD(n+1)$ group, then $H_{n+1}(M_v) = 0$. Now the long exact homology sequence of the pair $(M_v, \partial M_v)$ shows that ∂M_v consists of a single torus T . As T is orientable, the image of $\pi_1(T)$ in $G(v)$ is contained in $G(v)_0$. Thus there is an index 2 subgroup G' of G whose intersection with $G(v)$ is $G(v)_0$, and which is naturally a $PD(n+2)$ pair $(G', \partial G')$. Let M' denote the corresponding model space for this pair, and apply part 1) of the lemma to the appropriate vertex space of M' . This will imply that the inclusion of $T = \partial M_v$ into M_v induces an injection of fundamental groups with image $G(v)_0$, as required.

3) If $G(v)$ is orientable and T denotes a torus in ∂M_v , then the pair $(M_v, \partial M_v)$ is homotopy equivalent to $(T \times I, T \times \partial I)$. If T consists of a circular chain of untwisted annuli, then the quotient $\pi_1(T)/L$ is isomorphic to \mathbb{Z} , and we take the orbifold X_v to be the annulus. If T contains two twisted annuli, then the quotient $\pi_1(T)/L$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, and we take the orbifold X_v to be the product $C \times I$, where C is the quotient of S^1 by a reflection involution.

If $G(v)$ is non-orientable, then ∂M_v consists of a single torus T whose inclusion into M_v induces an injection of fundamental groups with image $G(v)_0$. If T consists of a circular chain of untwisted annuli, then the quotient $\pi_1(T)/L$ is isomorphic to \mathbb{Z} , and we take the orbifold X_v to be the Moebius band. If T contains two twisted annuli, then the quotient $\pi_1(T)/L$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, and we take the orbifold X_v to be "a twisted I -bundle" over C , which we describe as follows. Recall that the Moebius band is a twisted I -bundle over the circle S^1 . Let σ denote the reflection involution of S^1 with quotient C . Then σ extends to an involution $\bar{\sigma}$ of this I -bundle over S^1 , and X_v is the quotient of this action. Note that $\bar{\sigma}$ preserves each fibre over the fixed points of σ , and fixes

one of these fibres pointwise, while reflecting the other one. Thus the fixed set of $\bar{\sigma}$ consists of an interval, where the local picture of $\bar{\sigma}$ is a reflection, and of an isolated point, where the local picture is of a rotation through π . Thus X_v has underlying surface a disc D , the boundary of D contains a single mirror interval whose complement is thus a copy of C , and X_v also has an order 2 cone point in the interior of D . \square

Now we understand how " v meets ∂G " when v is of torus type, we can apply this to the case where v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of peripheral Seifert type to obtain the following result.

Corollary 2.8.2. *Let v be a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of peripheral Seifert type, and let X_v denote the base orbifold of v . Then there is a natural map from M_v to X_v such that ∂M_v maps onto ∂X_v . Further, each annulus in $\partial_0 M_v$ and $\partial_1 M_v$ maps to a 1-suborbifold of ∂X_v with non-empty boundary, by the natural map.*

Proof. Recall from Definition 2.3.6 that $\Gamma_{n,n+1}^c(G)$ can be refined by splitting at v to a graph of groups structure Γ' of G with a vertex v' of Γ' such that $G(v') = G(v)$ and v' is of Seifert type adapted to ∂G . The projection map $\Gamma' \rightarrow \Gamma$ sends v' to v and is an isomorphism apart from the fact that certain edges incident to v' are collapsed to v . Further if e is an edge of Γ' which is incident to v' and collapsed to v , then the other vertex w of e is of torus type. As $G(e)$ is a torus, w is of torus type 2) in Definition 2.3.12. We can correspondingly refine M to obtain the vertex space $M_{v'}$. As v' is of Seifert type adapted to ∂G , there is a natural projection of $M_{v'}$ to $X_{v'}$ with all the required properties. It also follows from Lemma 2.8.1 that the base orbifold of w is an annulus or $C \times I$. In turn this implies that X_v and $X_{v'}$ are isomorphic. Now it follows that there is a natural projection of M_v to X_v in which ∂M_v maps onto ∂X_v , and each annulus component of $\partial_0 M_v$ and of $\partial_1 M_v$ maps in the natural way to a 1-suborbifold of ∂X_v with non-empty boundary, as required. \square

We have shown that for any V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ of commensuriser type, but not of solid torus type, there is a base orbifold X_v such that ∂X_v is divided into the image of $\partial_0 M_v$, which we denote by $\partial_0 X_v$, and the image of $\partial_1 M_v$, which we denote by $\partial_1 X_v$. This precisely mirrors the picture in a 3-manifold of a peripheral Seifert type or torus type component of the characteristic submanifold. Now we can continue our discussion of the double $D\Gamma_{n,n+1}^c$. As discussed above, this gives rise to a single vertex space DM_v of DM obtained by doubling M_v along $\partial_0 M_v$. It is clear that DM_v has a natural map to the 2-orbifold DX_v obtained by doubling X_v along $\partial_0 X_v$. It is easy to see that $\pi_1^{orb}(DX_v)$ is not virtually

cyclic, so the corresponding V_0 -vertex V of $D\Gamma_{n,n+1}^c$ is of $VPCn$ -by-Fuchsian type.

If v is a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of I -bundle type, it is of $VPC(n-1)$ -by-Fuchsian type. Let K denote the $VPC(n-1)$ normal subgroup of $G(v)$. As before, we let X_v denote the compact 2-orbifold whose orbifold fundamental group is $G(v)/K$, and whose boundary corresponds to the edges of $\Gamma_{n,n+1}^c(G)$ which are incident to v . Thus doubling M_v along $\partial_0 M_v$ yields a V_0 -vertex V of $D\Gamma_{n,n+1}^c$ of $VPCn$ -by-Fuchsian type with the same base 2-orbifold X_v .

The case of V_0 -vertices of $\Gamma_{n,n+1}^c(G)$ of solid torus type seems to be different, and we will need a separate and more subtle argument. Let v be a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of solid torus type. Thus $G(v)$ is $VPCn$ and has a $VPCn$ subgroup L such that all the annuli in $\partial_0 M_v$ and in $\partial_1 M_v$ have boundary with fundamental group L .

Lemma 2.8.3. *Let v be a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$ of solid torus type, and let L be as above. Then there is a compact 2-orbifold X_v , equal to a cone or the quotient of a cone by a reflection, such that $G(v)$ is L -by- $\pi_1^{orb}(X_v)$, and a natural map from M_v to X_v such that ∂M_v maps onto ∂X_v . Further, each annulus in $\partial_0 M_v$ and $\partial_1 M_v$ maps to a 1-suborbifold of ∂X_v with non-empty boundary, by the natural map.*

Remark 2.8.4. *The orbifold fundamental group of X_v must be finite in this case, as L has finite index in $G(v)$. Note also that the definition of solid torus type, Definition 2.3.8, did not include the statement that L is normal in $G(v)$. This seems to be quite nontrivial.*

Proof. As before there is an excision isomorphism between $H_{n+2}(M_v, \partial M_v)$ and $H_{n+2}(M, \partial M) \cong \mathbb{Z}$. As $G(v)$ is $VPCn$, we know that $H_{n+1}(M_v)$ is zero. Now the long exact sequence of the pair $(M_v, \partial M_v)$ shows that ∂M_v consists of a single torus T . This torus is a union of annuli in $\partial_0 M_v$ and $\partial_1 M_v$, and all these annuli have boundary with fundamental group L . Thus L is a normal subgroup of $\pi_1(T)$ with quotient \mathbb{Z} or $\mathbb{Z}_2 * \mathbb{Z}_2$. However the inclusion of T into M_v cannot induce an injective map of fundamental groups, as $G(v)$ is $VPCn$, but $\pi_1(T)$ is $VPC(n+1)$. In this case, we need a more complicated argument to find the base orbifold for v .

Let M_V denote the double of M_v along $\partial_0 M_v$. The double of each annulus in $\partial_1 M_v$ is a torus component of ∂M_V . Recall that any torus in DG is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$. As pairs of these tori in ∂M_V are joined by an annulus in $\partial_0 M_v$, it follows from the proof of Lemma 2.5.13 that all components of ∂M_V are enclosed by a single V_0 -vertex W of $\Gamma_{n+1}^c(DG)$. Now let A be an

annulus in $\partial_0 M_v$. As A is an annulus with ends in W , which is a V_0 -vertex of $\Gamma_{n+1}^c(DG)$, it follows that A cannot cross any component of ∂M_W . Thus either V itself is enclosed by W , or some component T of ∂M_W is enclosed by V and is not peripheral in V . Suppose there is such a component T of ∂M_W . As T cannot cross any annulus in $\partial_0 M_v$, it follows that T is enclosed by v or τv , which is impossible, as $G(v)$ is $VPCn$. We conclude that V must be enclosed by W . If W is isolated, it follows that V is also isolated and hence that v must be isolated, contradicting the assumption that v is of solid torus type. If W is of special Seifert type, it follows that V must be isolated or also of special Seifert type. We have just shown that V cannot be isolated, so it must be of special Seifert type. As v is of solid torus type, the only possibility is that it is of special solid torus type, with a single incident edge carrying a subgroup of index 2 in $G(v)$. In this case, we can take X_v to be a cone with cone point labeled 2, and with $\partial_0 X_v$ and $\partial_1 X_v$ each consisting of a single arc in ∂X_v . If W is $VPCn$ -by-Fuchsian, it follows that L is the normal $VPCn$ subgroup of $G(W)$, and that $G(V)$ is the pre-image in $G(W)$ of a suborbifold X of the base orbifold X_W of W . Now we can choose a projection map $p : M_V \rightarrow X$, so that ∂M_V maps to ∂X , and each annulus in $\partial_1 M_v$, and its translate by τ , projects in the natural way to a 1-orbifold contained in ∂X . The involution τ of DM induces an involution of $G(V)$ which is the identity on L , and so induces a proper homotopy equivalence of X of order 2. The Nielsen Realization Theorem [10] implies that there is an involution of X in the given homotopy class, which we again denote by τ . Recall that each component of ∂M_V is the double of an annulus in $\partial_1 M_v$. It follows that τ induces an involution on each component of ∂M_V which interchanges the two annuli. Hence the involution τ on X acts by a reflection on each component of ∂X , and so fixes one or two points of each such component. We now need to consider how $p : M_V \rightarrow X$ maps each annulus Λ of $\partial_0 M_v$ into X . We already know that each component of $\partial \Lambda$ is also a component of the boundary of an annulus in $\partial_1 M_v$, and so is mapped to a point of ∂X fixed by τ .

If Λ is untwisted, it follows that we can homotop p restricted to Λ to factor through the natural projection of Λ to the unit interval. This yields a path λ in X joining two points of ∂X fixed by τ . We now give X a hyperbolic metric such that ∂X consists of geodesics and τ is an isometry, and then homotop λ rel $\partial \lambda$ to the unique geodesic in its homotopy class. As $\tau \Lambda = \Lambda$, it follows that $\tau \lambda$ is homotopic rel $\partial \lambda$ to λ . Now the uniqueness of hyperbolic geodesics in a homotopy class implies that $\tau \lambda = \lambda$. It follows that λ is contained in the fixed set of τ , and that λ must be a simple geodesic.

If Λ is twisted, then p maps $\partial \Lambda$ to a point a of ∂X fixed by τ . As $\pi_1(\Lambda)$

contains $\pi_1(\partial\Lambda) = L$ with index 2, it follows that $p_*\pi_1(\Lambda)$ is a subgroup of $\pi_1^{orb}(X)$ of order 2. Such a subgroup of $\pi_1^{orb}(X)$ must be carried by a mirror m of X or by a cone point w of X with even number attached. In the first case, we can homotope the map from Λ to X to have image a path λ joining a to m , and then further homotop λ to be the shortest geodesic in its homotopy class. As τ fixes Λ , it follows that τ must preserve m , and that $\tau\lambda$ is homotopic to λ fixing a and keeping the other end of λ in m . Now the uniqueness of hyperbolic geodesics in a homotopy class implies that $\tau\lambda = \lambda$. It follows that λ is contained in the fixed set of τ , and that λ must be a simple geodesic. In the second case, we can homotope the map from Λ to X to have image a path λ joining a to w , and then further homotop λ to be the shortest geodesic in its homotopy class. As τ fixes Λ , it follows that τ must fix w , and that $\tau\lambda$ is homotopic to λ rel $\partial\lambda$. Now the uniqueness of hyperbolic geodesics in a homotopy class implies again that $\tau\lambda = \lambda$, so that λ must be contained in the fixed set of τ , and λ must be a simple geodesic. In both cases, the image of λ is a 1-suborbifold of X , isomorphic to Q , the quotient of the unit interval by a reflection.

At this point, we have arranged that each annulus of $\partial_0 M_v$ maps to a "simple arc" in X which is contained in the fixed set of τ , so that distinct such arcs cannot cross. It is conceivable that two of these arcs coming from twisted annuli could share an endpoint at a cone point of X , but that would imply the two twisted annuli in question carried the same subgroup of DG , which is not possible. It follows that the image of ∂M_v in X is a connected 1-orbifold C , without boundary, embedded in X . Thus C is a circle or the quotient of a circle by reflection. The image of $\pi_1^{orb}(C)$ in $\pi_1^{orb}(X)$ must be finite, as it is contained in $\pi_1(M_v)/L$, which is the image of $\pi_1(M_v)$. It follows that C bounds a suborbifold Z of X with finite orbifold fundamental group, which can only be a cone or the quotient of a cone by a reflection. This is the required base orbifold X_v for v . \square

Now we come to an important result about the V_1 -vertices of $D\Gamma_{n,n+1}^c$.

Lemma 2.8.5. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, such that G is not VPC . Let V be a V_1 -vertex of $D\Gamma_{n,n+1}^c$, denote $G(V)$ by K , and let ∂K denote the family of subgroups of K associated to the edges of $D\Gamma_{n,n+1}^c$ incident to V . Then $(K, \partial K)$ is an orientable atoroidal $PD(n+2)$ pair.*

Proof. The fact that $(K, \partial K)$ is an orientable $PD(n+2)$ pair follows from [1]. Now suppose that T is a torus in $(K, \partial K)$. We need to show that T is conjugate into a group in ∂K . Recall from the above discussion of DM and $D\Gamma_{n,n+1}^c$, that V must be obtained from a V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$. If V is isolated, the required

result is immediate. If V is equal to a V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$, then T is a torus in G . Now any torus in G is conjugate into a group in ∂G or is essential and so enclosed by some V_0 -vertex of $\Gamma_{n,n+1}^c(G)$. In either case, it must be conjugate into a group in ∂K . If V is obtained by doubling from some V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$, there are three possible cases up to conjugacy, that T is an essential torus in G or \overline{G} , that T is contained in a group in ∂G , or that T is decomposed into essential annuli lying in $(G, \partial G)$ or $(\overline{G}, \partial \overline{G})$ and enclosed by v or by τv . The first case is again not possible. In the second case, T would be peripheral in V , as required. Now we consider the third case in which T is decomposed into essential annuli lying in $(G, \partial G)$ or $(\overline{G}, \partial \overline{G})$ and enclosed by v or by τv . We note that any essential annulus in $(G, \partial G)$ is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}^c(G)$. Thus if an essential annulus in $(G, \partial G)$ is enclosed by a V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$, it must be a cover of an annulus associated to an edge of $\Gamma_{n,n+1}^c(G)$ incident to v . Thus the annuli into which T is decomposed are all covers of edge annuli in v or τv . If v is a non-isolated V_1 -vertex of $\Gamma_{n,n+1}^c(G)$, it is not possible to have two distinct edge annuli whose boundaries carry the same group L . It follows that T is a subgroup of the double of a single edge annulus of v , so that T is a subgroup of an edge torus of V , as required. \square

Next we discuss more carefully the V_0 -vertices of $D\Gamma_{n,n+1}$. These are all obtained from V_0 -vertices of $\Gamma_{n,n+1}(G)$ except for those isolated vertices obtained from torus groups in ∂G enclosed by a V_1 -vertex of $\Gamma_{n,n+1}(G)$.

If V is obtained from a V_0 -vertex v of $\Gamma_{n,n+1}(G)$ of interior type, then V is equal to v or τv , so is isolated, of special Seifert type or of interior Seifert type. In the last case, if L denotes the $VPCn$ normal subgroup of $G(V)$ with Fuchsian quotient, then L is contained in G or \overline{G} , but is not contained in any group in ∂G .

If V is obtained from a V_0 -vertex v of $\Gamma_{n,n+1}(G)$ of peripheral type, then M_V is the double of M_v along $\partial_0 M_v$. If v is isolated, then V is also isolated. If v is of commensuriser type, there is a $VPCn$ subgroup L of G such that $G(v) = \text{Comm}_G(L)$ and all edge annuli of v have boundary which carries L . Thus L is contained in G and also contained in groups in ∂G . Further Lemma 2.8.1, Corollary 2.8.2 and Lemma 2.8.3 together show that $G(v)$ is L -by- $\pi_1^{orb}(X_v)$, where X_v is the base orbifold of v . Hence $G(V)$ is L -by- $\pi_1^{orb}(DX_v)$, where DX_v is the double of X_v along $\partial_0 X_v$. In almost all cases $\pi_1^{orb}(DX_v)$ is not virtually cyclic, and V is of $VPCn$ -by-Fuchsian type, where the normal $VPCn$ subgroup is again L . The only exception occurs when v is of special solid torus type, with a single incident edge carrying a subgroup of $G(v)$ of index 2. In this case, V is of special Seifert type.

Finally if V is obtained from a V_0 -vertex v of $\Gamma_{n,n+1}(G)$ of I -bundle type, then V is of Seifert type, but this time the normal $VPCn$ subgroup L of $G(V)$ is not conjugate into G or \overline{G} .

Now we can state our doubling result which is precisely analogous to the situation in 3-manifold theory.

Theorem 2.8.6. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, such that G is not VPC . Then the double $D\Gamma_{n,n+1}^c$ of $\Gamma_{n,n+1}^c(G)$ is equal to the decomposition $\Gamma_{n+1}^c(DG)$ of DG .*

Remark 2.8.7. *The double $D\Gamma_{n,n+1}$ of $\Gamma_{n,n+1}(G)$ need not be equal to the decomposition $\Gamma_{n+1}(DG)$ of DG . For example, consider the case when G is the fundamental group of a 3-manifold M , and $\Gamma_{n,n+1}^c(G)$ has a V_0 -vertex v of special solid torus type, with three incident edges each carrying $G(v)$. Thus the characteristic submanifold of M has a component X which is a solid torus which meets ∂M in three annuli all of degree 1 in X . Doubling M yields a component DX of the characteristic submanifold of DM , which is the double of X along $X \cap \partial M$. Thus DX is the product of a pair of pants with the circle. Now X corresponds to a V_1 -vertex of $\Gamma_{n,n+1}(G)$, but DX corresponds to a V_0 -vertex of $\Gamma_{n+1}(DG)$.*

Proof. The outline of our proof is to show that each V_0 -vertex of $D\Gamma_{n,n+1}^c$ is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$, and that each V_0 -vertex of $\Gamma_{n+1}^c(DG)$ is enclosed by some V_0 -vertex of $D\Gamma_{n,n+1}^c$. Assuming these two facts, we can deduce the theorem as follows. Let V be a V_0 -vertex of $D\Gamma_{n,n+1}^c$ enclosed by the V_0 -vertex W of $\Gamma_{n+1}^c(DG)$. As W is enclosed by a V_0 -vertex V' of $D\Gamma_{n,n+1}^c$, it follows that V is enclosed by V' . As $D\Gamma_{n,n+1}^c$ is reduced, it follows that $V = V'$, and hence that $G(V) = G(W)$. Let $\partial G(V)$ denote the family of subgroups of $G(V)$ associated to the edges of $D\Gamma_{n,n+1}^c$ incident to V , and similarly for W . The facts that V is enclosed by W , and $G(V) = G(W)$ implies that the $PD(n+2)$ pairs $(G(V), \partial G(V))$ and $(G(W), \partial G(W))$ are isomorphic. Similarly if W is a V_0 -vertex of $\Gamma_{n+1}^c(DG)$ enclosed by a V_0 -vertex V of $D\Gamma_{n,n+1}^c$, it follows that the $PD(n+2)$ pairs $(G(V), \partial G(V))$ and $(G(W), \partial G(W))$ are isomorphic. Together these facts imply that $D\Gamma_{n,n+1}^c$ is equal to $\Gamma_{n+1}^c(DG)$, as required.

First we will show that each V_0 -vertex of $D\Gamma_{n,n+1}^c$ is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$.

Our discussion immediately before this theorem shows that if V is a V_0 -vertex of $D\Gamma_{n,n+1}^c$, it is isolated, of special Seifert type, or of $VPCn$ -by-Fuchsian type. Now any torus in DG is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$, and crossing tori must be enclosed by the same V_0 -vertex of $\Gamma_{n+1}^c(DG)$. Thus if V

is not of special Seifert type, it follows that V is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$. If V is of special Seifert type, we know that $G(V)$ is conjugate into some V_0 -vertex group of $\Gamma_{n+1}^c(DG)$, so that again V is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$. Thus each V_0 -vertex of $D\Gamma_{n,n+1}^c$ is enclosed by some V_0 -vertex of $\Gamma_{n+1}^c(DG)$, as required.

It remains to show that each V_0 -vertex W of $\Gamma_{n+1}^c(DG)D\Gamma_{n,n+1}^c$ is enclosed by some V_0 -vertex of $D\Gamma_{n,n+1}^c$.

If W is isolated, let T denote an edge torus of W . As T is an edge torus of $\Gamma_{n+1}^c(DG)$, it crosses no torus in DG , and so is enclosed by some vertex of $D\Gamma_{n,n+1}^c$. Lemma 2.8.5 tells us that V_1 -vertices of $D\Gamma_{n,n+1}^c$ are atoroidal, so it follows that T , and hence W , is enclosed by some V_0 -vertex of $D\Gamma_{n,n+1}^c$, as required.

If W is of special Seifert type, we let T denote the edge torus of W . Again T is enclosed by some V_0 -vertex V of $D\Gamma_{n,n+1}^c$. As $G(W)$ contains T with index 2, it follows that W is enclosed by some vertex V' of $D\Gamma_{n,n+1}^c$. If V' is a V_1 -vertex, it follows that T is peripheral in both V and in V' . Now Lemma 2.2.7 shows that V' must be of special Seifert type. As no V_1 -vertex of $\Gamma_{n,n+1}^c(G)$ can be of special Seifert type, V' must be obtained by doubling from a V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$. This implies that v must be of special solid torus type. As no V_1 -vertex of $\Gamma_{n,n+1}^c(G)$ can be of special solid torus type, it follows that V' is a V_0 -vertex of $D\Gamma_{n,n+1}^c$ which encloses W , as required.

For the rest of this proof, we will assume that W is of $VPCn$ -by-Fuchsian type with normal $VPCn$ subgroup L . Thus Lemma 2.5.10 tells us that $G(W) = N_{DG}(L)$.

Let T denote the universal covering DG -tree of the graph of groups Δ determined by the doubling of G along ∂G . Thus Δ has two vertices with associated groups G and \bar{G} , and has edges corresponding to the groups in ∂G . We consider the actions of L and $G(W) = N_{DG}(L)$ on T , and the various cases which arise.

Case 1: L fixes a vertex z of T .

By a conjugation and possibly interchanging G and \bar{G} , we can assume that $G(z) = G$. In particular, $L \subset G$. Let T' denote the subtree of T consisting of all edges and vertices of T fixed by L . The action of $N_{DG}(L)$ on T must preserve T' .

Case 1a): $T' = \{z\}$.

Thus $N_{DG}(L)$ fixes z . This implies that $N_{DG}(L) \subset G$. In particular, any torus enclosed by W is also enclosed by the vertex U of Δ with associated group G . As W is filled by crossing tori, it follows that W itself is enclosed by U , and hence that W is enclosed by some V_0 -vertex Z of $\Gamma_{n,n+1}^c(G)$. Let \bar{Z} denote the

corresponding V_0 -vertex of $D\Gamma_{n,n+1}^c$, so that \bar{Z} is either equal to Z or obtained by doubling Z . In either case, W is enclosed by \bar{Z} , as required.

Case 1b): z has valence 1 in T' .

This implies that there are no essential annuli in $(G, \partial G)$ which carry L , and the same holds for \bar{G} . Thus every vertex of T' has valence 1 in T' , which implies that T' is equal to a single edge e of T . As $N_{DG}(L)$ preserves T' and cannot interchange the ends of e , we deduce that $N_{DG}(L)$ fixes e . Now the stabilizer of e is a group K in ∂G and so is an orientable $PD(n+1)$ group, and $N_{DG}(L)$ contains a torus subgroup Σ which is also $PD(n+1)$. It follows that Σ has finite index in K , so that K is $VPC(n+1)$, and hence also a torus. Hence W must be isolated, contradicting our assumption.

Case 1c): z has valence 2 in T' .

This implies that all vertices of T' have valence 2 in T' , so that T' is a line. As $N_{DG}(L)$ preserves this line, it follows that there is a map from $N_{DG}(L)$ to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$ whose kernel K fixes every point of T' . Note that $N_{DG}(L) \cap G = N_G(L)$. In the first case, $N_G(L)$ equals K , and in the second case, $N_G(L)$ contains K with index 2. As L fixes only two edges of T incident to z , there is a unique essential annulus in $(G, \partial G)$ which carries L . It follows that there is a V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$, such that $G(v) = N_G(L)$, and that v is isolated or of special solid torus type, with a single incident edge carrying a subgroup of index 2. Let V denote the V_0 -vertex of $D\Gamma_{n,n+1}^c$ obtained by doubling v . Then $G(V)$ contains $N_G(L)$, and contains an element which acts on T' by a translation of length 2. Hence $G(V)$ is equal to $G(W)$. In particular, W is enclosed by the V_0 -vertex V of $D\Gamma_{n,n+1}^c$, as required.

Case 1d): z has valence at least 3 in T' .

As L fixes three distinct edges of T incident to z , it follows that there are three distinct annuli in $(G, \partial G)$ whose boundaries carry L . Hence there is a V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ of special solid torus type or of commensuriser type, such that $G(v) = \text{Comm}_G(L) = N_G(L) = N_{DG}(L) \cap G$. Hence for each edge e of T' incident to v , with stabilizer S , we have $N_{DG}(L) \cap S = G(v) \cap S$. Let V denote the double of v , so that $G(V) \subset N_{DG}(L) = G(W)$. The quotient of T' by $G(V)$ has two vertices, so the same holds for the quotient of T' by $G(W)$. As both groups act on T' with the same edge and vertex stabilizers, it follows that $G(V) = G(W)$. In particular, W is enclosed by the V_0 -vertex V of $D\Gamma_{n,n+1}^c$, as required.

Case 2: L fixes no vertex of T .

In this case, there is a unique minimal L -subtree Λ of T . As L is VPC ,

Lemma 2.1.10 tells us that Λ is a line. The uniqueness of Λ implies that $N_{DG}(L)$ preserves Λ . As in Case 1c), there is a map from $N_{DG}(L)$ to \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$ whose kernel K fixes every point of Λ . And $N_{DG}(L) \cap G$ is either equal to K or contains K with index 2. As $G(W) = N_{DG}(L)$, we know that any torus Σ enclosed by W must intersect L in a subgroup of finite index in L . Hence the $VPC(n+1)$ group Σ acts on Λ , and $\Sigma \cap K$ is a $VPCn$ subgroup of Σ which fixes the vertex z and preserves the two edges of Λ incident to z . This determines an essential annulus in $(G, \partial G)$ which carries $\Sigma \cap K$. Every torus enclosed by W determines an essential annulus in $(G, \partial G)$ preserving the same two edges of Λ incident to z . As W is not isolated or of special Seifert type, there are infinitely many distinct such annuli, so that $N_{DG}(L) \cap G = G(v)$, for some V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ of I -bundle type. Let V be obtained by doubling v . As $G(V)$ contains $N_{DG}(L) \cap G$ and acts on Λ by a translation of length 2, it follows that $G(V) = G(W)$. In particular, W is enclosed by the V_0 -vertex V of $D\Gamma_{n,n+1}^c$, as required. \square

The machinery which we have just developed can be used to give simple proofs of some other results. For example, we have the following result.

Theorem 2.8.8. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, such that G is not VPC . Let \mathcal{F} denote the family of all essential annuli in $(G, \partial G)$, and let \mathcal{F}_n denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup. Then $\Gamma(\mathcal{F} : G)$, the algebraic regular neighbourhood of \mathcal{F} in G , is equal to $\Gamma_n(G)$, the algebraic regular neighbourhood of \mathcal{F}_n in G .*

Proof. An essential annulus A in $(G, \partial G)$ is enclosed by some V_0 -vertex v of $\Gamma_n(G)$. The discussion in this section shows that if v is of I -bundle type, then A determines a loop in the base orbifold of v , and if v is of commensuriser type, then A determines an arc in the base orbifold of v . Further this yields a bijection between equivalence classes of annuli in $(G, \partial G)$ and loops and arcs in base orbifolds of V_0 -vertices of $\Gamma_n(G)$. As any compact 2-orbifold is filled by essential (possibly singular) loops, and is also filled by essential (possibly singular) arcs, it follows that $\Gamma(\mathcal{F} : G)$ is equal to $\Gamma_n(G)$, as required. \square

2.9 Concluding remarks and problems

In this section, we briefly discuss some problems which arise from our work in this paper.

In the previous section, we showed, in Theorem [2.8.6](#), that if $(G, \partial G)$ is an orientable $PD(n+2)$ pair, such that G is not VPC , then the double $D\Gamma_{n,n+1}^c$ of $\Gamma_{n,n+1}^c(G)$ is equal to the decomposition $\Gamma_{n+1}^c(DG)$ of DG . This precisely mirrors the situation in 3-manifold theory. But in that theory, one can deduce the JSJ decomposition theorem for 3-manifolds with boundary from the corresponding result for closed manifolds by "undoubling". This is a great simplification of the direct proofs. Our first problem is to decide whether an analogous argument is possible in the setting of $PD(n+2)$ pairs.

Problem 2.9.1. *Can the main result of this paper, Theorem [2.3.14](#), be deduced from the properties of the decomposition $\Gamma_{n+1}^c(DG)$ of DG , by some "undoubling" argument?*

Our main result in this paper, Theorem [2.3.14](#), shows that the situation for $PD(n+2)$ pairs is very similar to that for 3-manifolds with boundary. Now in 3-manifold theory, the characteristic submanifold of a Haken 3-manifold has the enclosing property for Seifert pairs. Thus it is reasonable to ask the following.

Problem 2.9.2. *Do our results in this paper imply a result for $PD3$ pairs analogous to the enclosing property for Seifert pairs in 3-manifolds.*

This is Conjecture 10.4 of Wall's survey article [\[34\]](#). However we are unable to answer this question. Even a precise formulation of the statement requires a theory of Poincaré duality triads, and/or of Poincaré duality pairs with compressible boundary, neither of which has been developed so far.

Since one of the main results in the JSJ theory of 3-manifolds is Johannson's Deformation Theorem [\[8\]](#), it is natural to ask the following.

Problem 2.9.3. *Is there a result for Poincaré duality pairs which is analogous to Johannson's Deformation Theorem?*

Some of our discussion and definitions have been rather topological in order to suit the setting of group pairs and duality. A more algebraic description of the decomposition $\Gamma_{n,n+1}^c(G)$ should involve Poincaré duality triads rather than pairs, since a vertex group of $\Gamma_{n,n+1}^c(G)$ has two distinct important types of subgroups, namely the edge groups and the intersection groups with ∂G . We feel that a reformulation in these terms and a strengthening of the statements would be necessary in order to formulate the analogue of Johannson's Deformation Theorem. In [\[33\]](#), Wall studied the notion of triads of Poincaré complexes. It is natural to ask whether our algebraic decompositions can be realised using

finite aspherical Poincaré complexes. Suppose that G is the fundamental group of such a complex X of dimension $n + 2$ and that G splits over a $VPCn$ group H . A basic question is whether X splits over a subcomplex Y with $\pi_1(Y) = H$. One expects to have to change the complex X to achieve this, so it seems better to ask whether there is a finite complex X' of the same simple homotopy type as X , such that X' splits over a suitable subcomplex Y . This involves the study of some obstruction groups most studied by Waldhausen [32]. For a general torsion free $VPCn$ group H , these obstruction groups are not well understood, so the problem seems difficult. But if we assume that H is torsion free and polycyclic, then Waldhausen's work shows that the obstruction groups are zero, so that we can find such a complex X' as desired. However, even if we try to restrict our attention to almost invariant subsets of G over torsion free polycyclic subgroups, it seems possible that the edge groups of the decompositions we obtain may be VPC . If the edge groups are polycyclic, as happens in the 3-dimensional case, then the analogue of Johansson's Deformation Theorem can be formulated. We expect this analogue to be correct for 3-dimensional Poincaré duality pairs, and finite aspherical Poincaré 3-complexes. In general it may be true only when the peripheral groups are polycyclic.

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Chapter 3

Comparing decompositions of Poincaré duality pairs

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Abstract. Analogues of JSJ decompositions were developed for Poincaré duality pairs in [19]. These decompositions depend only on the group. Our focus will be on describing the edge splittings of these decompositions more precisely. We use our results to compare these decompositions with two other closely related decompositions.

3.1 Introduction

In this paper, we consider algebraic analogues of previous work in the topology of 3-manifolds related to the JSJ decomposition introduced by Jaco and Shalen [8] and Johansson [9]. In [8] and [9], the authors considered a compact orientable Haken 3-manifold M with incompressible boundary, and constructed the characteristic submanifold $V(M)$ as a maximal Seifert pair embedded in M . The frontier of $V(M)$ is a family of disjoint essential annuli and tori in M , which decompose M into pieces either in $V(M)$ or its complement. In [12], [13], [14], [16] and [20], the emphasis turned to annuli and tori rather than Seifert pairs. In [12], the authors gave a new approach to constructing this decomposition of M in which embedded essential annuli and tori were the main subject of interest. They defined an embedded essential annulus or torus in M to be *canonical* if it can be isotoped to be disjoint from any other embedded essential annulus

or torus in M . This led to a finer decomposition of M , which they called the Waldhausen decomposition (W-decomposition), from which the JSJ decomposition can be obtained in a natural way. In [16], the interest was in possibly singular essential annuli and tori in M . The authors defined an embedded essential annulus or torus in M to be *topologically canonical* if it has intersection number zero with any (possibly singular) essential annulus or torus in $(M, \partial M)$. Most canonical annuli and tori in a 3-manifold are also topologically canonical, and the exceptions can be precisely described. The existence of these exceptions explains why the W-decomposition of M is in general finer than the JSJ decomposition. In [16], the authors also defined an algebraic analogue in which a splitting of $\pi_1(M)$ given by an essential embedded annulus or torus in M is *algebraically canonical* if it has intersection number zero with any almost invariant subset of $\pi_1(M)$ which is over \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}$. (See [15] for a discussion of the idea of intersection numbers of almost invariant sets.) They showed that topologically canonical splittings are not quite the same as algebraically canonical ones, and gave some examples to demonstrate this. If M has empty boundary, there is no difference.

In [20], (which is a revised version of [19]), an analogue of the JSJ decomposition of 3-manifolds was developed for orientable $PD(n+2)$ pairs, with $n \geq 1$. The decomposition for a $PD(n+2)$ pair $(G, \partial G)$ is simply the decomposition $\Gamma_{n,n+1}(G)$ of [17], which is defined for many almost finitely presented groups G . (If ∂G is empty, so that G is a $PD(n+2)$ group, then $\Gamma_{n,n+1}(G)$ is equal to the decomposition $\Gamma_{n+1}(G)$.) See [18] for corrections to [17]. Thus this decomposition depends only on the group G and not on ∂G . In the case when ∂G is empty, Kropholler [10] had earlier obtained this decomposition. In [17], the decomposition $\Gamma_{n,n+1}(G)$ was constructed as the regular neighbourhood of the family consisting of all almost invariant (a.i.) subsets of G over $VPCn$ subgroups together with all a.i. subsets of G over $VPC(n+1)$ subgroups which do not cross any a.i. subset of G over a $VPCn$ subgroup. (A group is $VPCn$ if it is virtually polycyclic of Hirsch length n .) This regular neighbourhood is reduced, meaning that two adjacent vertices cannot both be isolated, except when the graph is a loop with just these two vertices. (A vertex w of a G -tree T is called *isolated* if it has valence 2, and these two edges have the same stabilizer as w . The image of w in $G \backslash T$ is also called *isolated*.) In general, the edge groups of $\Gamma_{n,n+1}$ may not even be finitely generated, but it is shown in [20] that, in the case of $PD(n+2)$ pairs, the edge groups are all either $VPCn$ or $VPC(n+1)$.

It was shown in [20] that if G is the fundamental group of a compact orientable Haken 3-manifold M with incompressible boundary, then $\Gamma_{1,2}(G)$ dif-

fers from the JSJ decomposition of M only in some small Seifert pieces which have no crossing annuli or tori. One can easily move from $\Gamma_{1,2}(G)$ to its completion $\Gamma_{1,2}^c(G)$, and this completion corresponds to the JSJ decomposition of M . The difference between $\Gamma_{1,2}(G)$ and its completion $\Gamma_{1,2}^c(G)$ is related to special properties of small Seifert fibre spaces. If the boundary of M is empty, then similar comments apply to $\Gamma_2(G)$ and its completion $\Gamma_2^c(G)$. In the general case of a $PD(n+2)$ pair $(G, \partial G)$ we denote the completion of $\Gamma_{n,n+1}(G)$ by $\Gamma_{n,n+1}^c(G)$ or just $\Gamma_{n,n+1}^c$ when the group G is clear from the context. The completion $\Gamma_{n,n+1}^c(G)$ is essentially obtained from $\Gamma_{n,n+1}(G)$ by re-labelling some V_1 -vertices as V_0 -vertices and then adding isolated V_1 -vertices as needed to keep the graph bipartite. In particular, the edge splittings of $\Gamma_{n,n+1}$ are the same as the edge splittings of $\Gamma_{n,n+1}^c$. Again the completed decomposition depends only on the group G and not on ∂G .

Our focus in this paper will be on the edge splittings of these decompositions of a group G . This is closely related to the approach in [16] in the case of 3-manifolds. Our main result, Theorem 3.4.1, is similar to that in [16] but is in the setting of the decomposition $\Gamma_{n,n+1}(G)$ of a $PD(n+2)$ pair $(G, \partial G)$. Our result is more detailed than that in [16], and gives a precise description of the special cases which arise. These results are new even in the setting of 3-manifolds, and they yield a substantial refinement of the results in [16].

There are other natural approaches to finding JSJ decompositions of a $PD(n+2)$ pair $(G, \partial G)$. The analogue of [12] would be to consider a maximal family of splittings of G by annuli and tori which cross no other such splitting. (In this paper, we use the word "cross" to mean "has non-zero intersection number with".) Another approach would be simply to consider the regular neighbourhood of the family of all almost invariant subsets of G which are over $VPCn$ or $VPC(n+1)$ subgroups. For general groups, neither of these decompositions need exist. However, in this paper, we use the results of [20] and of Theorem 3.4.1 to show that both decompositions exist in the setting of Poincaré duality pairs, and we compare these three different decompositions. The differences between them leads to a detailed study of various small 2-dimensional orbifolds and fibrations over them by $VPCn$ groups. We think that this clarifies how these various natural decomposition come about. We also discuss the special case of $PD3$ pairs where the descriptions are somewhat simpler. This seems to make an analogue of Johansson's Deformation Theorem possible for $PD3$ pairs, and also seems relevant to some questions raised by Wall in sections 6 and 10 of [23].

In section 3.2, we describe the notions of annuli, tori, and their associated almost invariant sets as in [20]. We will also recall some constructions and results

from [20] which we will use. In section 3.3, we describe examples generalizing those given in [16]. We also completely characterize all such examples. In section 3.4, we prove our main result, Theorem 3.4.1. In section 3.5, we compare three different versions of JSJ decompositions of PD -pairs. In section 3.6, we discuss how our results become substantially simpler when applied to the case of $PD3$ pairs, and we compare our results with those in [12]. In section 3.7, we discuss some related questions.

We will also use earlier definitions and results from [1], [2], [3] and [10]. There are two survey articles from around 2000, [4] and [23], which contain a number of problems related to PD groups and pairs.

3.2 Preliminaries

We will consider orientable $PD(n+2)$ pairs $(G, \partial G)$ and first describe, following [20], annuli and tori in $(G, \partial G)$ and their associated almost invariant sets.

Let H be a $VPC(n+1)$ subgroup of G . Note that as G is a PD group, it is torsion free. Hence H is also torsion free, and so is a $PD(n+1)$ group. The double DG of G is an orientable $PD(n+2)$ group, and so the pair (DG, H) has two ends if H is orientable, and only one end otherwise. In the first case, DG contains two complementary nontrivial H -almost invariant subsets X and X^* , and any nontrivial H -almost invariant subset of DG is equivalent to one of these. Let Y denote the intersection $X \cap G$. Thus Y and its complement Y^* in G are H -almost invariant subsets of G . Further they are nontrivial unless H is peripheral in $(G, \partial G)$, i.e. H is conjugate into a group in ∂G . We say that Y is *dual to H* . If H is an orientable $VPC(n+1)$ subgroup of G , we call it a torus in G . Note that the H -almost invariant set Y dual to H is automatically adapted to ∂G . Conversely, suppose that H is a $VPC(n+1)$ subgroup of G , and Y is a nontrivial H -almost invariant subset of G which is adapted to ∂G . Then Y extends to a nontrivial H -almost invariant subset of DG . It follows that H must be orientable and hence a torus in G .

The case of annuli requires more work. An annulus in a $PD(n+2)$ pair is a certain type of orientable $PD(n+1)$ pair. We need to consider two types of annulus. One type is $\Lambda_H = (H, \{H, H\})$, where H is an orientable PDn group which is also $VPCn$. We call this an untwisted annulus. The other type is $\Lambda_H = (H, H_0)$, where H is a non-orientable PDn group which is $VPCn$, and H_0 is the orientation subgroup of H . We call this a twisted annulus. Corresponding to these, we have $K(\pi, 1)$ spaces which we denote by $(A, \partial A)$. Similarly, we

denote the $K(\pi, 1)$ pair corresponding to $(G, \partial G)$ by $(M, \partial M)$. Note that when $n = 1$, the only $PD1$ group is \mathbb{Z} , and this is orientable. Thus twisted annuli do not appear in the theory of 3-manifolds. For simplicity, we assume that G is finitely presented so that we can identify certain cohomology groups of G with the cohomology groups with compact supports of various covers of M . In the general case, when G is almost finitely presented, we have to take the appropriate ‘finitely supported’ cohomology groups of covers of M .

An annulus in $(G, \partial G)$ is an injective homomorphism of group pairs $\Theta : \Lambda_H \rightarrow (G, \partial G)$. This means that Θ maps H to G and also maps each group in $\partial\Lambda_H$ to a conjugate of some group in ∂G . Such Θ induces a continuous map $\theta : (A, \partial A) \rightarrow (M, \partial M)$. Note that in the untwisted case, such a map θ is determined up to homotopy by choosing a copy of H in two conjugates of groups in ∂G , such that the two copies of H are conjugate in G . And in the twisted case, θ is determined up to homotopy by choosing a copy of H in G and a conjugate of some group in ∂G such that the intersection of H with this conjugate contains H_0 . Thus an annulus can be thought of purely algebraically. We call the annulus ‘essential’ if θ cannot be homotoped relative to ∂A into ∂M . It is clear that the essentiality of an annulus is also a purely algebraic property. An untwisted annulus is essential if and only if the images of the two boundary groups are not conjugate in a group in ∂G . And a twisted annulus is essential if and only if H_0 lies in a boundary group K in ∂G , and $H \cap K = H_0$.

We next show how to associate an almost invariant set to an essential annulus. Consider the lift $\theta_H : (A, \partial A) \rightarrow (M_H, \partial M_H)$ to the cover M_H of M with fundamental group H . Let S be the component of ∂M_H containing $\theta_H(\partial_0 A)$, where $\partial_0 A$ is one specified component of ∂A in the untwisted case and is ∂A in the twisted case.

In the untwisted case, since $\theta_H(A)$ cannot be homotoped rel ∂A into ∂M , the other component $\partial_1 A$ must be mapped by θ_H into some other component T of ∂M_H . Thus both S and T have fundamental groups isomorphic to H and the images of the fundamental cycle of H generate $H_n(S)$ and $H_n(T)$, both with \mathbb{Z} and \mathbb{Z}_2 coefficients. Let $[A]$ denote the fundamental cycle of A in $H_{n+1}(A, \partial A)$. Then in the boundary map

$$H_{n+1}(M_H, \partial M_H) \rightarrow H_n(\partial M_H) \simeq H_n(S) \oplus H_n(T) \oplus \cdots$$

we see that the projection of the image of $[A]$ to each of the first two direct summands of $H_n(\partial M_H)$ is a generator.

In the twisted case, $\pi_1(S)$ must be isomorphic to H_0 since θ cannot be homotoped into ∂M relative to ∂A . Thus the projection of the image of $[A]$ to the

summand $H_n(S)$ is a generator. So, with \mathbb{Z} or \mathbb{Z}_2 coefficients, we have $(\theta_H)_*([A])$ is non-zero in $H_{n+1}(M_H, \partial M_H)$. Also the image with \mathbb{Z}_2 -coefficients is the specialisation of the image with \mathbb{Z} -coefficients. Denote these images by α and $\bar{\alpha}$, and denote the duals of these images in $H_c^1(M_H; \mathbb{Z})$ and $H_c^1(M_H; \mathbb{Z}_2)$ by β and $\bar{\beta}$. Again $\bar{\beta}$ is the specialisation of β .

Next, we want to relate $\bar{\beta}$ to the ends of the pair (G, H) . We have that H is of infinite index in G , so that $H_f^0(M_H; \mathbb{Z}_2) = 0$ (finite cohomology is used as in section 7.4 of [3]). Thus $H_f^1(M_H; \mathbb{Z}_2)$ fits into the exact sequence:

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H_e^0(M_H; \mathbb{Z}_2) \rightarrow H_f^1(M_H; \mathbb{Z}_2) \xrightarrow{\bar{r}} H^1(M_H; \mathbb{Z}_2).$$

Here H_e^0 is the 0-cohomology of the space of ends and the last map is the restriction map from finite cohomology to ordinary cohomology. Group theoretically, the above sequence identifies with the following. Let $P[H \backslash G]$ denote the power set of right cosets Hg of H in G , and let $E[H \backslash G]$ denote $P[H \backslash G]/\mathbb{Z}_2[H \backslash G]$. The above sequence can be identified with

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H^0(G; E[H \backslash G]) \xrightarrow{\delta} H^1(G; \mathbb{Z}_2[H \backslash G]) \xrightarrow{\bar{r}} H^1(H; \mathbb{Z}_2).$$

Thus $\bar{\beta}$ gives an element of $H^1(G; \mathbb{Z}_2[H \backslash G])$ which we continue to denote by $\bar{\beta}$. In order to show that this element gives a nontrivial almost invariant set, we need to know that it is non-zero in $H^1(G; \mathbb{Z}_2[H \backslash G])$, and is in the kernel of \bar{r} . We already know that $\bar{\beta}$ is non-zero since we started with an essential annulus. Thus it remains to show that $\bar{r}(\bar{\beta}) = 0$. Consider the following diagram:

$$\begin{array}{ccc} H^1(G; \mathbb{Z}[H \backslash G]) & \xrightarrow{r} & H^1(H; \mathbb{Z}) \\ \downarrow \rho & & \downarrow \rho \\ H^1(G; \mathbb{Z}_2[H \backslash G]) & \xrightarrow{\bar{r}} & H^1(H; \mathbb{Z}_2) \end{array}$$

In Theorem 2 of [21], Swarup showed that r is the zero map. Since $\bar{\beta} = \rho(\beta)$ it follows that $\bar{r}(\bar{\beta}) = 0$, although in general \bar{r} is not the zero map. Thus we see that $\bar{\beta} = \delta(e)$ for some element e of $H^0(G; E[H \backslash G])$. Since the kernel of δ is just \mathbb{Z}_2 , the element e defines a nontrivial H -almost invariant subset Y of G up to equivalence and complementation. This completes our association of an almost invariant set Y with an essential annulus θ . We say that Y is dual to θ . It turns out that given a nontrivial almost invariant subset X of G which is over a $VPCn$ group H , there is a subgroup H' of finite index in H such that X is a finite sum of almost invariant sets over H' each dual to an annulus.

For future reference, we give some more terminology. As usual $(G, \partial G)$ is an orientable $PD(n+2)$ pair. If the almost invariant subset of G dual to an essential annulus or torus is associated to a splitting σ , we will say that σ is *dual* to the same essential annulus or torus. If Γ is a graph of groups structure for G , and v is a vertex of Γ , then an essential annulus or torus in $(G, \partial G)$ is enclosed by v if the dual almost invariant subset of G is enclosed by v . Finally if θ and ϕ are each an essential annulus or torus in $(G, \partial G)$, we will say that θ and ϕ cross if the dual almost invariant subsets of G cross.

In [17], the authors considered an almost finitely presented group G and an integer n such that G has no nontrivial almost invariant subsets over $VPCk$ subgroups for $k < n$. Then it was shown that the family $\mathcal{F}_{n,n+1}$ of all equivalence classes of almost invariant subsets of G over $VPCn$ groups, and all n -canonical almost invariant subsets over $VPC(n+1)$ groups has an algebraic regular neighbourhood, denoted $\Gamma_{n,n+1}(G)$. In this setting a H -almost invariant subset of G is n -canonical if it does not cross any almost invariant subset over a $VPCn$ subgroup. If $(G, \partial G)$ is a $PD(n+2)$ pair, it was shown by Kropholler and Roller (Lemma 4.3 of [10]) that G has no nontrivial almost invariant subsets over $VPCk$ subgroups for $k < n$, so that the decomposition $\Gamma_{n,n+1}(G)$ exists. In [20], the authors showed that almost invariant subsets of G over $VPC(n+1)$ subgroups which do not cross any almost invariant subset over a $VPCn$ subgroup are automatically adapted to ∂G . Further, if we enlarge the family $\mathcal{F}_{n,n+1}$ to include all almost invariant sets over $VPC(n+1)$ subgroups which are adapted to ∂G , the new family $\mathcal{G}_{n,n+1}$ has the same regular neighbourhood $\Gamma_{n,n+1}(G)$.

If M is a compact orientable Haken 3-manifold with incompressible (i.e. π_1 -injective) boundary, the characteristic submanifold $V(M)$ of M is a compact submanifold whose frontier consists of incompressible annuli and tori in M . This decomposition of M into pieces is called the JSJ decomposition. The components of $V(M)$ are Seifert fibre spaces or I -bundles. Cutting M along the frontier of $V(M)$ yields a graph of groups structure $\Gamma(M)$ for $G = \pi_1(M)$ whose edge groups are isomorphic to \mathbb{Z} or to $\mathbb{Z} \times \mathbb{Z}$. This graph is bipartite as each component of the frontier of $V(M)$ lies in the boundary of a component of $V(M)$ and a component of the complement. In [20], the authors showed that if $(G, \partial G)$ is a $PD(n+2)$ pair, then $\Gamma_{n,n+1}(G)$ has many properties in common with $\Gamma(M)$, with V_0 -vertices of $\Gamma_{n,n+1}(G)$ corresponding to the components of $V(M)$. For the complete details the reader is referred to [20], but we will need to recall some of the definitions for use in this paper.

In [20], an important part is played by groups which are $VPCn$ -by-Fuchsian. Such a group has a $VPCn$ normal subgroup whose quotient is the orbifold fun-

damental group of a compact 2-orbifold. Further the quotient is assumed not to be virtually cyclic. A V_0 -vertex v of $\Gamma_{n,n+1}$ such that $G(v)$ is $VPCn$ -by-Fuchsian corresponds to a component W of $V(M)$ which is a Seifert fibre space. Topologically there are different cases depending on how W meets ∂M , and extra conditions are imposed on the edges of $\Gamma_{n,n+1}(G)$ which are incident to v to reflect this. This is what is meant by saying that v is of *Seifert type*. If v is a V_0 -vertex of $\Gamma_{n,n+1}$ such that $G(v)$ is $VPCn$ -by- $\pi_1^{orb}(X)$, where X is a 2-orbifold with virtually cyclic fundamental group, we say that v is of *solid torus type* if $\pi_1^{orb}(X)$ is finite, and of *torus type* otherwise. The terminology reflects the type of the corresponding components of $V(M)$. Again conditions need to be imposed on the edges of $\Gamma_{n,n+1}(G)$ which are incident to v .

There are some other important special cases. In [20], the authors defined a V_1 -vertex of $\Gamma_{n,n+1}(G)$ to be of *special Seifert type* if it has only one incident edge e which is dual to an essential torus, and $G(e)$ is of index 2 in $G(v)$. Also a V_1 -vertex of $\Gamma_{n,n+1}(G)$ is of *special solid torus type*, if v is of solid torus type and does not enclose any crossing annuli. In Lemma 8.5 of [20], the authors gave a complete list of possible such vertices. The authors also considered the completion $\Gamma_{n,n+1}^c$ of $\Gamma_{n,n+1}$. This is obtained from $\Gamma_{n,n+1}$ by re-labelling as V_0 -vertices those V_1 -vertices of special Seifert type or of special solid torus type, then adding isolated V_1 -vertices to keep the graph bipartite. If the result is not reduced, we reduce it by collapsing edges.

Now the main theorem of [20] can be stated.

Theorem 3.2.1 (Theorem 3.14 of [20]). *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . Let $\mathcal{F}_{n,n+1}$ denote the family of equivalence classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup, together with the equivalence classes of all n -canonical almost invariant subsets of G which are over a $VPC(n+1)$ subgroup. Finally let $\Gamma_{n,n+1}$ denote the reduced algebraic regular neighbourhood of $\mathcal{F}_{n,n+1}$ in G , and let $\Gamma_{n,n+1}^c$ denote the completion of $\Gamma_{n,n+1}$. Thus $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ are bipartite graphs of groups structures for G , with vertices of V_0 -type and of V_1 -type.*

Then $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ have the following properties:

1. *Each V_0 -vertex v of $\Gamma_{n,n+1}$ satisfies one of the following conditions:*
 - (a) *v is isolated, and $G(v)$ is VPC of length n or $n+1$, and the edge splittings associated to the two edges incident to v are dual to essential annuli or tori in G .*

- (b) v is of $VPC(n-1)$ -by-Fuchsian type, and is of I -bundle type.
 - (c) v is of $VPCn$ -by-Fuchsian type, and is of interior Seifert type.
 - (d) v is of commensuriser type. Further v is of Seifert type, or of torus type, or of solid torus type.
2. The V_0 -vertices of $\Gamma_{n,n+1}^c$ obtained by the completion process are of special Seifert type or of special solid torus type.
 3. Each edge splitting of $\Gamma_{n,n+1}$ and of $\Gamma_{n,n+1}^c$ is dual to an essential annulus or torus in G .
 4. Any nontrivial almost invariant subset of G over a $VPC(n+1)$ group and adapted to ∂G is enclosed by some V_0 -vertex of $\Gamma_{n,n+1}$, and also by some V_0 -vertex of $\Gamma_{n,n+1}^c$.
 5. If H is a $VPC(n+1)$ subgroup of G which is not conjugate into ∂G , then H is conjugate into a V_0 -vertex group of $\Gamma_{n,n+1}^c$.

Remark 3.2.2. Recall that a vertex w of a G -tree T is called isolated if it has valence 2, and these two edges have the same stabilizer as w . The image of w in $G \backslash T$ is also called isolated.

Notice that any vertex v of $\Gamma_{n,n+1}$ or $\Gamma_{n,n+1}^c$ has two types of "boundary" subgroups. The first type comes from the edge groups of the decomposition and the family of all these subgroups will be denoted by $\partial_1 v$. The second type comes from the decomposition of ∂G by edges of the decompositions and this family will be denoted by $\partial_0 v$. The first type gives us $PD(n+1)$ pairs in $(G, \partial G)$, namely annuli or tori, and the second type gives us $PD(n+1)$ pairs which are contained in ∂G . In the three-dimensional topological case, $\partial_0 v$ and $\partial_1 v$ correspond to surfaces which can be amalgamated to yield the boundary of the 3-manifold $M(v)$ which corresponds to v . But this boundary may be compressible, and so need not yield a $PD3$ -pair. In the general case we get a triple $(G(v); \partial_0 v, \partial_1 v)$ which corresponds to a Poincaré triad ([22]) but this theory in the case of groups has not been worked out.

We should also discuss the reason for excluding VPC groups from consideration in Theorem 3.2.1. For simplicity we will consider the case when ∂G is empty, so that G is a $PD(n+2)$ group. Thus $\mathcal{F}_{n,n+1}$ consists of the equivalence classes of all almost invariant subsets of G which are over a $VPC(n+1)$ subgroup, so that $\Gamma_{n,n+1}(G) = \Gamma_{n+1}(G)$. As G is VPC and $PD(n+2)$, it must be

$VPC(n+2)$. As ∂G is empty, cases 1b) and 1d) of Theorem 3.2.1 cannot arise. Also as G is VPC , the condition of being of $VPCn$ -by-Fuchsian type in case 1c) can never occur. It should be replaced by the condition of being $VPCn$ -by- $VPC2$, to have a statement with some chance of holding. By the definition of VPC , any $VPC(n+2)$ group G contains some $VPC(n+1)$ subgroup, and hence must contain a torus T . If G admits a second torus T' which crosses T , then $\Gamma_{n+1}(G)$ consists of a single V_0 -vertex. But this vertex need not satisfy the modified condition 1c) in the statement of Theorem 3.2.1. For example, in section 7 of [7], the author gives two examples of torsion free $VPC4$ groups which are orientable $PD4$ groups, and do not contain any normal $VPC2$ subgroup. As these examples are finite extensions of \mathbb{Z}^4 , they contain many subgroups isomorphic to \mathbb{Z}^3 , and hence many tori, so that $\Gamma_3(G)$ consists of a single V_0 -vertex, which cannot satisfy condition 1a) or the modified condition 1c) in the statement of Theorem 3.2.1. Note that torsion free $VPC3$ groups which are orientable $PD3$ groups, do satisfy the modified version of Theorem 3.2.1. For any such group is the fundamental group of a closed orientable 3-manifold M which admits a geometric structure modeled on E^3 , Nil or $Solv$. In the first two cases, M is a Seifert fibre space, and $\Gamma_2(G)$ consists of a single V_0 -vertex of $VPC1$ -by- $VPC2$ type. In the third case, either $\Gamma_2(G)$ consists of a single isolated V_0 -vertex and a single isolated V_1 -vertex joined by two edges, so that $\Gamma_2(G)$ is a loop, or $\Gamma_2(G)$ consists of a single isolated V_0 -vertex, joined to two V_1 -vertices of special Seifert type.

We recall Definition 5.1 and Proposition 5.3 from [20].

Definition 3.2.3. *An orientable $PD(n+2)$ pair $(G, \partial G)$ is atoroidal if any orientable $VPC(n+1)$ subgroup of G is conjugate into one of the groups in ∂G .*

Proposition 3.2.4. *Let $(G, \partial G)$ be an orientable atoroidal $PD(n+2)$ pair, where $n \geq 1$. Let A and B be $VPC(n+1)$ groups in ∂G , possibly $A = B$. Let S and T be $VPCn$ subgroups of A and B respectively, and let g be an element of G such that $gSg^{-1} = T$. Then one of the following holds:*

1. A and B are the same element of ∂G , and $g \in A$.
2. A and B are distinct elements of ∂G , are the only groups in ∂G , and $A = G = B$. Thus $(G, \partial G)$ is the trivial pair $(G, \{G, G\})$.
3. A and B are the same element of ∂G . Further A is the only group in ∂G , and has index 2 in G .

In [20], the above proposition was applied to the V_1 -vertices of the torus decomposition of a $PD(n+2)$ pair $(G, \partial G)$. More precisely, let V be a V_1 -vertex of the torus decomposition $T_{n+1}(G, \partial G)$, and let K denote the associated group $G(V)$. Let $\partial_1 K$ denote the family of subgroups of K associated to the edges of $T_{n+1}(G, \partial G)$ incident to V , let $\partial_0 K$ denote the family of subgroups of K which lie in ∂G , and let ∂K denote the union $\partial_1 K \cup \partial_0 K$ of these two families. Then $(K, \partial K)$ is an orientable atoroidal $PD(n+2)$ pair. Each group in $\partial_1 K$ is $VPC(n+1)$, so the above proposition can be applied to any annulus in $(K, \partial K)$ with ends in $\partial_1 K$.

We now generalize this idea to apply to V_1 -vertices of $\Gamma_{n,n+1}^c(G)$.

Proposition 3.2.5. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, where $n \geq 1$. Let K be the group associated to a V_1 -vertex V of $\Gamma_{n,n+1}^c(G)$, and let $\partial_1 K$ denote the family of subgroups of K associated to the edges of $\Gamma_{n,n+1}^c(G)$ incident to V . Let A and B be groups in $\partial_1 K$, possibly $A = B$. Let S and T be $VPCn$ subgroups of A and B respectively, and let g be an element of K such that $gSg^{-1} = T$. Then one of the following holds:*

1. *A and B are the same element of $\partial_1 K$, and $g \in A$.*
2. *A and B are distinct elements of $\partial_1 K$, are the only groups in $\partial_1 K$, and $A = K = B$. Thus V is an isolated V_1 -vertex V of $\Gamma_{n,n+1}^c(G)$.*

Remark 3.2.6. *This result fails if we consider the uncompleted decomposition $\Gamma_{n,n+1}(G)$. For example, if V is a V_1 -vertex of $\Gamma_{n,n+1}(G)$ of special Seifert type, or of solid torus type such that V has valence 1, and the edge group has index 2 or 3 in $G(V)$, then V does not satisfy either of the conclusions 1)-2).*

Proof. Let $\partial_0 K$ denote the family of subgroups of K coming from the decomposition of ∂G induced by the edge splittings of $\Gamma_{n,n+1}^c(G)$. For later use, we note that any essential annulus in $(G, \partial G)$ is enclosed by a V_0 -vertex of $\Gamma_{n,n+1}^c(G)$. Thus if an essential annulus Λ in $(G, \partial G)$ is enclosed by the V_1 -vertex V with associated group K , it cannot be essential in K . This is because there is an edge e of $\Gamma_{n,n+1}^c(G)$ incident to V such that the associated edge splitting is dual to an annulus Λ' covered by Λ . Note that the group associated to e lies in $\partial_1 K$.

Now let ∂K denote the union of the two families $\partial_0 K$ and $\partial_1 K$. (Recall that groups in $\partial_1 K$ are $PD(n+1)$ pairs in $(G, \partial G)$, and groups in $\partial_0 K$ are $PD(n+1)$ pairs which are contained in ∂G .) The pair $(K, \partial K)$ is again atoroidal, in the sense that any orientable $VPC(n+1)$ subgroup of K is conjugate into one of

the groups in ∂K , but $(K, \partial K)$ need not be a PD -pair. This is because the groups in $\partial_0 K$ and $\partial_1 K$ need not be PD -groups. Now we let DK denote the double of K along the family Σ of groups in $\partial_0 K$ which are not tori, and let ∂DK denote the family consisting of the induced double of $\partial_1 K$ together with the double of the family of torus groups in $\partial_0 K$. Note that as $\partial_1 K$ consists of essential annuli and tori in $(G, \partial G)$, each group in the induced double of $\partial_1 K$ is $VPC(n+1)$. For the double of an essential annulus, this is proved in section 2 of [20]. Lemma 8.7 of [20] tells us that $(DK, \partial DK)$ is an orientable atoroidal $PD(n+2)$ pair.

Now we proceed as follows. The group A in $\partial_1 K$ yields the group A' in ∂DK , where A' equals A if A is a torus, and equals the double DA of A if A is an annulus. Similarly the group B in $\partial_1 K$ yields the group B' in ∂DK . The $VPCn$ subgroups S and T of A and B are subgroups of A' and B' respectively, and the element g of K such that $gSg^{-1} = T$ lies in DK . Now we apply Proposition 3.2.4 to obtain one of the three cases listed there. Case 1) of Proposition 3.2.4 implies that case 1) of Proposition 3.2.5 holds, and case 2) of Proposition 3.2.4 implies that case 2) of Proposition 3.2.5 holds. Finally case 3) of Proposition 3.2.4 implies that either V is of special Seifert type or of special solid torus type. Neither case can occur as such vertices cannot be V_1 -vertices of $\Gamma_{n,n+1}^c(G)$. This completes the proof of Proposition 3.2.5 \square

3.3 Examples of almost invariant sets

The discussion in [20] was mostly about almost invariant sets which are adapted to the boundary. However, in [16], Scott gave examples of almost invariant sets over orientable $VPC2$ subgroups of a $PD3$ pair which are not adapted to the boundary. This gave rise to the concept of special canonical torus which was used to show that the JSJ-decomposition of orientable 3-manifolds is algebraic, meaning that it depends only on the fundamental group of the manifold, not the boundary. As discussed earlier, we will say that an embedded essential annulus or torus in a 3-manifold M with incompressible boundary is *topologically canonical* if it has intersection number zero with any (possibly singular) essential annulus or torus in $(M, \partial M)$. We will say that a splitting of $\pi_1(M)$ given by an essential annulus or torus is *algebraically canonical* if it has intersection number zero with any almost invariant subset of $\pi_1(M)$ which is over \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}$. See [15] for a discussion of the idea of intersection numbers. Now we recall Scott's example.

Example 3.3.1 (Scott's example). This is Example 2.13 of [16]. Let F be an orientable surface with at least two boundary components and let C denote one of the boundary components. Thus $\pi_1(F)$ is free, and $\pi_1(C)$ is a free factor of $\pi_1(F)$. If the rank of $\pi_1(F)$ is at least 3, then it is easy to see that there is a nontrivial splitting of $\pi_1(F)$ as an amalgamated free product over $\pi_1(C)$. Similar considerations apply to express $\pi_1(F)$ as an HNN extension if it has rank 2.

We now take two copies F_1, F_2 of F and consider the two 3-manifolds $M_i = F_i \times S^1$, each with a boundary component T_i corresponding to $C_i \times S^1$. Form a 3-manifold M by gluing the M_i 's along T_i so that the fibrations do not match. The resulting torus T is a topologically canonical torus in the JSJ splitting of M . If each $\pi_1(F_i)$ has rank at least 3, we have $\pi_1(M_i) = A_i *_{H_i} B_i, i = 1, 2$, where $H_i = \pi_1(T_i)$. If G denotes $\pi_1(M)$, and H denotes the subgroup $H_1 = H_2$, and $A = A_1 *_{H} A_2, B = B_1 *_{H} B_2$, we have a splitting $G = A *_{H} B$ of G that crosses the splitting associated to T . Thus although T is topologically canonical, it is not algebraically canonical. Notice that embedded essential annuli in M_1 and M_2 , disjoint from T , yield splittings of G over the fibres of M_1 and M_2 , so that G also has splittings over incommensurable cyclic subgroups of H .

We construct some more examples. Consider $M_i = T_i \times I \cup N_i$, where N_i is an orientable 3-manifold attached to $T_i \times \{1\}$ along at least two disjoint annuli in ∂N_i . Now form M by identifying the M_i 's along $T_i \times \{0\}$, and let T denote the torus $T_1 \times \{0\} = T_2 \times \{0\}$. Assume that the annuli in $T_1 \times \{1\}$ and $T_2 \times \{1\}$ used to construct M_1 and M_2 carry incommensurable subgroups of $\pi_1(T) = H$. Thus $G = \pi_1(M)$ splits over incommensurable cyclic subgroups of H . Again, T is a topologically canonical torus in the JSJ decomposition of M . Now we form H -almost invariant subsets of $G = \pi_1(M)$ as follows. Consider the cover \widetilde{M}_H of M with $\pi_1(\widetilde{M}_H) = H$, so that each $T_i \times I$ lifts to \widetilde{M}_H , and the pre-image \widetilde{N}_i of each N_i is disconnected. Let $C_i, i = 1, 2$, be one of the annuli in this lift of $T_i \times \{1\}$ used to construct M_i , and let N'_i be the component of \widetilde{N}_i attached to C_i . For $i = 1, 2$, let X_i be the set of vertices of \widetilde{M}_H in $T_i \times [0, 1]$ together with the vertices in N'_i . This gives us two sets X_1, X_2 on different sides of T . To X_1 we add the vertices in $\widetilde{N}_2 - N'_2$ and to X_2 we add the vertices in $\widetilde{N}_1 - N'_1$ to obtain two H -almost invariant sets Y_1, Y_2 . Clearly $Y_2 = Y_1^*$ and crosses the almost invariant set X determined by T , namely X consists of all vertices of \widetilde{M}_H on one side of T .

We can mix the above two examples to get one manifold of each type on each side of T . These types fall under the heading of V_0 -vertices of commensuriser

type of Theorem [3.2.1](#). The first examples are special cases of the so called peripheral Seifert type in [\[20\]](#), that is, those Seifert type pieces in the decompositions $\Gamma_{n,n+1}$ and $\Gamma_{n,n+1}^c$ of a $PD(n+2)$ pair $(G, \partial G)$ which "intersect the boundary". The second are called toral type 2), in which one component of $\partial(T \times I)$ is an edge of the decomposition and the other boundary component intersects ∂G in a number of parallel annuli. Note that in [\[20\]](#), the definition of Seifert type requires the base orbifold to have fundamental group which is not virtually cyclic. The terminology torus type is used if this base group is virtually cyclic. These examples suggest the following definition.

Definition 3.3.2. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, such that G is not VPC . A splitting of G over a $VPC(n+1)$ subgroup H is called a **special canonical torus** if it has intersection number zero with any essential annulus in $(G, \partial G)$, and G splits over incommensurable $VPCn$ subgroups of H .*

Remark 3.3.3. *In Proposition [3.3.5](#), we will prove that a special canonical torus in G is an edge splitting of $\Gamma_{n,n+1}(G)$ or $\Gamma_{n,n+1}^c(G)$, so that this concept depends only on G , and not on ∂G . Recall that the edge splittings of $\Gamma_{n,n+1}(G)$ are the same as those of $\Gamma_{n,n+1}^c(G)$. As these edge splittings are each dual to an essential annulus or torus in $(G, \partial G)$, it follows that a special canonical torus in G is dual to an essential torus in $(G, \partial G)$, which partly justifies the terminology. Note that if ∂G is empty, so that G is a $PD(n+2)$ group, then G cannot split over a $VPCn$ subgroup. Thus special canonical tori do not exist in this case.*

Essentially the same definition was used in [\[16\]](#). However, the above examples are not the only possibilities. Again let M_1 denote the 3-manifold $F_1 \times S^1$, where F_1 is an orientable surface with at least two boundary components. Also let M_2 denote the orientable 3-manifold which is a twisted I -bundle over the Klein bottle. Form a 3-manifold M by gluing the boundary torus T of M_2 to one of the boundary tori of M_1 . In this case there are two distinct Seifert fibrations on M_2 , reflecting the fact that the Klein bottle itself has two distinct Seifert fibrations. But so long as the gluing is chosen not to match the fibration of M_1 with either fibration of M_2 , the torus T will again be a topologically canonical torus in the JSJ splitting of M . It will also not be algebraically canonical but will be a special canonical torus. The easiest way to see these facts is to note that M is double covered by the union of two copies of M_1 , glued along a boundary torus so that their fibrations do not match. Thus M is double covered by one of Scott's examples.

Next we need the following technical result.

Lemma 3.3.4. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , let J and J' be $VPC(n+1)$ subgroups of G , and let f and f' be edges of the universal covering G -tree T of $\Gamma_{n,n+1}^c(G)$, with stabilizers J and J' respectively. If J and J' are commensurable, then $J = J'$ and one of the following cases holds:*

1. $f = f'$.
2. $f \cap f'$ is an isolated vertex.
3. $f \cap f'$ is a V_0 -vertex w of valence 2 whose stabiliser contains an element interchanging f and f' , and so contains J with index 2.
4. There are consecutive adjacent edges f, b, b', f' of T such that $b \cap f$ and $b' \cap f'$ are isolated V_1 -vertices, and $b \cap b'$ is a V_0 -vertex w of valence 2 whose stabiliser contains an element interchanging b and b' , and so contains J with index 2.

Proof. As above, we will start with a $K(\pi, 1)$ pair $(M, \partial M)$ and a decomposition of $(M, \partial M)$ mimicking the decomposition $\Gamma_{n,n+1}^c$. If $f = f'$, we have case 1) of the lemma, so for the rest of the proof we will assume that $f \neq f'$.

The edges f and f' determine splittings of G over J and J' , so that these are tori in $(G, \partial G)$. Let L denote the intersection $J \cap J'$, so that L is also $VPC(n+1)$, and let Σ denote a torus with fundamental group L . Consider the $PD(n+2)$ pair $(K, \partial K)$ obtained by cutting $(G, \partial G)$ along these two splittings, and let $(N, \partial N)$ be obtained from M in the corresponding way. (It is possible that f and f' yield a single splitting.) The path in T between f and f' determines (up to homotopy) a map $F : (\Sigma \times I, \Sigma \times \partial I) \rightarrow (N, \partial N)$. We let N_0 denote the component of N which contains the image of F , and consider the induced map $(\Sigma \times I, \Sigma \times \partial I) \rightarrow (N_0, \partial N_0)$ which we continue to denote by F . The degree of F on each component of $\Sigma \times \partial I$ is non-zero, and if $F(\Sigma \times \partial I)$ is contained in a single component of ∂N_0 , these degrees add. Thus the degree of F is non-zero. It follows that $(N_0, \partial N_0)$ has a finite cover to which F lifts by a map which is an isomorphism of fundamental groups. In particular, it follows that the boundary of this cover consists of two tori with fundamental groups equal to L . Hence either ∂N_0 consists of two tori with the same fundamental group as N_0 , or ∂N_0 consists of a single torus with fundamental group J , and J has index 2 in $K = \pi_1(N_0)$. In either case, it follows that $J = J'$. In the first case, the path λ joining f and f' in T has stabilizer J , and each vertex on that path must be isolated. As $\Gamma_{n,n+1}^c(G)$ is reduced, we must have case 2) of the lemma. In the second case, the

stabilizer of λ contains J with index 2, and contains a reflection. Thus there is a vertex w of λ of valence 2 whose stabiliser contains an element interchanging the two incident edges, and so contains J with index 2. Further, as $\Gamma_{n,n+1}^c(G)$ is reduced, either w equals $f \cap f'$ or there are consecutive adjacent edges f, b, b', f' of T such that $b \cap f$ and $b' \cap f'$ are isolated vertices, and w equals $b \cap b'$. In either case, this implies that the image of w in $\Gamma_{n,n+1}^c(G)$ is of special Seifert type, and so is a V_0 -vertex. Thus we must have cases 3) or 4) of the lemma. \square

Now we can give an alternative description of special canonical tori in terms of $\Gamma_{n,n+1}^c(G)$.

Proposition 3.3.5. *For an orientable $PD(n+2)$ pair $(G, \partial G)$ such that G is not VPC , a splitting α of G over a $VPC(n+1)$ subgroup H is a special canonical torus if and only if the following conditions hold:*

1. α is an edge splitting of $\Gamma_{n,n+1}^c(G)$.
2. The V_1 -vertex w of α is isolated.
3. Each of the V_0 -vertices adjacent to w is of peripheral Seifert type, of toral type 2), or of special Seifert type.
4. At most one V_0 -vertex can be of special Seifert type.
5. If the two edges incident to w form a loop, there is only one adjacent V_0 -vertex. In this case, that vertex must be of peripheral Seifert type.

(The concepts of peripheral Seifert type, and toral type 2) are discussed immediately preceding Definition [3.3.2](#), and “special Seifert type” was defined before Theorem [3.2.1](#). The reader is referred to [\[20\]](#) for full details.)

Proof. First suppose that α is a splitting of G over a $VPC(n+1)$ subgroup H which satisfies conditions 1)-5) of the Proposition. As α is an edge splitting of $\Gamma_{n,n+1}^c(G)$, it has intersection number zero with any essential annulus in $(G, \partial G)$. It remains to show that G splits over incommensurable $VPCn$ subgroups of H . By condition 2), the V_1 -vertex w of α is isolated. Suppose there is no V_0 -vertex adjacent to w of special Seifert type. Then each of the adjacent V_0 -vertices meets ∂G in annuli (and/or tori in the case of peripheral Seifert type) and choosing an essential embedded annulus in the V_0 -vertex and with boundary in ∂G determines a splitting of G over the $VPCn$ subgroup of H carried by the fibres. These

subgroups of H must be incommensurable, as otherwise there would be an annulus in $(G, \partial G)$ which crosses α , contradicting the fact that any edge splitting of $\Gamma_{n,n+1}^c$ crosses no annulus in $(G, \partial G)$. Thus α is a special canonical torus in this case. Next suppose there is a V_0 -vertex adjacent to w of special Seifert type. Denote its associated group by \overline{H} , and recall that H is a subgroup of index 2 in \overline{H} . As before, choosing an essential embedded annulus in the other V_0 -vertex with boundary in ∂G determines a splitting of G over the $VPCn$ subgroup L of H carried by the fibres. Now let g denote an element of $\overline{H} - H$. We also have a splitting of G over L^g . As g normalises H , this is also a subgroup of H . Finally L and L^g must be incommensurable subgroups of H , for otherwise there would be an annulus in $(G, \partial G)$ which crosses α , contradicting the fact that α is an edge splitting of $\Gamma_{n,n+1}^c$. Thus again α is a special canonical torus.

Now suppose that α is a special canonical torus. By Lemma 6.2 of [20], the fact that α has intersection number zero with any essential annulus in $(G, \partial G)$ implies that α is adapted to ∂G . Thus the splitting α is dual to a torus in $(G, \partial G)$, and so must be enclosed by some V_0 -vertex of $\Gamma_{n,n+1}^c(G)$. As G splits over a $VPCn$ subgroup $L \subset H$, the pair $(G, \partial G)$ admits an essential annulus with group L , and any such annulus must be enclosed by a V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$, of commensuriser type, so that $G(v)$ contains H . Thus α is enclosed by v . As α has intersection number zero with any essential annulus in $(G, \partial G)$, it follows that α must be the splitting of G associated to an edge of $\Gamma_{n,n+1}^c(G)$ incident to v . This proves that α satisfies condition 1) of the proposition. Further v must be of peripheral Seifert type or of toral type 2). Now we use the hypothesis that G splits over two incommensurable $VPCn$ subgroups L and L' of H . So the pair $(G, \partial G)$ admits an essential annulus with group L' , which is enclosed by a V_0 -vertex v' of $\Gamma_{n,n+1}^c(G)$ which also has an incident edge with splitting α . Further v' must be of peripheral Seifert type or of toral type 2).

If v and v' are distinct, the incident edges with splitting α must also be distinct. As neither of v and v' is isolated or of special Seifert type, Lemma 3.3.4 implies that v and v' must be separated by an isolated V_1 -vertex, so that α satisfies conditions 2)-5) of the Proposition, as required.

If v and v' coincide, there must be $g \in G$ such that $L' = L^g$. There are now two cases, depending on whether or not there are two distinct edges incident to v with associated splitting α . Again we will apply Lemma 3.3.4 and use the fact that v is not isolated nor of special Seifert type. If there are two distinct such edges, Lemma 3.3.4 implies that they meet in an isolated V_1 -vertex. In addition, v cannot be of torus type 2), as such a V_0 -vertex can have at most one incident edge dual to a torus. It follows that α satisfies conditions 2)-5) of the Proposition.

If there is only one such edge, Lemma 3.3.4 implies that there is a V_0 -vertex v'' of special Seifert type which is adjacent to v and separated from v by an isolated V_1 -vertex w . Again this implies that α satisfies conditions 2)-5) of the Proposition, as required. \square

We can now apply Proposition 3.3.5 to obtain the following result.

Proposition 3.3.6. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC. Let α be a special canonical torus in $(G, \partial G)$ with group H . Then α crosses some almost invariant subset of G over a subgroup of finite index in H .*

Proof. From the definition of a special canonical torus, G splits over incommensurable subgroups L and L' of H . Let X be the H -almost invariant subset of G determined (up to equivalence and complementation) by α . We will apply Conditions 1)-5) of Proposition 3.3.5. Thus α is an edge splitting of $\Gamma_{n,n+1}^c(G)$, and the V_1 -vertex w of α is isolated.

If there are two distinct V_0 -vertices v and v' adjacent to w , neither of special Seifert type, we can assume that the splitting of G over L is enclosed by v , and that the splitting of G over L' is enclosed by v' . Thus there is an edge of $\Gamma_{n,n+1}^c(G)$ incident to v with associated splitting dual to an annulus with group L , and there is an edge of $\Gamma_{n,n+1}^c(G)$ incident to v' with associated splitting dual to an annulus with group L' . Let Y denote the L -almost invariant subset of G determined by the edge splitting of G over L , and let Y' denote the L' -almost invariant subset of G determined by the edge splitting of G over L' . By replacing each of X , Y and Y' by its complement if needed, we can arrange that $Y \subset X$ and $Y' \subset X^*$. Then $H(Y \cup Y')$ is a H -almost invariant subset of G . This subset crosses X , and hence crosses α , unless we are in one of the exceptional cases where $HY = X$, or $HY' = X^*$. Now for any $h \in H$, the set hY is equal to or disjoint from Y . Thus if P is a proper subgroup of finite index in H which contains L , then PY and $X - PY$ are both H -infinite. Similarly if P' is a proper subgroup of finite index in H which contains L' , then $P'Y'$ and $X^* - P'Y'$ are both H -infinite. Thus, if Q denotes $P \cap P'$, then Q has finite index in H , and $Q(Y \cup Y')$ is a Q -almost invariant subset of G , which crosses X , and hence crosses α , as required.

If there are two distinct V_0 -vertices z and z' adjacent to w , and if z' is of special Seifert type, there is a homomorphism $G(z') \rightarrow \mathbb{Z}_2$, with kernel H . This extends to a homomorphism $G \rightarrow \mathbb{Z}_2$, which is trivial on all vertex groups other than $G(z')$. Let K denote the kernel of this homomorphism, so that K is of index 2 in G . This naturally has the structure of a $PD(n+2)$ pair, and there is a natural map $\Gamma_{n,n+1}^c(K) \rightarrow \Gamma_{n,n+1}^c(G)$. The pre-image of z' is an isolated

V_1 -vertex of $\Gamma_{n,n+1}^c(K)$. The adjacent V_0 -vertices consist of two copies of z . Hence the preceding paragraph yields a subgroup Q of finite index in H , and a Q -almost invariant subset of K which crosses α . It follows that there is also a Q -almost invariant subset of G which crosses α , as required.

If the two edges incident to w form a loop, so there is only one adjacent V_0 -vertex z , Condition 5) tells us that z must be of peripheral Seifert type. This loop determines a natural map from G to \mathbb{Z} , which is trivial on all vertex groups, and hence determines a natural map from G to \mathbb{Z}_2 , which is trivial on all vertex groups. The kernel is a subgroup K of G of index 2 which is still naturally a $PD(n+2)$ pair, and again there is an edge splitting over H of $\Gamma_{n,n+1}^c(K)$ whose V_1 -vertex is isolated. The adjacent V_0 -vertices now consist of two copies of z . As before, this yields a subgroup Q of finite index in H , and a Q -almost invariant subset of G which crosses α , as required. \square

We note that in Example [3.3.1](#), the special canonical torus with group H crosses a splitting over the same group H . Thus it seems reasonable to ask the following.

Problem 3.3.7. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . Let α be a special canonical torus in $(G, \partial G)$ with group H . When is it true that α crosses a splitting of G over H ?*

Proposition [3.3.6](#) does nothing to answer this question. But the argument does show that in most cases, a special canonical torus with group H crosses an almost invariant set over the same group H . For the special case in the argument when $HY = X$ can only occur if v is of torus type 2) and also has only one incident edge with associated splitting dual to an annulus. A similar statement for v' holds if $HY' = X^*$. We believe that the above problem has a positive answer except possibly in these exceptional cases. In the exceptional cases, it seems possible that a special canonical torus with group H may not cross any H -almost invariant subset of G .

Here is a construction which shows that in many cases, a special canonical torus α in $(G, \partial G)$ with group H crosses a splitting of G over H . Suppose that there are two distinct V_0 -vertices v and v' adjacent to w , and that each of v and v' has at least two incident edges in addition to the edge joining it to w . Let e be an edge of $\Gamma_{n,n+1}^c(G)$ incident to v with associated splitting dual to an annulus with group L , and let e' be an edge of $\Gamma_{n,n+1}^c(G)$ incident to v' with associated splitting dual to an annulus with group L' . Now construct a new graph of groups structure for G by sliding the end of e at v along the two edges joining v to v' , and

also sliding the end of e' at v' along the two edges joining v' to v . Let β denote the splitting of G determined by each of the two edges joining v and v' to w . Clearly β is a splitting of G over H , and by considering the universal covering G -trees of $\Gamma_{n,n+1}^c(G)$ and of the new graph of groups, it is easy to see that it must cross α .

3.4 Proof of the main result

In this section, we prove the main theorem below and then a result about enclosings of all almost invariant sets over $VPC(n+1)$ groups. First we need to introduce yet another version of the term "canonical", which generalizes the term "algebraically canonical" discussed in the introduction. Let $\mathcal{E}_{n,n+1}(G)$ denote the collection of all a.i. subsets of G which are over a $VPCn$ or $VPC(n+1)$ subgroup. We will say that an element of $\mathcal{E}_{n,n+1}$ is *canonical* if it has intersection number zero with every element of $\mathcal{E}_{n,n+1}$.

Theorem 3.4.1 (Main result). *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . The edge splittings of $\Gamma_{n,n+1}(G)$ and of $\Gamma_{n,n+1}^c(G)$ are either canonical or are special canonical tori.*

Remark 3.4.2. *Proposition [3.3.6](#) shows that these two conditions are mutually exclusive. Note that if ∂G is empty, the decomposition $\Gamma_{n+1}(G)$ is an algebraic regular neighbourhood of all almost invariant subsets of G over a $VPC(n+1)$ subgroup, so that all the edge splittings of $\Gamma_{n+1}(G)$ and of $\Gamma_{n+1}^c(G)$ are canonical, by definition.*

We start by setting up some notation. The main step of the start of the argument is discussed in Section 6 of [\[20\]](#) in a different context. There the authors used it to show that n -canonical almost invariant sets over $VPC(n+1)$ groups are automatically adapted to the boundary. Here we use it differently.

Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair, and let T denote the universal covering G -tree of the graph of groups $\Gamma_{n,n+1}^c(G)$. Let $(M, \partial M)$ be a $K(\pi, 1)$ pair with a decomposition mimicking the decomposition $\Gamma_{n,n+1}^c$. This induces a decomposition of the universal cover $(\widetilde{M}, \partial\widetilde{M})$ of $(M, \partial M)$, and we have an equivariant map $\widetilde{M} \rightarrow T$ preserving the decompositions. If v is a vertex of $\Gamma_{n,n+1}^c$ or of T , the corresponding subspaces of M or of \widetilde{M} will be denoted by M_v or \widetilde{M}_v respectively, and similarly for edges.

Lemma 3.4.3. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let e be an edge of T such that the associated splitting of G is not canonical. Then there is an almost invariant set X over a $VPC(n+1)$ subgroup H of G which is not adapted to ∂G and crosses e .*

Proof. Recall that $\Gamma_{n,n+1}^c(G)$ is the completion of the reduced algebraic regular neighbourhood $\Gamma_{n,n+1}(G)$ of $\mathcal{F}_{n,n+1}$ in G , where $\mathcal{F}_{n,n+1}$ denotes the family of equivalence classes of all nontrivial almost invariant subsets of G which are over a $VPCn$ subgroup, together with the equivalence classes of all n -canonical almost invariant subsets of G which are over a $VPC(n+1)$ subgroup. In [20], the authors showed that n -canonical almost invariant subsets of G over $VPC(n+1)$ subgroups are automatically adapted to ∂G . Further, if we enlarge the family $\mathcal{F}_{n,n+1}$ to include all almost invariant sets over $VPC(n+1)$ subgroups which are adapted to ∂G , the new family $\mathcal{G}_{n,n+1}$ has the same regular neighbourhood $\Gamma_{n,n+1}(G)$. In particular, no set in $\mathcal{G}_{n,n+1}$ can cross any edge of $\Gamma_{n,n+1}^c(G)$. Thus our assumption on e implies that there must be an almost invariant set X over a $VPC(n+1)$ subgroup H of G which is not adapted to ∂G and crosses e , as required. \square

Lemma 3.4.4. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let X be an almost invariant set over a $VPC(n+1)$ subgroup H of G which is not adapted to ∂G . Then the following statements hold:*

1. *There is a group S with a conjugate in ∂G such that $L = H \cap S$ is $VPCn$, and $X \cap S$ and $X^* \cap S$ are both H -infinite.*
2. *There is a V_0 -vertex v of T of commensuriser type which encloses all L -almost invariant subsets of G . Further, $\widetilde{M}_v \cap \partial \widetilde{M}$ is non-empty, v is of Seifert type or of torus type, and $G(v) = \text{Comm}_G(L) = N_G(L)$ contains H .*

Proof. 1) Since X is not adapted to ∂G , there is a group S in ∂G , and $g \in G$ such that $X \cap gS$ and $X^* \cap gS$ are both H -infinite. By replacing S by a conjugate if needed, we can arrange that $X \cap S$ and $X^* \cap S$ are both H -infinite. Consider the component Σ of $\partial \widetilde{M}$ with stabilizer S , and identify X and X^* with subsets of the 0-skeleton \widetilde{M}_0 of \widetilde{M} . We have that the intersections of X and X^* with the 0-skeleton Σ_0 of Σ are H -infinite. Hence they are L -infinite, where $L = H \cap S$. Thus $e(S, L) \geq 2$. As S is $PD(n+1)$ and H is VPC , it follows that L is $VPCn$.

2) By replacing H by a subgroup of finite index if necessary, we can assume that L is normal in H with $L \setminus H$ infinite cyclic. Let $P_L : \widetilde{M} \rightarrow M_L$, and $P_H :$

$\widetilde{M} \rightarrow M_H$ denote the covering projections, and let Σ_L and Σ_H denote the images of Σ in ∂M_L and ∂M_H respectively. As $L \setminus H$ acts on M_L , there are infinitely many translates of Σ_L each with fundamental group L , and thus infinitely many essential annuli in M_L . In [20], it was shown that, if the number of essential annuli in M_L is at least 4, there is a V_0 -vertex v of T of commensuriser type which encloses all L -almost invariant subsets of G . Hence the stabilizer, $G(v)$, of v contains H , and \widetilde{M}_v intersects $\partial \widetilde{M}$. It is possible that $\widetilde{M}_v \cap \partial \widetilde{M}$ contains Σ . This happens if S is $VPC(n+1)$. In any case, v is a V_0 -vertex with $H \subset G(v)$, and $\widetilde{M}_v \cap \partial \widetilde{M}$ is non-empty, and L stabilizes $\widetilde{M}_v \cap \partial \widetilde{M}$. Thus v is either of Seifert type or of toral type (see Definition 3.12 of [20]). If v is of Seifert type, Lemma 5.10 of [20] tells us that $G(v) = \text{Comm}_G(L) = N_G(L)$. If v is of toral type, we use the fact that $G(v)$ is $VPC(n+1)$ and splits over L . Now Lemma 1.10 of [20] implies that L is normal in $G(v)$ with quotient \mathbb{Z} or $\mathbb{Z}_2 * \mathbb{Z}_2$. It follows that in this case also $G(v) = \text{Comm}_G(L) = N_G(L)$. \square

Note that if X crosses an edge e of T , Lemma 3.4.4 does not tell us that e is incident to the vertex v of T obtained in part 2) of the above lemma. However the next lemma assures us that X must cross some edge incident to v .

Lemma 3.4.5. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , let T denote the universal covering G -tree of the graph of groups $\Gamma_{n,n+1}^c(G)$, and let e be an edge of T . Let X be an almost invariant set over a $VPC(n+1)$ subgroup H of G which is not adapted to ∂G and crosses e .*

Using the notation of Lemma 3.4.4, if f is the first edge on the path in T from v to e , then $G(f)$ is $VPC(n+1)$, so the associated splitting of G is dual to a torus, and X crosses this torus.

Remark 3.4.6. *If v is of torus type, the fact that an edge incident to v determines a splitting of G dual to a torus means that v is of torus type 2).*

Proof. Let Z and Z^* denote the almost invariant sets associated to e chosen so that Z contains $G(v)$, and let Y and Y^* denote the almost invariant sets associated to f chosen so that $Y \subset Z$ and $Y^* \supset Z^*$. As X crosses Z , we know that $X \cap Z^*$ and $X^* \cap Z^*$ are both H -infinite. As $Y^* \supset Z^*$, it follows that $X \cap Y^*$ and $X^* \cap Y^*$ are also both H -infinite.

Now suppose that $G(f)$ is $VPCn$. As v is of Seifert type or of toral type, the edge group $G(f)$ must be commensurable with L . Thus δY is L -finite. As $H \subset G(v)$, and δX is H -finite, it follows that δX lies in a bounded neighbourhood of M_v . As $G(f)$ is commensurable with L , this implies that $\delta X \cap Y^*$ is L -finite.

As δY is also L -finite, it follows that $X \cap Y^*$ and $X^* \cap Y^*$ each have L -finite coboundary. For $\delta(X \cap Y^*) = (X \cap \delta Y^*) \cup (\delta X \cap Y^*)$. We conclude that each of $X \cap Y^*$ and $X^* \cap Y^*$ is a nontrivial L -almost invariant subset of G contained in Y^* . In particular, they cannot be enclosed by v , which is a contradiction. This contradiction shows that $G(f)$ must be $VPC(n+1)$, so that the associated splitting of G is dual to a torus, as required. It remains to show that X crosses this torus.

As $G(f)$ determines a torus, it follows that the component Σ of $\partial \widetilde{M}$ with stabilizer S cannot meet this torus, and so must lie on the same side of \widetilde{M}_f as does \widetilde{M}_v . In particular, $S \subset Y$. As $X \cap S$ and $X^* \cap S$ are both H -infinite, it follows that $X \cap Y$ and $X^* \cap Y$ are both H -infinite. Hence all four corners of the pair (X, Y) are H -infinite, so that X crosses Y , as required. \square

Next we will show that H and $G(f)$ must be commensurable subgroups of G .

We split G along the torus $G(f)$ to obtain a new $PD(n+2)$ pair $(G', \partial G')$ (by Theorem 8.1 of [1]) with $(G', \partial G')$ containing $G(w)$, where v and w are the vertices of f . Correspondingly, M is split along M_f to obtain a new space N containing M_w . Thus, \widetilde{M} is split along \widetilde{M}_f and its translates to obtain a new space \widetilde{N} containing \widetilde{M}_w . In particular, the boundary of \widetilde{N} consists of boundary components of \widetilde{M} together with translates of \widetilde{M}_f .

Lemma 3.4.7. *Using the notation of Lemma 3.4.5, suppose that H and $G(f)$ are not commensurable, and let L' denote $H \cap G(f)$. Then L' is $VPCn$ and contains L with finite index, and there is an essential annulus in N , carrying L' , which lifts to an annulus A in $P_{L'}(\widetilde{N})$ from $P_{L'}(\widetilde{M}_f)$ to a component of $P_{L'}(\partial \widetilde{N})$.*

Proof. Note that part 2) of Lemma 3.4.4 tells us that $G(v) = \text{Comm}_G(L) = N_G(L)$ contains H . As v is of Seifert type or of torus type, and $G(f)$ is a boundary torus of $G(v)$, it follows that $G(f)$ also contains L . In particular, the intersection $H \cap G(f)$ contains L . As H and $G(f)$ are not commensurable, it follows that $H \cap G(f) = L'$ is $VPCn$ and contains L with finite index.

As in the proof of Lemma 3.4.5, we consider the intersections $X \cap Y^*$ and $X^* \cap Y^*$. Both sets are invariant under $H \cap G(f) = L'$. Again we know that $\delta X \cap Y^*$ must be L -finite. Suppose that $X \cap \delta Y$ is L -finite. Then $X \cap Y^*$ has L -finite coboundary and so $X \cap Y^*$ is a nontrivial L -almost invariant set which is not enclosed by v , which is again a contradiction. Thus $X \cap \delta Y$ must be L -infinite, and similarly $X^* \cap \delta Y$ must be L -infinite. Note that the intersections of

$X \cap Y^*$ and $X^* \cap Y^*$ with the 0-skeleton of \tilde{N} have coboundaries (in \tilde{N}) equal to $\delta X \cap Y^*$ and $\delta X^* \cap Y^*$ respectively, each of which is L -finite. As $X \cap Y^*$ and $X^* \cap Y^*$ contain $X \cap \delta Y^*$ and $X^* \cap \delta Y^*$ respectively which are both L' -infinite, it follows that each determines a nontrivial L' -almost invariant subset of G' . Hence there are essential annuli in N , carrying L' , one of which lifts to an annulus A in $P_{L'}(\tilde{N})$ from $P_{L'}(\tilde{M}_f)$ to a component of $P_{L'}(\partial\tilde{N})$, as required. \square

Lemma 3.4.8. *Using the notation of Lemma 3.4.5, the subgroups H and $G(f)$ of G are commensurable.*

Proof. Suppose that H and $G(f)$ are not commensurable, and let $L' = H \cap G(f)$. By Lemma 3.4.7 there is an essential annulus A in $P_{L'}(\tilde{N})$ from $P_{L'}(\tilde{M}_f)$ to a component $\Sigma_{L'}$ of $P_{L'}(\partial\tilde{N})$.

As A is essential, $P_{L'}(\tilde{M}_f)$ and $\Sigma_{L'}$ must be distinct components of $P_{L'}(\partial\tilde{N})$. Recall that \tilde{N} contains \tilde{M}_w , where w is the V_1 -vertex of the edge f . It follows that A has a sub-annulus A' which lies in $P_{L'}(\tilde{M}_w)$, and joins distinct boundary components. Thus the vertex w has an incident edge g , distinct from f , such that $G(g)$ contains L' . As $G(g)$ is an edge group of $\Gamma_{n,n+1}^c(G)$, it must be $VPCn$ or $VPC(n+1)$. In either case, we apply Proposition 3.2.5. Note that T is the universal covering G -tree of the graph of groups $\Gamma_{n,n+1}^c(G)$, not of $\Gamma_{n,n+1}(G)$, so this proposition is applicable. As f and g are distinct, case 1) of the conclusion is not possible. It follows that w is an isolated vertex of $\Gamma_{n,n+1}^c(G)$. In particular, $G(g) = G(f)$ is $VPC(n+1)$. Let v' denote the V_0 -vertex at the other end of the edge g . The part A_1 of A in $M_{v'}$ is an essential annulus carrying L' in the pair $(G(v'), \partial G(v'))$. We need to recall from Theorem 3.2.1 the possible types of V_0 -vertex of $\Gamma_{n,n+1}^c(G)$.

If v' is isolated, this would contradict the fact that $\Gamma_{n,n+1}^c(G)$ is reduced.

If v' is of $VPC(n-1)$ -by-Fuchsian type, and is of I -bundle type, then each edge splitting for edges incident to v' would be dual to an annulus. As the edge splitting dual to g is a torus, this case cannot occur.

If v' is of $VPCn$ -by-Fuchsian type, and is of interior Seifert type, then the essential annulus A_1 in $(G(v'), \partial G(v'))$ projects to an annulus in the base 2-orbifold. As v' is of $VPCn$ -by-Fuchsian type, the fundamental group of this base orbifold is not virtually cyclic. It follows that it does not admit an essential annulus. We conclude that this projected annulus is inessential, which implies that L' is commensurable with the $VPCn$ fibre group of $G(v')$. But this means that $G(v')$ commensurises L' , and hence commensurises L , which is again a contradiction as $G(v) = Comm_G(L)$.

Finally, if v' is of commensuriser type, the existence of the essential annulus A_1 in $(G(v'), \partial G(v'))$ implies that $G(v')$ commensurises L' , and hence commensurises L , which is again a contradiction as $G(v) = \text{Comm}_G(L)$.

We have shown that all cases lead to a contradiction so that H and $G(f)$ must be commensurable, as required. \square

Combining the preceding lemmas and Remark [3.4.6](#), we have proved the following.

Lemma 3.4.9. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , let T denote the universal covering G -tree of the graph of groups $\Gamma_{n,n+1}^c(G)$, and let e be an edge of T such that the associated splitting of G is not canonical. Then the following statements hold:*

1. *There is an almost invariant set X over a $VPC(n+1)$ subgroup H of G which is not adapted to ∂G and crosses e .*
2. *There is a group S with a conjugate in ∂G such that $L = H \cap S$ is $VPCn$, and $X \cap S$ and $X^* \cap S$ are both H -infinite.*
3. *There is a V_0 -vertex v of T of commensuriser type which encloses all L -almost invariant subsets of G . Further, $\widetilde{M}_v \cap \partial \widetilde{M}$ is non-empty, v is of Seifert type or of toral type 2), and $G(v) = \text{Comm}_G(L) = N_G(L)$ contains H .*
4. *If f is the first edge on the path from v to e , then $G(f)$ is $VPC(n+1)$ and commensurable with H , and X crosses the torus splitting given by f .*

Now we can complete the proof of Theorem [3.4.1](#), that the edge splittings of $\Gamma_{n,n+1}(G)$ and of $\Gamma_{n,n+1}^c(G)$ are either canonical or are special canonical tori.

Proof. Let e be an edge of T such that the associated splitting of G is not canonical, and apply Lemma [3.4.9](#). Let Y and Y^* denote the almost invariant sets associated to f , chosen so that $G(v) \subset Y$.

As H and $G(f)$ are commensurable, and X crosses Y , the intersections $X \cap Y^*$ and $X^* \cap Y^*$ are nontrivial almost invariant sets over the $VPC(n+1)$ group $H' = H \cap G(f)$.

As H' is a torus in $(G, \partial G)$, up to equivalence, we have only two H' -almost invariant sets which are adapted to ∂G , namely Y and Y^* . Thus neither of $X \cap Y^*$ and $X^* \cap Y^*$ is adapted to ∂G . Let Z denote $X \cap Y^*$. Then Lemma [3.4.4](#) tells us that there is a group S' in ∂G such that $Z \cap S'$ and $Z^* \cap S'$ are both H' -infinite,

and $H' \cap S'$ is a VPC_n group K . Further there is a V_0 -vertex v' of T , which encloses all K -almost invariant subsets of G so that $G(v')$ contains H' , and $\widetilde{M}_{v'}$ intersects $\partial\widetilde{M}$.

If Z crosses some edge e' of T , Lemma 3.4.5 tells us that if f' is the first edge on the path from v' to e' , then $G(f')$ is $VPC(n+1)$ and commensurable with H' , and Z crosses the torus splitting given by f' . In particular, f and f' have commensurable stabilizers. Thus we can apply Lemma 3.3.4 to deduce that $G(f) = G(f')$. Cases 1) or 3) of that lemma would imply that $v = v'$, so that Z is enclosed by v . But this is impossible as $Z \subset Y^*$, and $G(v) \subset Y$. Thus we must have cases 2) or 4) of Lemma 3.3.4. Further, as v and v' are not isolated, in case 2), $f \cap f'$ must be a V_1 -vertex.

If Z crosses no edge of T , then Z is enclosed by some V_0 -vertex v' , which again cannot be v . Thus H' is a subgroup of $G(v)$ and of $G(v')$, and hence of the edge f' incident to v' and on the path in T from v to v' . Again we must have cases 2) or 4) of Lemma 3.3.4 and in case 2), $f \cap f'$ must be a V_1 -vertex.

Now Proposition 3.3.5 implies that in all cases, the splitting determined by f is a special canonical torus, and so is the splitting determined by f' (and the splittings determined by b and b' in case 4) of Lemma 3.3.4).

Finally, we will show that the original edge e that was crossed by X must equal one of f or f' (or b or b' in case 4) of Lemma 3.3.4). In all cases, it follows that the splitting determined by e is a special canonical torus.

Recall that $Z = X \cap Y^*$, and that $Z \cap S'$ and $Z^* \cap S'$ are both H' -infinite. We claim that $X \cap S'$ and $X^* \cap S'$ are also both H' -infinite. As $X \cap S'$ contains $Z \cap S'$, the first part of the claim is clear. As $G(f)$ determines a torus, it follows that the component of $\partial\widetilde{M}$ with stabilizer S' cannot meet this torus, and so must lie on the same side of \widetilde{M}_f as does $\widetilde{M}_{v'}$. In particular, $S' \subset Y^*$, so that $Y \cap S'$ is H' -finite. Now $Z^* = X^* \cup (X \cap Y)$, so it follows that $X^* \cap S'$ is also H' -infinite, completing the proof of the claim.

Next we show that f is the only edge incident to v which is crossed by X . For suppose that X crosses an edge f'' incident to v . Lemma 3.4.9 shows that $G(f'')$ is $VPC(n+1)$ and commensurable with H . Now we apply Lemma 3.3.4 to the pair (f, f'') . Thus $G(f) = G(f'')$, and we must have case 1), 2), 3) or 4) of that lemma. As f and f'' have the same V_0 -vertex which is not isolated nor of special Seifert type, this is impossible unless $f = f''$.

Next suppose that X crosses some edge e'' of T . Note that X is H' -almost invariant, that $K = H' \cap S'$ is VPC_n , and $X \cap S$ and $X^* \cap S$ are both H' -infinite. Further v' is a V_0 -vertex of T of commensuriser type which encloses all

K -almost invariant subsets of G . Now we apply Lemma 3.4.5 with v' and e'' in place of v and e . This shows that if f'' is the first edge on the path from v' to e'' , then $G(f'')$ is $VPC(n+1)$ and commensurable with H' , and X crosses the torus splitting given by f'' . Now we can argue as in the preceding paragraph to show that f' is the only edge incident to v' which is crossed by X .

As X crosses the edge e , we conclude that f is the first edge of the path joining v to e , and that f' is the first edge of the path joining v' to e . It follows that e must lie between v and v' , so that the edge e of T must be equal to f or f' (or b or b'), as required. It follows that each edge splitting of $\Gamma_{n,n+1}^c(G)$ is either canonical or is a special canonical torus, thus completing the proof of Theorem 3.4.1 \square

The following result is an easy consequence of the above arguments. Recall that a H -almost invariant subset X of a group G is enclosed by a vertex v of a G -tree T , if for every edge e incident to v , we have either $X \leq Z_e$ or $X^* \leq Z_e$, where Z_e and Z_e^* are the almost invariant subsets of G associated to e chosen so that v lies in Z_e . There is a natural extension of this idea as follows. If T' is a subtree of T , we will say that X is enclosed by T' , if for every edge e incident to T' , but not contained in T' , we have either $X \leq Z_e$ or $X^* \leq Z_e$, where Z_e and Z_e^* are the almost invariant subsets of G associated to e chosen so that T' is contained in Z_e .

Proposition 3.4.10. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . Then any almost invariant subset of G over a $VPC(n+1)$ subgroup H is either enclosed by a V_0 -vertex of T , or is enclosed by an interval whose endpoints are the V_0 -vertices on opposite sides of a special canonical torus edge. Further H is commensurable with that splitting torus group.*

Proof. Let X be an almost invariant set over a $VPC(n+1)$ subgroup H of G . If X is adapted to ∂G , then X is enclosed by some V_0 -vertex. If X is not adapted to ∂G , there are two cases depending on whether or not it crosses some edge of T . If it crosses no edge of T , then X is enclosed by some vertex, and hence by a V_0 -vertex. If X crosses some edge of T , we apply all the results above. This yields two V_0 -vertices v and v' on opposite sides of a special canonical torus edge, and X crosses no edges of T apart from the edges f and f' (and b and b' in case 4) of Lemma 3.3.4 between v and v' . Further H is commensurable with $G(f)$. Thus X is enclosed by the interval whose endpoints are v and v' , the edges of this interval all have associated the same special canonical torus, and H is commensurable with that splitting torus group, as required. \square

3.5 Comparisons

As promised in the introduction, we can now give the comparisons of the JSJ decomposition of a $PD(n+2)$ pair $(G, \partial G)$ with two other naturally defined decompositions. The easiest to handle is the algebraic regular neighbourhood of the set $\mathcal{E}_{n,n+1}(G)$ (the collection of all a.i. subsets of G which are over a VPC_n or $VPC(n+1)$ subgroup). Note that it is not at all obvious that this family of almost invariant subsets of G has a regular neighbourhood. Such a regular neighbourhood does not exist for general groups, as discussed in [17]. However Proposition 3.4.10 implies that the decomposition of G obtained by collapsing to a point each interval whose endpoints are the V_0 -vertices on opposite sides of a special canonical torus edge is the regular neighbourhood of $\mathcal{E}_{n,n+1}(G)$. We have shown the following result.

Theorem 3.5.1. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let $\mathcal{E}_{n,n+1}(G)$ denote the collection of all a.i. subsets of G which are over a VPC_n or $VPC(n+1)$ subgroup. Then the regular neighbourhood of $\mathcal{E}_{n,n+1}(G)$ in G exists and is obtained from $\Gamma_{n,n+1}(G)$ by collapsing to a point each interval whose endpoints are the V_0 -vertices on opposite sides of a special canonical torus edge.*

Remark 3.5.2. *If ∂G is empty, the decomposition $\Gamma_{n+1}(G)$ is equal to the regular neighbourhood of $\mathcal{E}_{n,n+1}(G)$ in G , as $\mathcal{E}_{n,n+1}(G)$ is equal to the collection of all a.i. subsets of G which are over a $VPC(n+1)$ subgroup.*

Next we turn to the analogue of the topological decomposition of 3-manifolds obtained in [12]. One considers the family $\mathcal{S}_{n,n+1}(G)$, which consists of all a.i. subsets of G which are dual to splittings of G over annuli or tori in $(G, \partial G)$. We claim that this family also has a regular neighbourhood, and that the edge splittings are those in $\mathcal{S}_{n,n+1}(G)$ which cross no element of $\mathcal{S}_{n,n+1}(G)$. As in [12], we call this the Waldhausen decomposition or W -decomposition. Again it is not at all obvious that this family of almost invariant subsets of G has a regular neighbourhood. Such a regular neighbourhood does not exist for general groups, as discussed in [17].

We will say that an element of the family $\mathcal{S}_{n,n+1}(G)$ which crosses no element of $\mathcal{S}_{n,n+1}(G)$ is *isolated* in $\mathcal{S}_{n,n+1}(G)$. We start by describing the isolated elements of $\mathcal{S}_{n,n+1}(G)$, and showing that there are only finitely many such elements. Trivially, the edge splittings of $\Gamma_{n,n+1}(G)$, minus special canonical tori, are all isolated elements of $\mathcal{S}_{n,n+1}(G)$. We have the following result.

Lemma 3.5.3. *Let $(G, \partial G)$ be an orientable $PD(n + 2)$ pair such that G is not VPC , and let α be an isolated element of $\mathcal{S}_{n,n+1}(G)$, which is not an edge splitting of $\Gamma_{n,n+1}(G)$. Then α is enclosed by a V_0 -vertex of $\Gamma_{n,n+1}(G)$ of commensuriser type.*

Proof. As α is a splitting of G dual to an annulus or torus, it must be enclosed by some V_0 -vertex v of $\Gamma_{n,n+1}(G)$. Theorem [3.2.1](#) tells us that a V_0 -vertex of $\Gamma_{n,n+1}(G)$ must be isolated, of I -bundle type, of interior Seifert type, or of commensuriser type. Thus it suffices to show that the first three cases cannot occur.

If v is isolated, any splitting enclosed by v is equal to an edge splitting of $\Gamma_{n,n+1}(G)$. Hence this case cannot occur.

If v is of I -bundle type, and so of $VPC(n - 1)$ -by-Fuchsian type, then α must be a splitting dual to an annulus. Let K denote the $VPCn$ group carried by the splitting annulus, and let X_v denote the base 2-orbifold of v . Then the image of K in the fundamental group of X_v is $VPC(\geq 1)$. As a Fuchsian group cannot have a $VPC2$ subgroup, it follows that the image of K must be $VPC1$. As α is not an edge splitting of $\Gamma_{n,n+1}(G)$, and crosses no such splitting, it determines a splitting of $G(v)$ which is adapted to $\partial_1 v$. This then yields a splitting over a $VPC1$ subgroup of the fundamental group of X_v , which is adapted to the boundary ∂X_v . As discussed in section 5.1.2 of [\[6\]](#), this splitting is dual to a "simple closed curve" γ in X_v , meaning that γ is a connected, closed 1-dimensional suborbifold of X_v . Thus either γ is a circle or it is the quotient of a circle by a reflection involution. The assumption that α is not an edge splitting of $\Gamma_{n,n+1}(G)$, means that γ cannot be homotopic to a boundary curve of X_v . Hence there is another "simple closed curve" δ in X_v whose intersection number with γ is non-zero (Corollary 5.10 of [\[6\]](#)). This determines a splitting of G dual to an annulus whose intersection number with α is non-zero, contradicting our assumption that α crosses no element of $\mathcal{S}_{n,n+1}(G)$. We conclude that this case cannot occur.

If v is of interior Seifert type, and so of $VPCn$ -by-Fuchsian type, then α must be a splitting dual to a torus. As in the preceding paragraph, this torus yields an essential "simple closed curve" in the base 2-orbifold of v . As in that paragraph, this implies that there is a splitting of G dual to a torus whose intersection number with α is non-zero, contradicting our assumption that α crosses no element of $\mathcal{S}_{n,n+1}(G)$. We conclude that this case also cannot occur, which completes the proof of the lemma. \square

Theorem [3.2.1](#) tells us that if v is a V_0 -vertex of $\Gamma_{n,n+1}(G)$ of commensuriser type, then v is of peripheral Seifert type, or of torus type, or of solid torus type. In the penultimate section of [\[20\]](#), the authors discussed the structure of

such vertices in great detail. They showed that in each of these cases, $G(v)$ is $VPCn$ -by- Γ , where Γ is the fundamental group of a compact 2-dimensional orbifold X_v . The group Γ is finite if v is of solid torus type, is virtually infinite cyclic if v is of torus type, and is not virtually cyclic if v is of Seifert type. Recall that any vertex v of $\Gamma_{n,n+1}$ has two types of "boundary" subgroups. The first type comes from the edge groups of the decomposition and the family of all these subgroups will be denoted by $\partial_1 v$. The second type comes from the decomposition of ∂G by edges of the decompositions and this family will be denoted by $\partial_0 v$. The first type gives us $PD(n+1)$ pairs in $(G, \partial G)$, namely annuli or tori, and the second type gives us $PD(n+1)$ pairs which are contained in ∂G . If v is of commensuriser type, these families of subgroups determine a division of the boundary ∂X_v of X_v into suborbifolds $\partial_0 X_v$ and $\partial_1 X_v$, where $\partial_0 X_v$ equals the closure of $\partial X_v - \partial_1 X_v$. Note that $\partial_0 X_v$ must be non-empty. It is possible that $\partial_1 X_v$ may be empty, but this happens if and only if $\Gamma_{n,n+1}(G)$ consists of the single vertex v , so that $G = G(v)$. Next the authors of [20] show that one can double $G(v)$ along $\partial_0 v$ which is "the intersection of $G(v)$ with ∂G ". The new object $DG(v)$ is the fundamental group of a V_0 -vertex of $\Gamma_{n+1}(DG)$. It is $VPCn$ -by- $D\Gamma$, where $D\Gamma$ is the fundamental group of DX_v , the double of X_v along $\partial_0 X_v$. As we are assuming that G is not VPC , it follows that DG is also not VPC , so that $\pi_1^{orb}(DX_v)$ cannot contain a $VPC2$ subgroup.

In the proof of Lemma 3.5.3, we used the close connection between splittings of G over annuli or tori enclosed by a V_0 -vertex v and "simple closed curves" in the base 2-orbifold X_v . Now we will extend these ideas. We will need the idea of a "simple arc" λ in the pair $(X_v, \partial_0 X_v)$. This means that λ is a connected 1-dimensional suborbifold of X_v with non-empty boundary, so that either λ is an interval or it is the quotient of an interval by a reflection involution. Further λ has boundary contained in $\partial_0 X_v$.

We will say that a "simple closed curve" γ in the 2-orbifold X_v is *essential* in $(X_v, \partial_0 X_v)$ if $\pi_1^{orb}(\gamma)$ injects into $\pi_1^{orb}(X_v)$ and γ is not homotopic into $\partial_0 X_v$ or into $\partial_1 X_v$. Also we will say that a "simple arc" λ in $(X_v, \partial_0 X_v)$ is *essential* in $(X_v, \partial_0 X_v)$ if λ cannot be homotoped into $\partial_0 X_v$ nor into $\partial_1 X_v$ while keeping $\partial\lambda$ in $\partial_0 X_v$.

Lemma 3.5.4. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let α be an isolated element of $\mathcal{S}_{n,n+1}(G)$, which is not an edge splitting of $\Gamma_{n,n+1}(G)$, and is enclosed by a V_0 -vertex v of $\Gamma_{n,n+1}(G)$ of commensuriser type, and with base 2-orbifold X_v . Then the following hold:*

1. *If α is a splitting of G dual to a torus, it determines a "simple closed curve"*

C in X_v which is essential in $(X_v, \partial_0 X_v)$, and this yields a bijection between splittings of G dual to a torus which are enclosed by v and not an edge torus of v , and "simple closed curves" in X_v which are essential in $(X_v, \partial_0 X_v)$.

2. If α is a splitting of G dual to an annulus, it determines a "simple arc" λ in $(X_v, \partial_0 X_v)$ which is essential in $(X_v, \partial_0 X_v)$, and this yields a bijection between splittings of G dual to an annulus which are enclosed by v and not an edge annulus of v , and "simple arcs" in $(X_v, \partial_0 X_v)$ which are essential in $(X_v, \partial_0 X_v)$.

Proof. 1) Note that although α is not an edge splitting of $\Gamma_{n,n+1}(G)$, it need not be the case that it determines a splitting of $G(v)$. This is clear if v is of torus type, but the same difficulty can arise if v is of Seifert type. We resolve this problem by working with the double $DG(v)$ which is the fundamental group of a V_0 -vertex V of $\Gamma_{n+1}(DG)$. Now α gives a splitting of DG dual to a torus which is enclosed by V , and is not an edge splitting of $\Gamma_{n+1}(DG)$. As in the proof of Lemma 3.5.3, this splitting is dual to a "simple closed curve" γ in DX_v which cannot be homotopic to a boundary curve of DX_v . As α is enclosed by v , it follows that γ can be chosen to lie in X_v . Further γ must be essential in $(X_v, \partial_0 X_v)$, as DX_v is the double of X_v along $\partial_0 X_v$. Now reversing the arguments yields the required bijection.

2) Let A denote the annulus in $(G, \partial G)$ which induces the splitting α of G , let DA denote the torus in DG obtained by doubling A along ∂A , and let $D\alpha$ denote the splitting of DG induced by DA . As α is not an edge splitting of $\Gamma_{n,n+1}(G)$, it follows that $D\alpha$ is not an edge splitting of $\Gamma_{n+1}(DG)$. Now as in part 1), Da determines a splitting of $DG(v)$ dual to the torus DA which yields a "simple closed curve" γ in DX_v which cannot be homotopic to a boundary curve of DX_v . This splitting of $DG(v)$ is invariant under the involution interchanging the two copies of $G(v)$, so the curve γ can be chosen to be invariant under the involution of DX_v interchanging the two copies of X_v . Thus γ is the double of a "simple arc" λ in the base orbifold X_v . Further λ has boundary contained in $\partial_0 X_v$, reflecting the fact that the boundary of the annulus lies in ∂G . As γ is not homotopic to a boundary curve of DX_v , it follows that λ is essential in $(X_v, \partial_0 X_v)$. One can easily reverse this argument to obtain the required bijection. \square

In the preceding lemma, as α crosses no element of $\mathcal{S}_{n,n+1}(G)$, it follows that γ and λ cross no simple closed curve in X_v which is essential in $(X_v, \partial_0 X_v)$, and cross no essential simple arc in $(X_v, \partial_0 X_v)$. We will say that such γ and λ are *isolated*.

Note that in the above discussions, any compact 2-orbifold with non-empty boundary can occur as X_v , which is not the case in the setting of 3-manifolds. See Lemma 3.6.1 and the discussion at the end of section 3.6. Note also that in general it is possible that an arc or closed curve in X_v can be essential in $(X_v, \partial_0 X_v)$, but inessential in $(X_v, \partial X_v)$, meaning that it can be homotoped into ∂X_v while keeping $\partial\lambda$ in ∂X_v .

Lemma 3.5.5. *Let X be a compact 2-orbifold, and let $\partial_1 X$ denote a possibly empty compact suborbifold of ∂X . Let $\partial_0 X$ denote the closure of $\partial X - \partial_1 X$.*

1. *If the Euler characteristic $\chi(X) \leq 0$, and λ is an isolated essential "simple arc" in $(X, \partial_0 X)$, then λ is essential in $(X, \partial X)$.*
2. *If $\chi(X) \leq 0$, and C is an isolated "simple closed curve" in X which is essential in $(X, \partial_0 X)$, then C is essential in $(X, \partial X)$.*
3. *If $\chi(X) > 0$, there cannot be any "simple closed curve" in X which is essential in $(X, \partial_0 X)$. Also there cannot be any isolated "simple arc" in X which is essential in $(X, \partial_0 X)$.*

Remark 3.5.6. *Recall that here a "simple closed curve" is either a circle or the orbifold quotient of the circle by a reflection, and a similar comment applies to the phrase "simple arc". In each case, the quotient by a reflection is a non-orientable 1-orbifold*

Proof. 1) Suppose that λ is orientable, so that $\partial\lambda$ consists of two points, and is not essential in $(X, \partial X)$. Thus λ is parallel to an arc μ contained in some component C of ∂X . Of course, the ends of μ lie in $\partial_0 X$. As λ is essential in $(X, \partial_0 X)$, the arc μ cannot be contained in $\partial_0 X$. There must be at least two components of $\partial_1 X$ in the interior of μ , so there must be at least one component D of $\partial_0 X$ in the interior of μ . There can be no mirrors in X , as otherwise we could join D to a mirror to obtain an essential "simple arc" in $(X, \partial_0 X)$ which crosses λ . As X has no mirrors, it follows that C is a circle. But now there is an arc λ' with both ends in D which is parallel to an arc μ' in C , such that $\mu \cup \mu' = C$, and λ' is also essential in $(X, \partial_0 X)$ and crosses λ . This contradicts the hypothesis that λ is isolated, which proves the required result in the case when λ is orientable.

Next suppose that λ is not orientable, so that $\partial\lambda$ consists of one point, and is not essential in $(X, \partial X)$. Thus λ is homotopic to an isomorphic 1-orbifold μ contained in some component C of ∂X . Note that the reflector point of μ must

be a reflector point of C , which must be an intersection point of C with a mirror component m of X . Of course, $\partial\mu$ lies in ∂_0X . As λ is essential in (X, ∂_0X) , the orbifold μ cannot be contained in ∂_0X , nor in ∂_1X . It follows that there must be a component D of ∂_0X in μ other than the component which contains $\partial\mu$. Note that D may contain the reflector point of μ . If X has a mirror other than m , we could join D to such a mirror to obtain an essential "simple arc" in (X, ∂_0X) which crosses λ . It follows that m is the only mirror in X . In particular, C and m must together form a boundary component of the surface underlying X . But now there is an arc λ' with both ends in D which is parallel to an arc μ' of $C \cup m$, such that $\mu \cup \mu' = C \cup m$, and λ' is also essential in (X, ∂_0X) and crosses λ . This again contradicts the hypothesis that λ is isolated, which proves the required result in the case when λ is not orientable.

2) Suppose that C is not essential in $(X, \partial X)$, so that C is homotopic to a boundary component S of X . As C is essential in (X, ∂_0X) , it follows that S is not contained in ∂_0X or in ∂_1X . If X has negative Euler characteristic, there is a simple arc μ in X with both ends in S which is essential in $(X, \partial X)$, and so crosses C . By choosing the ends of μ to lie in ∂_0X , we obtain a contradiction. If X has zero Euler characteristic, and is orientable, it must be an annulus or $D^2(2, 2)$, the 2-disk with two interior cone points each labeled 2, as X has non-empty boundary. Note that $D^2(2, 2)$ is double covered by the annulus. Thus in general, if X has zero Euler characteristic, it is covered by the annulus. In particular ∂X has 1 or 2 components. In the first case, there is again an essential simple arc λ in $(X, \partial X)$ with boundary in ∂_0X which must cross C . See Figures 3.1e), f), h), i) and j). In the second case, the two boundary components are homotopic, so that C is homotopic to each. Thus neither is contained in ∂_0X or in ∂_1X , and there is a simple arc in X with ends in ∂_0X which joins these two boundary components, and so must be essential and cross C . See Figures 3.1a), b), c), d) and g). These contradictions complete the proof that C must be essential in $(X, \partial X)$, as required.

3) As $\chi(X) > 0$, the orbifold fundamental group of X must be finite, so that X cannot contain any "simple closed curve" which is essential in (X, ∂_0X) .

If $\chi(X) > 0$, and ∂X is non-empty, the universal orbifold cover of X must be the 2-disk, so that X is either a cone or the quotient of a cone by a reflection. Let $D^2(p)$ denote the 2-orbifold with underlying surface the 2-disk and with a single interior cone point of order $p \geq 1$, and let Y_p denote the quotient of $D^2(p)$ by a reflection. Note that $D^2(1)$ is simply the 2-disk. The underlying surface of Y_p is a disk D , and the boundary ∂Y_p consists of a single interval in ∂D with reflector ends. If $p = 1$, the rest of ∂D is a single mirror, and if $p \geq 2$, the rest of

∂D is divided into two mirrors separated by a boundary cone point, labeled p . Let $|\partial_0 X|$ denote the number of components of $\partial_0 X$. In all cases, there is a number k depending on X such that if $|\partial_0 X| < k$, then there are no essential "simple arcs" in $(X, \partial_0 X)$, and if $|\partial_0 X| \geq k$, then X contains such simple arcs, but no such arc can be isolated. If X is the disk $D^2(1)$, then $k = 4$. If X is a cone $D^2(p)$, $p \geq 2$, then $k = 3$. If X is Y_1 , then $k = 3$, and if X is Y_p , $p \geq 2$, then $k = 2$. Note that if X is $D^2(p)$, the existence of such k is clear. For if ρ denotes an orientation preserving homeomorphism of $(X, \partial_0 X)$ which sends each component of $\partial_0 X$ and $\partial_1 X$ to the next such component, and if there is an essential simple arc λ in $(X, \partial_0 X)$, then $\rho(\lambda)$ must cross λ . \square

Now we can prove the following result.

Lemma 3.5.7. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC . A splitting of G dual to an annulus or torus of $(G, \partial G)$ which is isolated in $\mathcal{S}_{n,n+1}(G)$ is either an edge splitting of $\Gamma_{n,n+1}(G)$ or is dual to an annulus enclosed by a V_0 -vertex of commensuriser type, which is not of solid torus type. Further the family of all splittings of G dual to an annulus or torus which are isolated in $\mathcal{S}_{n,n+1}(G)$ is finite.*

Proof. If α is a splitting dual to an annulus or torus and is not an edge splitting of $\Gamma_{n,n+1}(G)$, Lemma 3.5.3 tells us that α is enclosed by a V_0 -vertex v of commensuriser type.

If α is a splitting dual to a torus, it determines an isolated simple closed curve C in the base orbifold X_v of v , and C is essential in $(X_v, \partial_0 X_v)$. Thus part 2) of Lemma 3.5.5 implies that C is essential in $(X_v, \partial X_v)$. Now Corollary 5.10 of [6] implies there is some simple closed curve in X which crosses C , which is a contradiction. It follows that a splitting of G dual to a torus of $(G, \partial G)$ which crosses no element of $\mathcal{S}_{n,n+1}(G)$ must be an edge splitting of $\Gamma_{n,n+1}(G)$.

If α is a splitting dual to an annulus, it determines an isolated essential simple arc in $(X_v, \partial_0 X_v)$, which must be essential in $(X_v, \partial X_v)$ by part 1) of Lemma 3.5.5. As the number of disjoint non-parallel such arcs in X_v is finite, and as $\Gamma_{n,n+1}(G)$ has only finitely many vertices, the result follows. Finally part 3) of Lemma 3.5.5 implies that v cannot be of solid torus type. \square

Now we can proceed to give a complete description of the exceptional splittings of G . These are splittings of G dual to an annulus of $(G, \partial G)$ which are isolated in $\mathcal{S}_{n,n+1}(G)$ and are not edge splittings of $\Gamma_{n,n+1}(G)$. Thus each exceptional splitting is enclosed by some V_0 -vertex v of commensuriser type, which

is not of solid torus type. Let X_v be the base orbifold of v . Then our annulus determines an isolated essential simple arc λ in $(X_v, \partial_0 X_v)$, and so λ is essential in $(X_v, \partial X_v)$, by Lemma 3.5.5

In order to give a complete list of cases, the following lemma will be very useful.

Lemma 3.5.8. *Let X be a compact 2-orbifold, with Euler characteristic $\chi(X) \leq 0$, and let $\partial_1 X$ denote a possibly empty compact suborbifold of ∂X . Let $\partial_0 X$ denote the closure of $\partial X - \partial_1 X$. Let λ be an isolated essential simple arc in $(X, \partial_0 X)$, such that a point of $\partial\lambda$ lies in a component C of ∂X . Then the following hold:*

1. *If $\chi(X) < 0$, then $C \subset \partial_0 X$.*
2. *If $\chi(X) = 0$, and ∂X is connected, then $C \subset \partial_0 X$.*

Remark 3.5.9. *It follows that in all cases, if X admits such an arc λ , and if ∂X is connected, then $\partial_1 X$ must be empty. If $\chi(X) < 0$, the same conclusion holds if ∂X has two components which are joined by λ .*

There are two orbifolds where the hypotheses of the lemma hold and $\chi(X) = 0$, and ∂X is not connected. In each of these cases, C need not be contained in $\partial_0 X$. See Figures 3.1a), 3.1b) and 3.1c).

Proof. By Lemma 3.5.5, λ is essential in $(X, \partial X)$.

1) As $\chi(X) < 0$, it is not possible to have two component of ∂X which are homotopic. Thus if C is not contained in $\partial_0 X$, a push off C' is essential in $(X, \partial_0 X)$ and crosses λ . This contradiction shows that C must be contained in $\partial_0 X$, as required.

2) As ∂X is connected, we can use the same argument as in part 1) to show that C must be contained in $\partial_0 X$, as required. \square

Now we will proceed to list all cases of $(X, \partial_0 X, \partial_1 X)$, where X is a compact 2-orbifold with $\chi(X) \leq 0$ and non-empty boundary, $\partial_1 X$ is a possibly empty compact suborbifold of ∂X , and $\partial_0 X$ is the closure of $\partial X - \partial_1 X$, and there is an isolated essential simple arc λ in $(X, \partial_0 X)$. In all cases, when such an arc exists, it is unique up to isotopy. As λ has at least one boundary point, which must lie in $\partial_0 X$, it follows that $\partial_0 X$ is non-empty. To find an isolated essential simple arc λ we have to find all essential simple arcs and omit those that cross others. In what follows we have not shown all these arcs, only the isolated ones. (We show some examples with all possible arcs in Figure 3.5.)

Recall from Lemma 3.5.4 that any isolated essential arc in the base orbifold X_v of a V_0 -vertex v of $\Gamma_{n,n+1}(G)$ of commensuriser type gives rise to an exceptional annulus in $\mathcal{S}_{n,n+1}(G)$, under the assumption that G is not VPC . In particular, this excludes the situation where $\pi_1^{orb}(X_v)$ is $VPC1$ and $\partial_1 X_v$ is empty. Thus in Figure 3.1 the isolated arcs shown in 3.1a), 3.1b) and 3.1c) are the only cases which are relevant to finding exceptional annuli.

However if G is $VPC(n+1)$ and is $VPCn$ -by- $\pi_1^{orb}(X)$ where X is a 2-orbifold such that $\pi_1^{orb}(X)$ is $VPC1$ and $\partial_1 X$ is empty, then an isolated arc in X_v still determines an essential annulus in $(G, \partial G)$, but that annulus need not be isolated. For example, consider the isolated arcs shown in Figures 3.1e) and 3.1f). The orientable 3-dimensional Seifert fibre spaces over the orbifolds in these two figures are each homeomorphic to the twisted I -bundle over the Klein bottle with orientable total space, and the annuli determined by the isolated arcs cross each other. For a discussion of this example, see page 15 in [17]. Higher dimensional examples can be obtained from this example by taking the product with circles.

In drawing the orbifold X , the pictured boundary consists of the orbifold boundary ∂X and mirrors. The mirrors are drawn in thick lines and ∂X in thin lines. We then proceed to the division of ∂X into $\partial_0 X$ and $\partial_1 X$. In the following pictures $\partial_0 X$ is still drawn in thin lines, $\partial_1 X$ in dashed lines, and the isolated arc λ in dotted lines. Figure 3.1 shows all examples with $\chi(X) = 0$. Each of the orbifolds in Figure 3.1 is covered by the annulus, and so has orbifold fundamental group which is $VPC1$.

We next consider the cases with $\chi(X) < 0$. Recall that λ is a "simple arc" in $(X, \partial_0 X)$ which is essential in $(X, \partial X)$ and crosses no essential "simple closed curve" in X . Corollary 5.10 of [6] tells us that X admits no essential "simple closed curves" at all. Thus X lies on the list of ten orbifolds given in Proposition 5.12 of [6]. However, two of these ten have no boundary. In Figure 3.2, we show the remaining eight orbifolds. For each of these eight orbifolds, we use Lemmas 3.5.5 and 3.5.8 to determine the possible decompositions of ∂X into $\partial_0 X$ and $\partial_1 X$ which admit an isolated essential arc, and we show all these cases in Figures 3.3 and 3.4. Figure 3.3 shows the cases where $\partial_1 X$ is empty, and Figure 3.4 shows the other cases.

Here is a verbal description of the eight orbifolds in Figure 3.2, and of the possible isolated essential simple arcs.

1. $X = D^2(p, q)$, the 2-disc with two interior cone points of orders $p, q \geq 2$ with at least one strictly larger than 2. There is an isolated essential simple

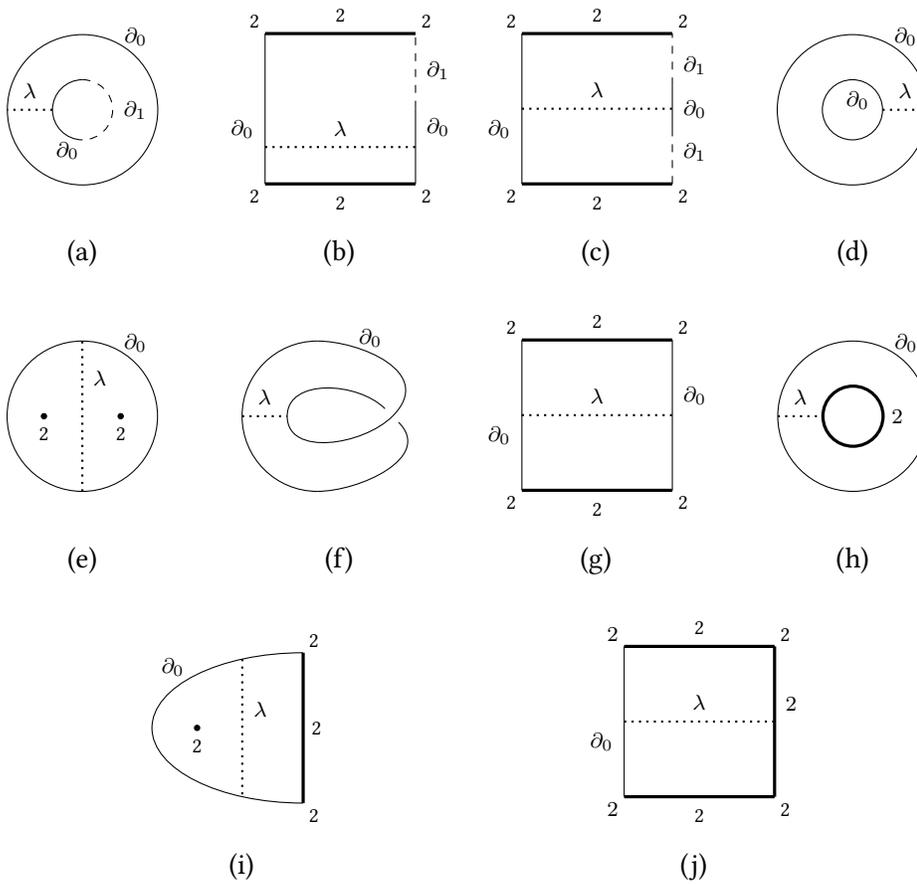


Figure 3.1: The case in which $\chi(X) = 0$

arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty.

2. $X = S^1 \times I(p)$, $p \geq 2$, the annulus with one interior cone point. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty or is a component of ∂X .
3. X is a pair of pants, with no singular points. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is a component of ∂X , or is the union of two components of ∂X .
4. The underlying surface of X is a disc D . The boundary of D contains one mirror interval, so that ∂X is the closure of the complement of this mirror interval in ∂D , and X has one interior cone point labeled p . There is an isolated essential arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty.
5. The underlying surface of X is an annulus A . The boundary of A contains one mirror interval, so that ∂X is the closure of the complement of this mirror interval in ∂A . There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty or is a component of ∂X .
6. The underlying surface of X is a disc D , and the boundary ∂X of X consists of a single interval in ∂D with reflector ends, and the rest of ∂D is divided into three mirrors separated by two boundary cone points, labeled $2p$ and $2q$. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty.
7. The underlying surface of X is a disc D . The boundary ∂X of X consists of two disjoint intervals in ∂D each with reflector ends, and the rest of ∂D is divided into a single mirror and two mirrors separated by a boundary cone point, labeled $2p$. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is empty or is a component of ∂X .
8. The underlying surface of X is a disc D . The boundary ∂X of X consists of three disjoint intervals in ∂D each with reflector ends, and the rest of ∂D consists of three mirrors. There is an isolated essential simple arc in $(X, \partial_0 X)$ iff $\partial_1 X$ is a component of ∂X , or is the union of two components of ∂X .

Thus when $\chi(X) < 0$, we have fourteen orbifolds with an isolated essential simple arc, of which the six shown in Figure 3.3 have $\partial_1 X$ empty. In these six cases, the group G (in Theorem 3.5.10) is $VPCn\text{-by-}\pi_1^{orb}(X)$.

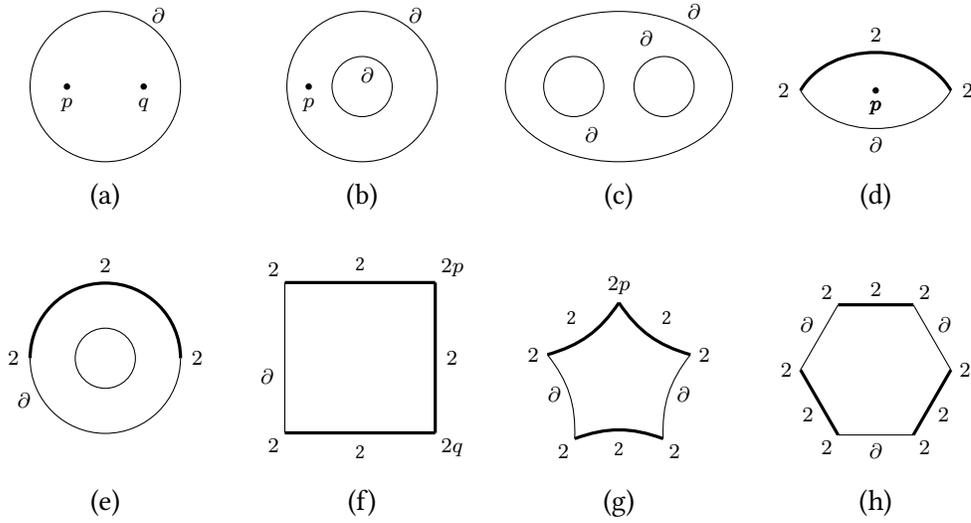


Figure 3.2: The eight orbifolds with $\chi < 0$, non-empty boundary, and no essential closed curves

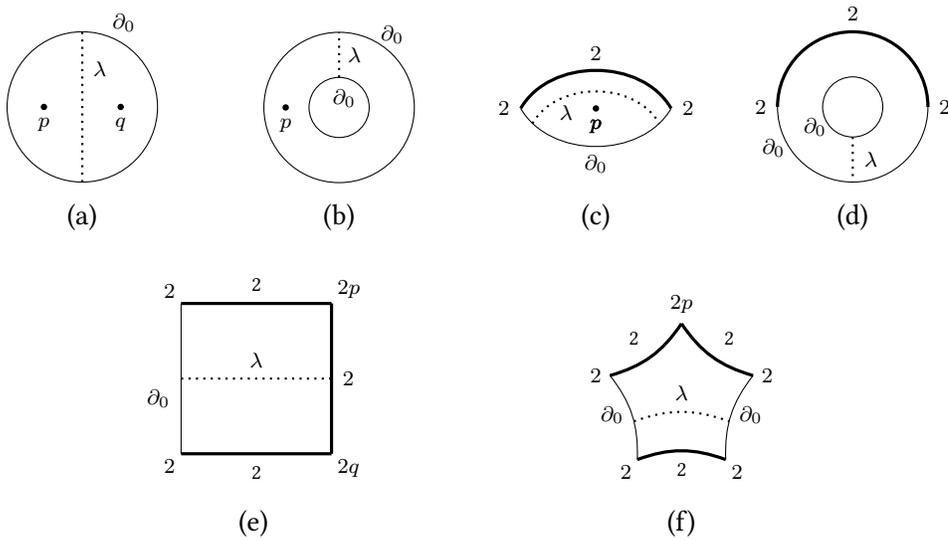


Figure 3.3: Cases with $\partial_1 X = \emptyset$

Finally we can show that the family $\mathcal{S}_{n,n+1}(G)$ has a regular neighbourhood which is a refinement $\Sigma_{n,n+1}(G)$ of $\Gamma_{n,n+1}(G)$. Every element of $\mathcal{S}_{n,n+1}(G)$ determines a simple closed curve or simple arc in the base 2-orbifold of one of the V_0 -vertices of $\Gamma_{n,n+1}(G)$. Now a connected compact 2-orbifold is filled by simple closed curves and simple arcs, unless it is one of the exceptional cases listed above. Thus cutting the base 2-orbifold along the exceptional arc yields 2-orbifolds which contain no isolated essential simple arc. However, in several cases the 2-orbifolds obtained by cutting along an isolated arc contain non-isolated essential simple arcs. Now splitting an exceptional V_0 -vertex v along the exceptional annulus yields a vertex or vertices with base orbifold obtained by cutting X_v along the isolated essential arc. These new vertices enclose elements of $\mathcal{S}_{n,n+1}(G)$ which correspond to simple arcs in the new base orbifolds. Thus these new vertices enclose elements of $\mathcal{S}_{n,n+1}(G)$ other than edge splittings of $\Gamma_{n,n+1}(G)$ if and only if the new base orbifold contains essential simple arcs.

Using the above notation, we can now describe the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $\mathcal{S}_{n,n+1}(G)$. It is a refinement of $\Gamma_{n,n+1}(G)$ which can be obtained essentially by splitting each exceptional V_0 -vertex of $\Gamma_{n,n+1}(G)$ along the exceptional annulus it contains. Each non-exceptional V_0 -vertex of $\Gamma_{n,n+1}(G)$ yields unchanged a V_0 -vertex of $\Sigma_{n,n+1}(G)$, and each V_1 -vertex of $\Gamma_{n,n+1}(G)$ yields unchanged a V_1 -vertex of $\Sigma_{n,n+1}(G)$. If v is an exceptional V_0 -vertex of $\Gamma_{n,n+1}(G)$, which contains a separating exceptional annulus, then v is split into two new vertices. If v is an exceptional V_0 -vertex of $\Gamma_{n,n+1}(G)$, which contains a non-separating exceptional annulus, then v is split into a single new vertex. If a new vertex encloses elements of $\mathcal{S}_{n,n+1}(G)$ other than edge splittings of $\Gamma_{n,n+1}(G)$ we label it as a V_0 -vertex. Otherwise, we label it as a V_1 -vertex. This yields a refinement of $\Gamma_{n,n+1}(G)$, but it may not be bipartite. By adding an isolated V_0 -vertex between adjacent V_1 -vertices, and an isolated V_1 -vertex between adjacent V_0 -vertices, and then reducing if needed, we can create a bipartite graph of groups which will be the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $\mathcal{S}_{n,n+1}(G)$. We have shown the following result.

Theorem 3.5.10. *Let $(G, \partial G)$ be an orientable $PD(n+2)$ pair such that G is not VPC , and let $\mathcal{S}_{n,n+1}(G)$ denote the family of all a.i. subsets of G which are dual to splittings of G over annuli or tori in $(G, \partial G)$. Then the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $\mathcal{S}_{n,n+1}(G)$ in G exists and is obtained from $\Gamma_{n,n+1}(G)$ by omitting special canonical tori and splitting each exceptional V_0 -vertex along the exceptional annulus it contains, as described above.*

Remark 3.5.11. *It follows from this theorem that if ∂G is empty, so that G is an ori-*

entable $PD(n+2)$ group, then the regular neighbourhood $\Sigma_{n,n+1}(G)$ of $\mathcal{S}_{n,n+1}(G)$ in G exists and is equal to $\Gamma_{n,n+1}(G) = \Gamma_{n+1}(G)$.

In Figures [3.1](#), [3.3](#) and [3.4](#) we drew only the essential isolated arcs. In Figure [3.5](#) we draw some examples with other possible essential arcs to illustrate that they do not lead to isolated arcs. We point out the corresponding figures from the text, but omit the labels.

3.6 Results in dimension 3

In this section, we consider the special case of $PD3$ pairs and compare our results in this case with the results of Neumann and Swarup in [\[12\]](#). In the previous section, a key role was played by the classification of compact 2-orbifolds with certain properties. In the case when $n = 1$, so that we are considering $PD3$ pairs, the following result greatly reduces the number of cases which need considering.

Lemma 3.6.1. *Let $(G, \partial G)$ be an orientable $PD3$ pair, let v be a V_0 -vertex of $\Gamma_{1,2}(G)$ which is of Seifert type or of commensuriser type, and let X denote the base 2-orbifold of v . Then X has no mirrors.*

Remark 3.6.2. *Note that we are allowing G to be VPC in the above statement. This result means that when $n = 1$, only Figures [3.1a](#)), [3.1d-f](#)), [3.3a,b](#)) and [3.4a-c](#)) are relevant to the results in this section. It also means that each V_0 -vertex of $\Gamma_{1,2}(G)$ can be regarded as a Seifert fibre space or an I -bundle.*

Proof. Recall that $G(v)$ is $VPC1$ -by- $\pi_1^{orb}(X)$, and that G is torsion free. Now a $VPC1$ group is a finite extension of \mathbb{Z} , so a torsion free $VPC1$ group is $PD1$ and must also be isomorphic to \mathbb{Z} . And a $VPC2$ group is a finite extension of $\mathbb{Z} \times \mathbb{Z}$, so a torsion free $VPC2$ group is $PD2$ and must be isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or to $\pi_1(K)$, where K denotes the Klein bottle. In particular, a torus in a $PD3$ pair must be isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

If X contains a mirror, there are three possibilities. The mirror must be a circle, or meet ∂X , or meet another mirror (or itself) in a corner reflector point. We will show that each of these cases is impossible, which implies that X has no mirrors, as required.

First we consider a component C of ∂X , which must be a circle or the quotient Q of a circle by a reflection. We know that $\pi_1^{orb}(C)$ is the image of a torus in ∂M_v . As $\pi_1^{orb}(Q) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, which is not abelian, $\pi_1(Q)$ cannot be a quotient of

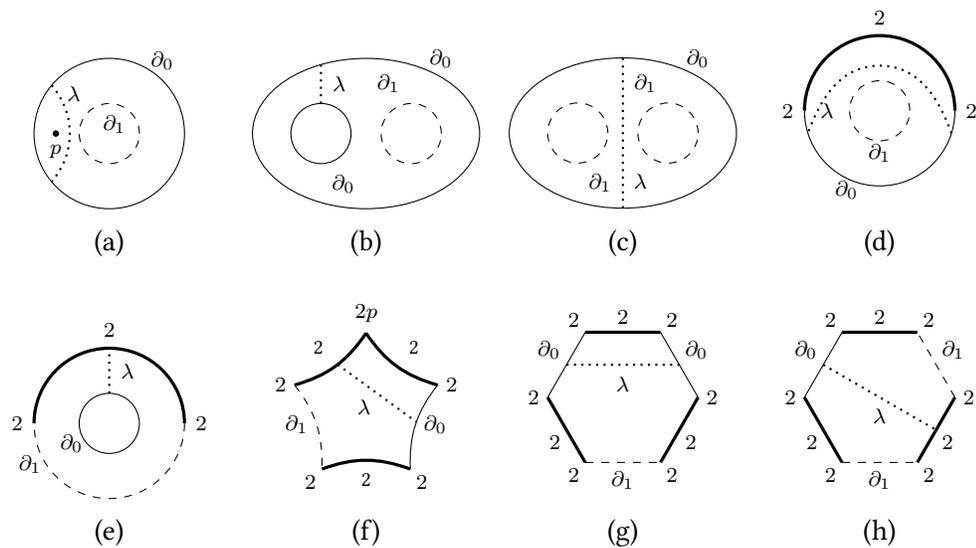


Figure 3.4: Cases with $\partial_1 X \neq \emptyset$

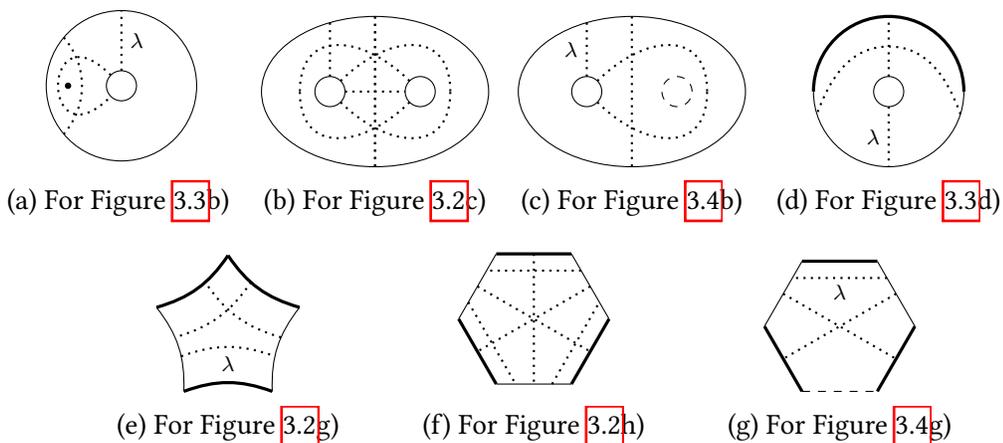


Figure 3.5: Note that $\partial_1 = \emptyset$ in all cases except (c) and (g)

the abelian group $\mathbb{Z} \times \mathbb{Z}$. It follows that all components of ∂X are circles. Hence no mirror of X can meet ∂X , as required.

Next suppose that X has a corner reflector. This yields a finite dihedral subgroup D of $\pi_1^{orb}(X)$, where the term dihedral group includes the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In particular D is not cyclic. But the pre-image of D in $G(v)$ is a torsion free $VPC1$ group and so is isomorphic to \mathbb{Z} , which implies that D must be cyclic. This contradiction shows that X cannot have corner reflectors, as required.

Finally suppose that X has a mirror m which is a circle. Then m has a neighbourhood orbifold Y in X with underlying space an annulus, such that ∂Y is equal to one boundary component C of the annulus and the other boundary component is m . We have $\pi_1^{orb}(C) = \pi_1(C) \cong \mathbb{Z}$, and $\pi_1^{orb}(m) = \pi_1(C) \times \mathbb{Z}_2$, and we let R and S denote the pre-images in $G(v)$ of $\pi_1(C)$ and $\pi_1^{orb}(m)$ respectively. Thus R is a subgroup of S of index 2. Each of R and S is a torsion free $VPC2$ group. As C determines a splitting of $\pi_1^{orb}(X)$ over $\pi_1(C)$ which is adapted to ∂X , this yields a splitting of $G(v)$ over R which is adapted to $\partial_1 v$, and hence determines a splitting of G over R which is adapted to ∂G . As the pair $(G, \partial G)$ is orientable, it follows that R is orientable, and so is a torus in G . Also the splitting of G over R yields one or two $PD3$ pairs with R as a boundary group. Now Lemma 2.2 of [11] implies that R is maximal among torus subgroups of G , so that S cannot be a torus. Thus S is isomorphic to $\pi_1(K)$. Now consider the presentation $\langle a, b : bab^{-1} = a^{-1} \rangle$ of $\pi_1(K)$. The kernel of the map $S \cong \pi_1(K) \rightarrow \pi_1^{orb}(m) \cong \mathbb{Z} \times \mathbb{Z}_2$ must be the subgroup A generated by a^2 , and this must also be the kernel of the map $R \rightarrow \pi_1(C) \cong \mathbb{Z}$. As R has index 2 in $\pi_1(K)$, it must be the orientation subgroup generated by a and b^2 . But then the quotient of R by A is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, which is a contradiction.

This completes the proof that X has no mirrors. \square

Now we can compare our results from section 3.5 with those of Neumann and Swarup in [12]. Recall that in section 3.5, we considered an orientable $PD(n+2)$ pair $(G, \partial G)$ such that G is not VPC and described the possible exceptional annuli. These are splittings of G dual to an annulus which are not edge splittings of $\Gamma_{n,n+1}^c(G)$. Each exceptional annulus is enclosed by some V_0 -vertex v of $\Gamma_{n,n+1}^c(G)$ of commensuriser type, and corresponds to an isolated arc λ in the base 2-orbifold X_v of v . In Figures 3.1a)-c), 3.3 and 3.4 we showed all possible such arcs and orbifolds. We will be interested in the special case when $n = 1$, so that $(G, \partial G)$ is an orientable $PD3$ pair. Lemma 3.6.1 tells us that those figures in which the orbifold X has a mirror, are not relevant in this case. This substantially reduces the number of possibilities. We only need to consider the six

isolated arcs shown in Figures [3.1a](#), [3.3a,b](#) and [3.4a-c](#)).

Recall from the beginning of section [3.2](#) that an annulus in a $PD(n+2)$ pair is a certain type of orientable $PD(n+1)$ pair, whose fundamental group is $VPCn$. When $n = 1$, a torsion free $VPC1$ group must be isomorphic to \mathbb{Z} , and this is an orientable $PD1$ group. Thus twisted annuli do not appear when considering $PD3$ pairs, and an untwisted annulus in our generalized sense is exactly the same as the ordinary annulus $S^1 \times I$. Further if our $PD3$ pair $(G, \partial G)$ comes from a compact orientable 3-manifold M , then there is a precise correspondence between $\Gamma_{1,2}^c(G)$ and the JSJ decomposition of M . Also any exceptional annuli in $(G, \partial G)$ correspond to embedded annuli in M which cross no other embedded essential annulus in M and are not splitting annuli of the JSJ decomposition of M . In [\[12\]](#), such annuli are called matched annuli, and the possibilities are listed in Lemma 3.4 of [\[12\]](#). We would expect this list to be the same as our list of six possible isolated arcs in Figures [3.1a](#), [3.3a,b](#) and [3.4a-c](#)), but there are some differences. The four isolated arcs shown in Figures [3.1a](#), [3.3a](#), [3.4a](#) and [3.4c](#)) yield the examples of matched annuli shown in Figure 5 of [\[12\]](#), but the two isolated arcs shown in [3.3b](#)) and [3.4b](#)) do not correspond to matched annuli shown in Figure 5 of [\[12\]](#). In [3.3b](#)), $\partial_1 X$ is empty which implies that $G = G(v)$, and that M is a Seifert fibre space, so this case is not of much interest. But in Figure [3.4b](#)), $\partial_1 X$ is non-empty, so the isolated arc λ in this figure corresponds to an interesting matched annulus in M . This seems to be an omission in [\[12\]](#). Figure [3.5c](#)) shows that the orbifold X in Figure [3.4b](#)) contains two essential arcs other than λ , but they cross, so neither is isolated. Cutting along the vertical one of the two crossing arcs in Figure [3.5c](#)) expresses X as the union of two orbifolds glued along a boundary arc. These are the first and second orbifolds shown in Figure 1 of [\[12\]](#). This is the unique case where gluing two of the orbifolds shown in Figure 1 of [\[12\]](#) yields an orbifold with no isolated essential arc. This possibility was omitted in the discussion in the second paragraph on page 35 of [\[12\]](#). Specifically the last sentence of that paragraph is incorrect.

The result of Lemma [3.6.1](#) fails in higher dimensions. Mirrors of all three types discussed in the proof of Lemma [3.6.1](#) can exist in all dimensions greater than 3. We discuss some examples in dimension 4. Again higher dimensional examples can be obtained by taking the product with circles.

Our starting point is that the orientable 3-manifold W which is a twisted I -bundle over the Klein bottle K is an example of a twisted 3-dimensional annulus, and the double DW of W is a 3-dimensional torus. Thus the orientable 4-manifold $DW \times I$ is the underlying space of a $PD4$ pair $(G, \partial G)$, and $\Gamma_{2,3}(G)$ consists of a single V_0 -vertex v , so $G = G(v)$, and v is of $VPC2$ -by-Fuchsian

type with base orbifold $Q \times I$, where Q is the quotient of the circle by a reflection. This orbifold has two mirrors each meeting the orbifold boundary in reflector points. If instead one considers the manifold $DW \times S^1$, one will have two mirrors each homeomorphic to a circle.

Finally, one can also give examples with corner reflectors as follows. A useful way to think about W is as the I -bundle over K associated to the $\partial I = S^0$ -bundle given by the double covering map $T \rightarrow K$. Note that this map is determined by a surjective homomorphism $\pi_1(K) \rightarrow \mathbb{Z}_2$. One way to construct W is to start with the product $T \times I$ of the 2-torus with the unit interval, and consider the involution (τ, σ) on $T \times I$, where τ is the free involution of T associated to the double covering map $T \rightarrow K$, and σ is the reflection of I . As τ is free, so is (τ, σ) , and the quotient of $T \times I$ by (τ, σ) is clearly W .

We will perform a similar construction starting with the product $T \times I \times I$, and using the natural homomorphism $\varphi : \pi_1(K) \rightarrow H_1(K; \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Only one of the three surjections $\pi_1(K) \rightarrow \mathbb{Z}_2$ yields an orientable double cover, and we will choose the basis of $H_1(K; \mathbb{Z}_2)$ so that projection onto each factor yields non-orientable double covers K' and K'' . Let T denote the torus which is the 4-fold cover of K corresponding to the kernel of φ . Thus $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts freely on T with quotient K . It also acts on $I \times I$ as the group generated by reflections in each factor, and we let X denote the quotient 2-orbifold of this action. The underlying space of X is a disc D , whose boundary is divided into two mirrors and an arc of ∂X . The product action on $T \times I \times I$ is free and orientation preserving, so that the quotient of $T \times I \times I$ by this action is an aspherical orientable 4-manifold Z , and Z has a natural projection to X . The pre-image in Z of each interior point of X is T . The pre-image of the corner reflector of X is K , and the pre-image of all other points of one mirror is K' and of the other mirror is K'' . Finally the pre-image of all other points of ∂X is T . Further the pre-image of ∂X , which is equal to ∂Z , consists of the union of the twisted I -bundle over K' with boundary T and the twisted I -bundle over K'' with boundary T , glued along T . The pre-image of one mirror is a twisted I -bundle over K with boundary K' , and the pre-image of the other mirror is a twisted I -bundle over K with boundary K'' . Thus $(Z, \partial Z)$ is the underlying space of a $PD4$ pair $(G, \partial G)$, and $\Gamma_{2,3}(G)$ consists of a single V_0 -vertex v , so $G = G(v)$, and v is of $VPC2$ -by-Fuchsian type with base orbifold X .

3.7 Some related questions

An unsatisfactory part of our work is that there is no algebraic treatment of the triple $(G(v), \partial_0 v, \partial_1 v)$ we discussed.

Problem 3.7.1. *Construct a theory of Poincaré triads for groups.*

There is a discussion by Wall in the case of complexes [22].

In Johansson's Deformation Theorem, he considers a homotopy equivalence $F : M \rightarrow M'$ between two Haken 3-manifolds with incompressible boundary. He shows that there is a bijection between the pieces of the JSJ decomposition of M and those of M' , and that F can be homotoped to send the pieces of M to the pieces of M' . In particular, the splitting annuli and tori of M are sent to splitting annuli and tori of M' . For the non-characteristic pieces of M , he shows one can further homotop F to arrange that the intersection with the boundary of M is mapped to the boundary of the corresponding piece of M' . Finally one can arrange that the restriction of F to each non-characteristic piece is a homeomorphism to the corresponding piece of M' . It is natural to ask whether there is an algebraic analogue of this. The natural analogue would be when one considers two $PD(n+2)$ pairs $(G, \partial G)$ and $(G', \partial G')$ with an isomorphism between G and G' . There is a bijection between the underlying graphs of $\Gamma_{n,n+1}^c(G)$ and $\Gamma_{n,n+1}^c(G')$, and one would like to know that for a V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$, the part $\partial_0 v$ of ∂v coming from ∂G can be deformed into $\partial_0 v'$ of the corresponding V_1 -vertex v' of $\Gamma_{n,n+1}^c(G')$. It seems reasonable this should hold when $n = 1$, but this seems far from clear when $n > 1$. The reason is that the proof of Johansson's Deformation Theorem depends on the non-existence of certain types of essential annulus in the non-characteristic pieces of M . In higher dimensions the analogous fact would be the non-existence of essential higher dimensional annuli, but that may not exclude the existence of essential maps of the 2-dimensional annulus.

Problem 3.7.2. *In the case of orientable $PD3$ pairs $(G, \partial G)$ and $(G', \partial' G)$, with G and G' isomorphic, for any V_1 -vertex v of $\Gamma_{n,n+1}^c(G)$, and the corresponding V_1 -vertex v' of $\Gamma_{n,n+1}^c(G')$, show that $\partial_0 v$ can be deformed into $\partial_0 v'$.*

In Theorem 8.1 of [1], Bieri and Eckmann proved a result which we have used several times. Namely that if a PDn pair is split along a $PD(n-1)$ subgroup relative to the boundary, then we again get PDn pairs.

Problem 3.7.3. *Is there an analogue of the Bieri-Eckmann Theorem when a PDn pair is split along a $PD(n-1)$ pair?*

Examples in dimension 3 show that if one splits a 3-manifold with incompressible boundary along a surface with non-empty boundary, the resulting manifold may have compressible boundary. Thus if one splits a $PD3$ pair along a $PD2$ pair, the resulting object need not be a $PD3$ pair. This again seems to need a theory of PD triples for groups. However, Gitik [5] has proven an analogue of the Bieri-Eckmann Theorem in the special case when splitting a PDn pair along a $PD(n - 1)$ pair does yield a PDn pair.

A related natural question is:

Problem 3.7.4. *Is there a theory of PD pairs when the maps from the boundary groups are not injective?*

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Chapter 4

A deformation theorem for Poincaré duality pairs in dimension 3

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Abstract. We prove the analogue of Johansson’s Deformation Theorem for $PD3$ pairs.

Dedicated to Walter Neumann on his 75th birthday

4.1 Introduction

In [10], the authors gave an analogue for $PD(n+2)$ pairs, [1], of the JSJ-decomposition of a 3-manifold. For a $PD(n+2)$ pair $(G, \partial G)$, this depends only on G , and is a bipartite graph of groups, which the authors denoted by $\Gamma_{n,n+1}^c(G)$. Note that this is not quite the same as the decomposition $\Gamma_{n,n+1}(G)$, which they also discuss in [10]. For brevity, we will denote $\Gamma_{n,n+1}^c(G)$ by Γ_G in this paper. In the case of the group G of an orientable 3-manifold M with incompressible boundary, $\Gamma_{1,2}(G)$ is the graph of the JSJ decomposition of G . The union of the manifolds in $V_0(G)$ is called the characteristic submanifold of M and the union of those intersecting the boundary is called the peripheral characteristic submanifold. This graph of groups has vertices of two types denoted V_0 and V_1 . The V_0 -vertices of Γ_G correspond to the components of the characteristic submanifold $V(M)$ of a 3-manifold M , and the V_1 -vertices of Γ_G correspond to components of $M - V(M)$.

The edges of Γ_G correspond to the annuli and tori which form the frontier of $V(M)$ in M . For each vertex v of Γ_G , the edges of the decomposition determine a family of subgroups of $G(v)$ which is denoted by $\partial_1 v$. The authors of [10] also showed that the decomposition of ∂G induced from Γ_G determines a family of subgroups of $G(v)$ denoted $\partial_0 v$. Each of the groups in $\partial_0 v$ and $\partial_1 v$ has a natural structure as a $PD(n+1)$ pair, and $\partial v = \partial_0 v \cup \partial_1 v$ naturally has the structure of a family of $PD(n+1)$ groups obtained by gluing the pairs in $\partial_0 v$ and $\partial_1 v$ along their boundaries. Thus ∂v forms a sort of boundary of v but since the groups in ∂v may not inject into $G(v)$, we do not have a notion of Poincaré duality for the pair $(G(v), \partial v)$. However, in the case of a 3-dimensional Poincaré duality pair $(G, \partial G)$, the family ∂v consists of closed surfaces, for each vertex v of Γ_G , and we showed in [7] that if v is a V_0 -vertex of Γ_G then the pair $(G(v), \partial v)$ is a 3-dimensional manifold.

In this paper, we will prove the following analogue of Johannson's Deformation Theorem [3] for $PD3$ pairs.

Theorem 4.1.1. [4.2.12] *Let $(G, \partial G)$ and $(H, \partial H)$ be two $PD3$ pairs with G isomorphic to H , and let Γ_G and Γ_H be the corresponding isomorphic bipartite graphs of groups. If v in Γ_G and w in Γ_H are corresponding V_1 -vertices, then the isomorphism carries $\partial_1 v$ to $\partial_1 w$, and $\partial_0 v$ to $\partial_0 w$, and ∂v isomorphically to ∂w .*

That $\partial_1 v$ is carried to $\partial_1 w$ follows from the fact that Γ_G depends only on G , and not on ∂G , as was shown in [10]. The new element is that $\partial_0 v$ is carried to $\partial_0 w$.

The approach here gives another proof of Johannson's Deformation Theorem in the 3-manifold case. The arguments are sketched at the end in section 3.

4.2 Proof of the Main result

For $PD3$ pairs $(G, \partial G)$ and $(H, \partial H)$, the decompositions Γ_G and Γ_H induce decompositions of ∂G and ∂H giving rise to $\partial_0 v$ and $\partial_0 w$. These decompositions are described in detail in [10] using $K(\pi, 1)$ spaces representing the group pairs. We take spaces $(M, \partial M)$ and $(N, \partial N)$ to represent the above pairs and start with a split homotopy equivalence $f : M \rightarrow N$ with inverse $g : N \rightarrow M$. As the edge spaces are now 2-dimensional annuli and tori, we can homotop f and g to be homeomorphisms on the edge spaces. For $i = 0, 1$, we denote by V_i the subspace of M which is the union of all V_i -vertex spaces, and let W_0 and W_1 denote the corresponding subspaces of N . Thus $M = V_0 \cup V_1$, and $N = W_0 \cup W_1$.

and $f(V_0) \subset W_0$, $f(V_1) \subset W_1$, and f is a homeomorphism on the edge spaces $V_0 \cap V_1$.

A crucial role in our arguments is played by the fact that an essential annulus in a $PD3$ pair $(G, \partial G)$ is enclosed by a V_0 -vertex of Γ_G . In particular, if an essential annulus in $(G, \partial G)$ is enclosed by a V_1 -vertex of Γ_G , it must be homotopic into an edge annulus of that vertex.

Now we establish some notation to be used throughout this section. We fix corresponding V_1 -vertices v and w of Γ_G and Γ_H respectively, let X be the component of V_1 corresponding to v , and let Y be the component of W_1 , corresponding to w . Thus f carries X to Y with $\partial_1 X$ going homeomorphically to $\partial_1 Y$, where $\partial_1 X$ and $\partial_1 Y$ denote the unions of the annuli and tori which form the edge surfaces of X and Y respectively. Also $\partial_0 X$ and $\partial_0 Y$ denote the unions of the surfaces $X \cap \partial M$ and $Y \cap \partial N$ respectively. Finally ∂X denotes the union $\partial_0 X \cup \partial_1 X$, and ∂Y denotes the union $\partial_0 Y \cup \partial_1 Y$.

For each component \bar{t} of $\partial_0 X$, we want to deform \bar{t} into $\partial_0 Y$, and then obtain a homeomorphism from ∂X to ∂Y . Let M_t denote the cover of M corresponding to $\pi_1(\bar{t})$, and let N_t denote the corresponding cover of N . Denote by t the lift of \bar{t} into M_t , and by X_t the component of the pre-image of X in M_t which contains t . Let Y_t denote the corresponding component of the pre-image of Y in N_t . Let $\partial_0 X_t$ denote the pre-image in X_t of $\partial_0 X$, and let $\partial_1 X_t$ denote the pre-image in X_t of $\partial_1 X$. Define $\partial_0 Y_t$ and $\partial_1 Y_t$ in the same way. Finally let f_t denote the induced homotopy equivalence from M_t to N_t .

Let A be an annulus component of $\partial_1 X$, and let B be the corresponding annulus component of $\partial_1 Y$. We are assuming that f maps A to B by a homeomorphism inducing the given isomorphism on the corresponding edge groups of Γ_G and Γ_H . But there are two possible isotopy classes for this homeomorphism, and we may need to alter the initial choice. We define a "flip" on A to be a map from M to itself which preserves each of V_0 and V_1 , is the identity outside some small neighbourhood of A , and whose restriction to A is a homeomorphism which interchanges the components of ∂A . Further a flip on A is homotopic to the identity map of M by a homotopy supported on a small neighbourhood of A . Thus by composing f with a flip on A , we can homotop f to change the initial choice of isotopy class of homeomorphism from A to B , and this homotopy is supported on a small neighbourhood of A .

We start by considering some special cases.

Lemma 4.2.1. *Suppose that one of the following holds:*

1. *A component \bar{t} of $\partial_0 X$ is an annulus.*

2. There is a component \bar{t} of $\partial_0 X$ such that \bar{t} is not closed, and $\pi_1(\bar{t})$ has finite index in $\pi_1(X)$.

Then v and w are isolated V_1 -vertices of Γ_G and Γ_H respectively. Further we can homotop $f : M \rightarrow N$ to arrange that it maps $\partial_1 X$ to $\partial_1 Y$ by a homeomorphism, and maps $\partial_0 X$ to $\partial_0 Y$ by a homeomorphism. Thus f also maps ∂X to ∂Y by a homeomorphism.

Proof. 1) As v is a V_1 -vertex of $\Gamma_H = \Gamma_{n,n+1}^c$, and \bar{t} is an annulus in X with boundary in $\partial_1 X$, we can apply Proposition 2.5 of [7]. This tells us that either $\partial\bar{t}$ lies in a single component s of $\partial_1 X$, and that \bar{t} is homotopic into s fixing $\partial\bar{t}$, or that v is isolated. The first case would imply that s is homotopic into \bar{t} fixing ∂s , and so is homotopic into ∂M fixing ∂s , which contradicts the fact that s is an essential annulus in $(M, \partial M)$. It follows that v is an isolated V_1 -vertex of Γ_G , as required. In particular, $\pi_1(X)$ is infinite cyclic. Hence $\pi_1(Y)$ is also infinite cyclic, so that any component of $\partial_0 Y$ must be an annulus. Now applying Proposition 2.5 of [7] again shows that Y is an isolated V_1 -vertex of Γ_H , as required. Thus by flipping on one of the annuli in $\partial_1 X$ if needed, we can homotop f to map $\partial_1 X$ to $\partial_1 Y$ by a homeomorphism, and simultaneously map $\partial_0 X$ to $\partial_0 Y$ by a homeomorphism. This completes the proof of part 1) of the lemma.

2) We will show that some component of $\partial_0 X$ is an annulus, so that the result follows from part 1). We use the above notation. As $\pi_1(\bar{t})$ has finite index in $\pi_1(X)$, it follows that X_t is compact. In particular, $\partial_0 X_t$ is also compact. As the inclusion of t in X_t is a homotopy equivalence, there is a retraction ρ of X_t to t .

If $\partial_0 X_t$ equals t , and A is an annulus component of $\partial_1 X_t$, the retraction ρ induces a map from A to t which sends ∂A by a homeomorphism to two components of ∂t . Hence the map from A to t is a proper map of degree 1, and so induces a surjection $\pi_1(A) \rightarrow \pi_1(t)$. This implies that t is an annulus, as required.

If $\partial_0 X_t$ is not equal to t , we let s be another component of $\partial_0 X_t$. Any loop λ in s is homotopic in X_t into t , and so determines an annulus in X_t . If s is not an annulus, we can choose λ to be an essential non-peripheral loop in s . It follows that the annulus in X_t determined by λ is π_1 -injective, and cannot be properly homotoped in X_t into $\partial_1 X_t$, nor into $\partial_0 X_t$, while keeping its boundary in $\partial_0 X_t$. It follows that this annulus is essential in $(M, \partial M)$ and cannot be homotoped into $\partial_1 X_t$, which contradicts the fact that any essential annulus in a $PD3$ pair $(G, \partial G)$ is enclosed by a V_0 -vertex of Γ_G . This contradiction shows that s must be an annulus, which completes the proof of part 2) of the lemma. \square

Next we consider the case in which \bar{t} is closed. This case was considered in [11] and the proof here is similar (see also [4]).

Lemma 4.2.2. *Using the above notation, if \bar{t} is a component of $\partial_0 X$ which is a closed surface, then we can homotop f to arrange that f maps \bar{t} to a component of $\partial_0 Y$ by a homeomorphism.*

Proof. Again we use the above notation. We claim that each component of $\partial M_t - t$ is contractible. For suppose a component r of $\partial M_t - t$ is not contractible. Then there is an annulus A from r to t , since $\pi_1(t) \rightarrow \pi_1(M_t)$ is an isomorphism. As r and t are distinct components of ∂M_t , the annulus A is essential in M_t , and so must be properly homotopic to an annulus in $\partial_1 X_t$. As t is closed, this is a contradiction which proves the claim.

The long exact homology sequence for the pair $(M_t, \partial M_t)$ with integer coefficients yields the exact sequence

$$H_2(\partial M_t) \rightarrow H_2(M_t) \rightarrow H_2(M_t, \partial M_t) \rightarrow H_1(\partial M_t) \rightarrow H_1(M_t).$$

As the inclusion of t into M_t is a homotopy equivalence, and the other components of ∂M_t are contractible, it follows that the first and last maps in this sequence are isomorphisms. Also $H_3(M_t) = H_3(t) = 0$. It follows that $H_3(M_t, \partial M_t) = 0$ and $H_2(M_t, \partial M_t) = 0$. The first equality tells us that M_t is not compact, and the second implies, by duality, that $H_c^1(M_t) = 0$. Since f_t is a proper homotopy equivalence from M_t to N_t , it follows that N_t is not compact and $H_c^1(N_t) = 0$. Hence $H_3(N_t, \partial N_t) = 0$, and by duality $H_2(N_t, \partial N_t) = 0$. As $H_2(N_t) \cong \mathbb{Z}$, the long exact homology sequence of the pair $(N_t, \partial N_t)$ shows that $H_2(\partial N_t) \cong \mathbb{Z}$. Thus there is exactly one closed component s of ∂N_t , and the induced map $H_2(s) \rightarrow H_2(N_t)$ is an isomorphism. Now let ρ denote a retraction $M_t \rightarrow t$, and consider the composite map $s \subset N_t \xrightarrow{g_t} M_t \xrightarrow{\rho} t$. Since each of these three maps induces an isomorphism on H_2 , the composite map $s \rightarrow t$ has degree 1. Thus the induced map $\pi_1(s) \rightarrow \pi_1(t)$ is surjective and hence an isomorphism. Thus s is a retract of N_t , and $f_t|_t : t \rightarrow N_t$ can be deformed into s .

We next want to show that s is actually in Y_t . Since each component of $\partial M_t - t$ is contractible, there are no annuli in $\partial_1 X_t$, and so no annuli in $\partial_1 Y_t$. If s is not in Y_t , the fact that s is homotopic into Y_t implies that there must be a non-contractible component of $\partial_1 Y_t$ which can only be a torus. Denote this torus by T , and note that as s is closed, $\pi_1(s)$ must be of finite index in $\pi_1(T)$. Consider the images \bar{s} and \bar{T} in N of s and T respectively. If we cut N along \bar{T} we get one or two $PD3$ pairs, depending on whether \bar{T} separates N .

Let $(K, \partial K)$ denote the pair such that ∂K contains \bar{s} . Then ∂K also contains one or two copies of \bar{T} , depending on whether \bar{T} separates N . Since \bar{s} and \bar{T} carry commensurable subgroups of $\pi_1(Y)$, it follows from Lemma 2.2 of [5] that $(K, \partial K)$ must be trivial, meaning that ∂K consists of two copies of K . Thus ∂K consists entirely of one copy of \bar{s} and one copy of \bar{T} , and each carries K . But this implies that \bar{T} separates N and splits $\pi_1(N)$ trivially, which is a contradiction. It follows that s must lie in Y_t , and hence that t can be deformed into s staying in Y_t . Thus we can homotop f to arrange that f maps \bar{t} to a component of $\partial_0 Y$ by a homeomorphism, as required. \square

The main part of the proof of Theorem 4.2.12 will be the remaining cases in which \bar{t} has boundary, which we handle with a sequence of propositions. By Lemma 4.2.1, we can assume that no component of $\partial_0 X$ is an annulus, and that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$. Using our previous notation, the cover X_t of X contains t , and X_t and M_t are non-compact.

Proposition 4.2.3. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then the following hold.*

1. *Each component of $\partial_1 X_t$ which covers an annulus component of $\partial_1 X$ is either an annulus meeting t in a single boundary component, or is contractible.*
2. *Let C be a non-contractible component of $\partial_0 X_t$, other than t . Then there is an annulus component A of $\partial_1 X_t$, such that one component of ∂A is contained in C , and the other component of ∂A is contained in t . Further $\pi_1(C)$ is infinite cyclic.*
3. *Each component of $\partial_1 X_t$ which covers a torus component of $\partial_1 X$ is contractible.*

Proof. 1) Suppose that C is an annulus component of $\partial_1 X_t$ which does not meet t . There is a π_1 -injective annulus A in X_t joining a component of ∂C to t . As ∂C lies in a component s of $\partial_0 X_t$, the annulus A is in $(M_t, \partial M_t)$. As s and t are distinct components of $\partial_0 X_t$, it follows that A is essential in $(M_t, \partial M_t)$, and so must be homotopic into $\partial_1 X_t$ while keeping ∂A in $\partial_0 X_t$. But this implies that either C meets t , or that s is an annulus, either of which contradicts our hypotheses. We conclude that any annulus component of $\partial_1 X_t$ must meet t .

Let A be a component of $\partial_1 X_t$ which meets t . Thus A is an annulus. Suppose that ∂A is contained in t . There is a retraction of X_t to t and it maps A to t sending

∂A by a homeomorphism to two components of ∂t . Hence this map $A \rightarrow t$ has degree 1, which implies that t is an annulus. This contradiction shows that each component of $\partial_1 X_t$ which meets t is a compact annulus which meets t in exactly one boundary component. This completes the proof of part 1).

2) As C is non-contractible, there is a π_1 -injective annulus joining C to t . This cannot be homotopic into $\partial_0 X_t$ while keeping its boundary in $\partial_0 X_t$, and so it must be homotopic into $\partial_1 X_t$ while keeping its boundary in $\partial_0 X_t$. This implies that there is an annulus component A of $\partial_1 X_t$, such that one component of ∂A is contained in C , and the other component of ∂A is contained in t . Further this argument shows that any loop in C is homotopic into $A \cap C$, showing that $\pi_1(C)$ must be infinite cyclic, as required.

3) Suppose there is a non-contractible component C of $\partial_1 X_t$ which covers a torus component T of $\partial_1 X$, and let H be an infinite cyclic subgroup of $\pi_1(C)$. As $\pi_1(T)$ normalises H , it follows that H is a subgroup of $\pi_1(t)$ which has infinite index in its normalizer. Consider the cover M_H of M , with $\pi_1(M_H) = H$, and let t_H denote the component of the pre-image of t with $\pi_1(t_H) = H$. As H has infinite index in its normalizer, M_H has infinitely many components of ∂M_H which contain translates of t_H and have fundamental group H . It follows that there are infinitely many distinct annuli in $(M_H, \partial M_H)$, all carrying H , so that there are crossing such annuli. As these annuli are all enclosed by the V_1 -vertex v of Γ_H , corresponding to X , this is a contradiction, which completes the proof of part 3). \square

Now let A_1, \dots, A_n denote the annuli of $\partial_1 X_t$ which meet t , and let $\partial A_i = \{a_1, a'_i\}$ with a_i in t . We have corresponding annuli B_1, \dots, B_n in $\partial_1 Y_t$ with $\partial B_i = \{b_i, b'_i\}$. We may assume that f_t carries A_i homeomorphically to B_i with a_i going to b_i initially. Note that in the covering projections $p_t : M_t \rightarrow M$, $q_t : N_t \rightarrow N$ some of these annuli may be identified. An annulus A in $\partial_1 X$ which meets \bar{t} lifts to two annuli in $\partial_1 X_t$ if $\partial A \subset \partial \bar{t}$, and lifts to one annulus otherwise.

Since the cover $p_t : M_t \rightarrow M$ is formed with respect to the image of $\pi_1(t)$, we have a clearer picture of the cover M_t than of N_t . We can use this to obtain some information about the homology of X_t , as follows. Each annulus A_i lies in a component, say P_i , of the closure of $M_t - X_t$. As the inclusion of X_t in M_t is a homotopy equivalence, it follows that X_t meets P_i only in A_i , and that the inclusion of A_i into P_i is a homotopy equivalence. Note that $\partial_0 X_t$ equals the intersection $\partial M_t \cap X_t$. We let $\partial_0 P_i$ denote the intersection $\partial M_t \cap P_i$. Proposition [4.2.3](#) tells us that if Θ is a component of $\partial_1 X_t$ other than the A_i 's, then Θ is

contractible. As the inclusion of X_t in M_t is a homotopy equivalence, it follows that Θ lies in a component, say P_Θ , of the closure of $M_t - X_t$, that X_t meets P_Θ only in Θ , and that P_Θ is contractible. We have that M_t is the union of X_t , the P_i 's, and the P_Θ 's, and that ∂M_t is the union of $\partial_0 X_t$, the $\partial_0 P_i$'s, and the $\partial_0 P_\Theta$'s.

Proposition 4.2.4. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then $H_2(X_t, \partial_0 X_t) \cong \mathbb{Z}^n$, and is freely generated by $[A_1], \dots, [A_n]$.*

Proof. Recall that $\partial_0 X_t$ consists of t , various components containing some a'_i , and perhaps some other components. Proposition 4.2.3 tells us that these extra components are contractible, and that any component of $\partial_0 X_t$ which contains some a'_i has infinite cyclic fundamental group. The long exact homology sequence of the pair $(X_t, \partial_0 X_t)$ implies that

$$0 \rightarrow H_2(X_t, \partial_0 X_t) \rightarrow H_1(\partial_0 X_t) \rightarrow H_1(X_t) \rightarrow 0$$

is exact, as $H_2(X_t) = H_2(t) = 0$, and $H_1(\partial_0 X_t) \cong H_1(t) \oplus \sum_{i=1}^n H_1(a'_i)$, and $H_1(t)$ maps isomorphically onto $H_1(X_t)$. Since a_i and a'_i map to the same elements in $H_1(X_t)$, the proposition follows. \square

Next we prove the following.

Proposition 4.2.5. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then $H_2(X_t, \partial_0 X_t) \rightarrow H_2(M_t, \partial M_t)$ is an isomorphism and both are freely generated by $[A_1], \dots, [A_n]$.*

Proof. Similar arguments to those in Proposition 4.2.3 tell us that the components of $\partial_0 P_i$ which do not contain a_i or a'_i are contractible, and the other components of $\partial_0 P_i$ have infinite cyclic fundamental group. Also, for each contractible component Θ of $\partial_1 X_t$, all components of $\partial_0 P_\Theta$ are contractible. It follows that the inclusion of A_i into P_i induces an isomorphism from $H_1(\partial A_i)$ to $H_1(\partial_0 P_i)$ and an injection from $H_0(\partial A_i)$ to $H_0(\partial_0 P_i)$, and similar statements hold when Θ is the universal cover of an annulus. If Θ is the universal cover of a torus, we note that $H_1(\partial_0 P_\Theta)$ is zero. As the inclusion of A_i into P_i is a homotopy equivalence, it follows from the long exact homology sequences of the pairs $(A_i, \partial A_i)$ and $(P_i, \partial_0 P_i)$ that $H_2(A_i, \partial A_i) \rightarrow H_2(P_i, \partial_0 P_i)$ is an isomorphism and $H_1(A_i, \partial A_i) \rightarrow H_1(P_i, \partial_0 P_i)$ is an injection. If Θ is the universal cover of an annulus or torus, then $H_2(\Theta, \partial \Theta) = H_2(P_\Theta, \partial_0 P_\Theta) = 0$,

and $H_1(\Theta, \partial\Theta) \rightarrow H_1(P_\Theta, \partial_0 P_\Theta)$ is an injection. Now we consider the Mayer-Vietoris sequence for the pair $(M_t, \partial M_t)$ expressed as the union of $(X_t, \partial_0 X_t)$ and $(\cup P_i, \cup \partial_0 P_i) \cup (\cup P_\Theta, \cup \partial_0 P_\Theta)$. We obtain the short exact sequence

$$0 \rightarrow \sum_{i=1}^n H_2(A_i, \partial A_i) \oplus \sum_{\Theta} H_2(\Theta, \partial\Theta) \rightarrow \sum_{i=1}^n H_2(P_i, \partial_0 P_i) \oplus \sum_{\Theta} H_2(P_\Theta, \partial_0 P_\Theta) \oplus H_2(X_t, \partial_0 X_t) \rightarrow H_2(M_t, \partial M_t) \rightarrow 0$$

where the first term is $H_3(M_t, \partial M_t)$ which is zero as M_t is not compact. The final term being zero reflects the fact that the boundary map from $H_2(M_t, \partial M_t)$ is zero, as the next map in the Mayer-Vietoris sequence is injective.

As $H_2(\Theta, \partial\Theta) = H_2(P_\Theta, \partial_0 P_\Theta) = 0$, and $H_2(A_i, \partial A_i) \rightarrow H_2(P_i, \partial_0 P_i)$ is an isomorphism, it follows that $H_2(X_t, \partial_0 X_t) \rightarrow H_2(M_t, \partial M_t)$ is an isomorphism, as required. \square

Next we want to apply the same arguments to N_t, Y_t and the components of the closure of $N_t - Y_t$. to obtain the analogous isomorphism but without the information about the generators. For Y_t we have only the following.

Proposition 4.2.6. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then $H_2(Y_t, \partial_0 Y_t) \rightarrow H_2(N_t, \partial N_t)$ is an isomorphism.*

Proof. Proposition [4.2.3](#) shows that the components of $\partial_1 X_t$ consist of the annuli A_1, \dots, A_n together with contractible components. As we have a graph of groups isomorphism between Γ_G and Γ_H , it follows that the components of $\partial_1 Y_t$ consist of the annuli B_1, \dots, B_n together with contractible components. Each annulus B_i lies in a component, say Q_i , of the closure of $N_t - Y_t$, and Y_t meets Q_i only in B_i . If Θ is one of these contractible components of $\partial_1 Y_t$, then Θ lies in a component, say Q_Θ , of the closure of $N_t - Y_t$, such that Y_t meets Q_Θ only in Θ , and Q_Θ is contractible. We have that N_t is the union of Y_t , the Q_i 's, and the Q_Θ 's, and that ∂N_t is the union of $\partial_0 Y_t$, the $\partial_0 Q_i$'s, and the $\partial_0 Q_\Theta$'s. Further, as in the proof of Proposition [4.2.5](#), the components of $\partial_0 Q_i$ which do not contain b_i or b'_i are contractible, and the other components of $\partial_0 Q_i$ have infinite cyclic fundamental group. Also, for each contractible component Θ of

$\partial_1 Y_t$, all components of $\partial_0 Q_\Theta$ are contractible. As the inclusion of B_i into Q_i is a homotopy equivalence, it follows from the long exact homology sequences of the pairs $(B_i, \partial B_i)$ and $(Q_i, \partial_0 Q_i)$ that $H_2(B_i, \partial B_i) \rightarrow H_2(Q_i, \partial_0 Q_i)$ is an isomorphism and $H_1(B_i, \partial B_i) \rightarrow H_1(Q_i, \partial_0 Q_i)$ is an injection. If Θ is the universal cover of an annulus or torus, then $H_2(\Theta, \partial\Theta) = H_2(Q_\Theta, \partial_0 Q_\Theta) = 0$, and $H_1(\Theta, \partial\Theta) \rightarrow H_1(Q_\Theta, \partial_0 Q_\Theta)$ is an injection. Now as in the proof of Proposition 4.2.5, we apply the Mayer-Vietoris sequence for the pair $(N_t, \partial N_t)$ expressed as the union of $(Y_t, \partial_0 Y_t)$ and $(\cup Q_i, \cup \partial_0 Q_i) \cup (\cup Q_\Theta, \cup \partial_0 Q_\Theta)$, to deduce the result. But note that although this argument is very similar to the proof of Proposition 4.2.5, the groups involved may be very different, as ∂M_t and ∂N_t may be different. \square

However duality implies that $H_2(M_t, \partial M_t) \cong H_c^1(M_t) \cong H_c^1(N_t) \cong H_2(N_t, \partial N_t)$. Thus

Proposition 4.2.7. *All the groups in Propositions 4.2.5 and 4.2.6 are free abelian of rank n .*

In the 3-manifold setting, the map $H_2(Y_t, \partial_0 Y_t) \rightarrow H_2(Y_t, \partial Y_t)$ is zero, and later we will be able to show this holds in the present setting. But we will begin by proving something weaker in Proposition 4.2.10 below.

We write A for the union of the A_i 's, and B for the union of the B_i 's. Note that we have homotopy equivalences of pairs $(M_t, A) \rightarrow (N_t, B)$ and $(M_t, \partial A) \rightarrow (N_t, \partial B)$.

Proposition 4.2.8. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then $H_2(M_t, \partial A)$ is free abelian of rank $n+1$, generated by $[t]$ and $[A_1], \dots, [A_n]$.*

Proof. Consider the long exact homology sequence of the triple $(M_t, \partial M_t, \partial A)$. As M_t is not compact, we have $H_3(M_t, \partial M_t) = 0$. Also Proposition 4.2.5 implies that the boundary map $H_2(M_t, \partial M_t) \rightarrow H_1(\partial M_t, \partial A)$ is zero. Thus we obtain the short exact sequence

$$0 \rightarrow H_2(\partial M_t, \partial A) \rightarrow H_2(M_t, \partial A) \rightarrow H_2(M_t, \partial M_t) \rightarrow 0.$$

Recall that ∂M_t is the union of $\partial_0 X_t$, $\cup \partial_0 P_i$ and $\cup \partial_0 P_\Sigma$. Further the proofs of Propositions 4.2.4 and 4.2.5 show that each component of $\partial_0 X_t$, $\cup \partial_0 P_i$ and $\cup \partial_0 P_\Sigma$ other than t , is either contractible and disjoint from t , or meets t in a

single boundary component and has infinite cyclic fundamental group. It follows that $H_2(\partial M_t, \partial A)$ is infinite cyclic generated by $[t]$. Now the above short exact sequence implies the result of the proposition. \square

The homotopy equivalence $(M_t, \partial A) \rightarrow (N_t, \partial B)$ immediately implies the following result.

Proposition 4.2.9. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then $H_2(N_t, \partial B)$ is free abelian of rank $n + 1$, generated by $[f(t)]$ and $[B_1], \dots, [B_n]$.*

Now we are able to show the following result.

Proposition 4.2.10. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then the map $H_2(Y_t, \partial_0 Y_t) \rightarrow H_2(Y_t, \partial Y_t)$ sends $[f(t)]$ to zero.*

Proof. Consider the long exact homology sequence of the triple $(N_t, \partial N_t, \partial B)$. As N_t is not compact, we have $H_3(N_t, \partial N_t) = 0$, as in the proof of Proposition [4.2.7](#). Thus we obtain the exact sequence

$$0 \rightarrow H_2(\partial N_t, \partial B) \rightarrow H_2(N_t, \partial B) \rightarrow H_2(N_t, \partial N_t).$$

We do not know that the boundary map $H_2(N_t, \partial N_t) \rightarrow H_1(\partial N_t, \partial B)$ is zero, nor do we know the rank of $H_2(\partial N_t, \partial B)$. However Proposition [4.2.7](#) tells us that $H_2(N_t, \partial N_t)$ is free abelian of rank n . Thus the image of the map $H_2(N_t, \partial B) \rightarrow H_2(N_t, \partial N_t)$ has rank at most n . In particular, the generators $[f(t)]$ and $[B_1], \dots, [B_n]$ of $H_2(N_t, \partial B)$ are mapped to dependent elements of $H_2(N_t, \partial N_t)$. Now Proposition [4.2.6](#) tells us that $H_2(Y_t, \partial_0 Y_t) \rightarrow H_2(N_t, \partial N_t)$ is an isomorphism. Thus the elements $[f(t)]$ and $[B_1], \dots, [B_n]$ are dependent elements of $H_2(Y_t, \partial_0 Y_t)$. As the map $H_2(Y_t, \partial_0 Y_t) \rightarrow H_2(Y_t, \partial Y_t)$ sends each $[B_i]$ to zero, it must also send $[f(t)]$ to zero, as required. \square

Now we come to the key argument.

Proposition 4.2.11. *Using the above notation, let \bar{t} be a component of $\partial_0 X$ with non-empty boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$, and suppose that no component of $\partial_0 X$ is an annulus. Then there is a component S of $\partial_0 Y_t$ whose boundary is contained in ∂B , and contains exactly one component from each ∂B_i , such that $\pi_1(S) \rightarrow \pi_1(N_t)$ is an isomorphism.*

Proof. As ∂t is the union of the a_i 's, it follows that $f(\partial t)$ is the union of the b_i 's. Thus Proposition 4.2.10 implies that there is a (possibly disconnected) compact surface S in ∂Y_t whose boundary is the union of the b_i 's. If S is not contained in $\partial_0 Y_t$, it must contain some B_i . By replacing S by the closure of $S - B_i$, we can replace S by a new (possibly disconnected) compact surface in ∂Y_t whose boundary is contained in ∂B , and contains exactly one component from each ∂B_i . By repeating this process as needed, we will eventually find a (possibly disconnected) compact surface S in $\partial_0 Y_t$ whose boundary is contained in ∂B , and contains exactly one component from each ∂B_i .

Let s be a component of S and consider the composite map $s \xrightarrow{g_t} X_t \xrightarrow{\rho} t$, where ρ is a retraction of X_t to t . The resulting map is π_1 -injective, carries each boundary component of s to a boundary component of t by a homeomorphism, and sends distinct components of ∂s to distinct components of ∂t . It follows that the composite map $s \rightarrow t$ has degree 1. Hence it is onto on π_1 , and so an isomorphism on π_1 . It follows that this map $s \rightarrow t$ is properly homotopic to a homeomorphism, so that s must be equal to S . Hence S satisfies the conclusion of the proposition, as required. \square

Now we are ready to complete the proof of Theorem 4.2.12.

Theorem 4.2.12. *Let $(G, \partial G)$ and $(H, \partial H)$ be two PD3 pairs with G isomorphic to H , and let Γ_G and Γ_H be the corresponding isomorphic bipartite graphs of groups. If v in Γ_G and w in Γ_H are corresponding V_1 -vertices, then the isomorphism carries $\partial_1 v$ to $\partial_1 w$, and $\partial_0 v$ to $\partial_0 w$, and ∂v isomorphically to ∂w .*

Proof. If there is a component \bar{t} of $\partial_0 X$, which is a closed surface, then the result was proved in Lemma 4.2.2.

If there is a component \bar{t} of $\partial_0 X$, which is an annulus, or if there is a component \bar{t} of $\partial_0 X$ such that \bar{t} is not closed, and $\pi_1(\bar{t})$ has finite index in $\pi_1(X)$, then the result was proved in Lemma 4.2.1.

Now we consider the remaining cases. Suppose that no component of $\partial_0 X$ is an annulus, and let \bar{t} be a component of $\partial_0 X$, which has boundary, such that $\pi_1(\bar{t})$ has infinite index in $\pi_1(X)$. Proposition 4.2.11 produces a component S of $\partial_0 Y_t$ such that ∂S consists of one and exactly one from each pair $\{b_j, b'_j\}$, and $\pi_1(S) \rightarrow \pi_1(N_t)$ is an isomorphism. Let \bar{S} denote the image in $\partial_0 Y$ of S .

Under our indexing, $f(a_i) = b_i$ and ∂S contains only one of $\{b_i, b'_i\}$. Thus by deforming along B_i if necessary, we can push $f_t(t)$ homeomorphically onto S . We will use flips to alter f to arrange that, after applying these flips, $f(\bar{t})$ can be deformed into \bar{S} while fixing $\partial \bar{t}$. By repeating for all components of $\partial_0 X$,

we will arrange that, after applying certain flips, $f(\partial X)$ can be deformed to a homeomorphism from ∂X to ∂Y , while fixing $\partial_1 X$.

There are two cases to consider, depending on whether two of the A_i have the same image in M , under p_t .

Case 1: $p_t(A_1) = p_t(A_2)$, and so $q_t(B_1) = q_t(B_2)$.

Recall that in our notation, $f_t(a_1) = b_1$ and $f_t(a_2) = b_2$. Thus under these identifications, we have $p_t(a'_1) = p_t(a_2)$, $p_t(a_1) = p_t(a'_2)$, $q_t(b'_1) = q_t(b_2)$, $q_t(b_1) = q_t(b'_2)$. Moreover, ∂S contains one of $\{b_1, b'_1\}$ and one of $\{b_2, b'_2\}$.

Case 1a: ∂S contains $b_1 = f_t(a_1)$.

We claim that in this case $b_2 = f_t(a_2)$ is in ∂S . If not ∂S contains b_1 and b'_2 which are identified under q_t . Thus, the image of ∂X_t contains the image of S with b_1 and b'_2 identified and also the image of B_1 and B_2 . Thus we have a branched surface at $q_t(b_1)$ with three branches, whereas $\partial X = \partial_0 X \cup \partial_1 X$ is a closed surface. Therefore, if ∂S contains b_1 , it also contains b_2 and so we can homotop $f_t(t)$ into S fixing ∂t . Hence we can homotop $f(\bar{t})$ into the image of S in N fixing $\partial \bar{t}$, as required.

Case 1b: ∂S contains $b'_1 = f_t(a'_1)$.

Arguing as in case 1a), it follows that ∂S must consist of b'_1 and b'_2 . We also have $f_t(a_1) = b_1$ and $f_t(a_2) = b_2$. Thus it is not possible to homotop $f_t(t)$ into S fixing ∂t . However, if we compose f with simultaneous flips on A_1 and A_2 , the images of a_1 and a'_1 are interchanged, as are the images of b_1 and b'_1 . We have now arranged that $f_t(a_1)$ and $f_t(a_2)$ are in ∂S , so that we can homotop $f_t(t)$ into S fixing ∂t . Hence we can homotop $f(\bar{t})$ into \bar{S} while fixing $\partial \bar{t}$, as required.

Case 2: $p_t(A_1) \neq p_t(A_2)$, and so $q_t(B_1) \neq q_t(B_2)$.

This case is easier. If $f(a_i) = b_i$ is in ∂S , then we leave f as it is. Otherwise, we flip on A_i to arrange that $f(a_i) = b_i$. After a finite number of such flips we have new maps f, f_t which send ∂t homeomorphically onto ∂S and which induce the previous map on the fundamental groups and thus map $\pi_1(t)$ isomorphically to $\pi_1(S)$. In particular, we can homotop $f_t(t)$ into S fixing ∂t , as required.

In conclusion, we have shown the following. We start with a given map $f : M \rightarrow N$ such that $f(V_0) \subset W_0$, $f(V_1) \subset W_1$, and f is a homeomorphism on the edge spaces $V_0 \cap V_1$. Suppose that $\partial_0 X$ has no annulus component, and also has no component \bar{t} such that \bar{t} is not closed, and $\pi_1(\bar{t})$ has finite index in $\pi_1(X)$. If \bar{t} is a component of $\partial_0 X$, then after applying flips on some of the annuli in $\partial_1 X$ which meet \bar{t} , we can homotop $f(\bar{t})$ fixing $\partial \bar{t}$, to arrange that f maps \bar{t} to a component \bar{S} of $\partial_0 Y$ by a homeomorphism.

Now we examine how to inductively extend the above procedure to the remaining components of $\partial_0 X$. Let $\bar{r} \neq \bar{t}$ be a component of $\partial_0 X$. As above, after

applying flips on some of the annuli in $\partial_1 X$ which meet \bar{r} , we can homotop $f(\bar{r})$ fixing $\partial\bar{r}$, to arrange that f maps \bar{r} to a component \bar{R} of $\partial_0 Y$ by a homeomorphism. We claim that \bar{R} cannot equal \bar{S} . For if they were equal, then \bar{r} and \bar{t} would be homotopic to each other in X . As neither is an annulus, choosing a non-peripheral curve in \bar{t} yields a π_1 -injective annulus joining \bar{r} and \bar{t} which cannot be homotopic into $\partial_1 X$ while keeping its boundary in $\partial_0 X$. This is an essential annulus in $(M, \partial M)$ which cannot be homotoped into a V_0 -vertex which is impossible.

We conclude that \bar{R} cannot equal \bar{S} , so that they are disjoint. Now there is no problem except possibly if there is an annulus component A of $\partial_1 X$ which meets both \bar{r} and \bar{t} . If we need to flip on A in order to map \bar{r} to \bar{R} , then before the flip, the circle $A \cap \bar{r}$ must be mapped to the same component of ∂B as $A \cap \bar{t}$, which is impossible, as the restriction of f to A is a homeomorphism.

Thus after a finite number of steps we can modify f by a homotopy, which is a product of flips, which preserves the decomposition, and thus carries each group in $\partial_1 X$ to a group in $\partial_1 Y$ and similarly $\partial_0 X$ to $\partial_0 Y$ and giving a homeomorphism ∂X to ∂Y . This completes the proof of Theorem [4.2.12](#). \square

4.3 Discussion of Johansson's Deformation Theorem

We sketch below how to adapt the arguments of the previous sections to a proof of Johansson's Deformation Theorem (JDT). The proof of JDT consists of three elements.

- 3.1 The existence of JSJ decomposition.
- 3.2 The enclosing property of the characteristic manifold, that is, that essential maps of annuli and tori can be homotoped into the characteristic manifold.
- 3.3 Deforming $\partial_0(v)$ for a V_1 vertex, that is a vertex outside the characteristic manifold, into $\partial_0(w)$ of the corresponding vertex of the target manifold.

An elementary construction of the JSJ decomposition is given in [\[6\]](#). Note that there is an omission in the matching lemma 3.4 of [\[6\]](#) which is corrected in Figure 4. of [\[7\]](#).

That the characteristic submanifold as constructed in [\[6\]](#) has the enclosing property was proved by Scott in [\[8\]](#).

This takes care of steps 3.1 and 3.2 above. We next outline how to adapt the arguments of section two to a proof of JDT.

For Lemma [4.2.1](#) part 1 we can use Proposition 3.2 of [\[6\]](#), or prove it directly. The crucial part of the proof as in [\[9\]](#) is that the map $H_2(v, \partial_0 v) \rightarrow H_2(v, \partial v)$ is trivial for a V_1 -vertex. This follows from Proposition [4.2.4](#). Thus Propositions [4.2.5](#) to [4.2.10](#) are unnecessary. The proofs after Proposition [4.2.10](#) proceed as before.

We note that only a version of the annulus theorem is needed in the proof of JDT. Thus we only need that the peripheral characteristic submanifold has the enclosing property for annuli. It is an open question whether the peripheral characteristic submanifold can be constructed by elementary methods as in [\[6\]](#). This is done in section 8 of [\[10\]](#) in the general case, but the proof is complicated. Wall has several questions about PD3 pairs in [\[13\]](#).

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APPENDIX

Errata for “Regular Neighbourhoods and Canonical
Decompositions for Groups”, Asterisque 289 (2003), by Peter
Scott and Gadde A. Swarup

There are two errors in the discussion of our construction of algebraic regular neighbourhoods. While they are easy to fix, this requires changes to the paper in several places. We discuss these errors first. Then we list the remaining errors in order of their occurrence.

Invertible almost invariant sets

The first problem is in our treatment of almost invariant sets which are invertible (see Definition 2.12). Before discussing the details, we need to briefly recall the construction in chapter 3. We have a finitely generated group G with finitely generated subgroups H_1, \dots, H_n and, for $1 \leq i \leq n$, we have a nontrivial H_i -almost invariant subset X_i of G . Recall that E denotes the set of all translates of all the X_i 's and X_i^* 's, and that an element U of E is isolated if it crosses no element of E . We construct the algebraic regular neighbourhood $\Gamma(X_1, \dots, X_n)$ in stages. First we consider the case when the X_i 's are in good position. This means that if U and V are any elements of E and two of the four sets $U^{(*)} \cap V^{(*)}$ are small, then one must be empty. In Theorem 3.8, we describe a graph of groups structure $\Theta(X_1, \dots, X_n)$ for G . If no X_i is simultaneously isolated and equivalent to an invertible almost invariant set, then $\Gamma(X_1, \dots, X_n)$ is defined to be $\Theta(X_1, \dots, X_n)$. If some X_i is isolated and is equivalent to an invertible almost invariant set Y_i , we claimed (see the last five lines of page 48) that we could replace each such X_i by an equivalent almost invariant set Z_i such that Z_i is not invertible and the new family is in good enough position. (Good enough position means that if U and V are any elements of E and two of the four sets $U^{(*)} \cap V^{(*)}$ are small, then either one must be empty, or some element of E crosses both U and V .) This allowed us to define $\Gamma(X_1, \dots, X_n)$ in general. Unfortunately this claim is incorrect. The error is in the existence part of Lemma 3.14 and is already clear in the case when $n = k = 1$.

Here is the statement of the existence part of Lemma 3.14 in this case.

Let G denote a finitely generated group, and let H be a finitely generated subgroup of G . Let X be a nontrivial H -almost invariant subset of G , such that X is isolated. Then there is an almost invariant set Z equivalent to X , such that Z is not invertible, and is in good enough position.

The following example shows that this statement is false.

Example Let G denote the group $\mathbb{Z}_2 * \mathbb{Z}_2$, let H denote the trivial subgroup and let σ denote the given splitting of G over H . Let X be a H -almost invariant subset of G associated to σ . As X is associated to a splitting, it crosses none of its translates and so is isolated in the set $E(X)$ which consists of all translates of X and X^* . Let Z be an almost invariant set which is equivalent to X . If Z is in good enough position, we claim that Z must be invertible. Thus we have the required counterexample to the above statement.

Here is the proof of the claim that Z must be invertible. The key point is that every translate of Z is equivalent to Z or Z^* , so that two translates of Z never cross. Thus the fact that Z is in good enough position implies that Z is in good position. It follows that $E(Z)$, the set of all translates of Z and Z^* , is nested. Now we can use Dunwoody's construction to produce a G -tree T whose oriented edges correspond to the elements of $E(Z)$. As Z is equivalent to X which is associated to the given splitting σ , the splitting of G determined by T must be conjugate to σ . In particular, the quotient graph of groups $G \backslash T$ has one edge and two vertices, with the edge group being trivial and the vertex groups having order 2. Thus T is a copy of the real line and the elements in G of order 2 act on T by reflections. Further each vertex of T is the fixed point of one of these reflections. Let s denote the oriented edge of T which corresponds to Z , choose the terminal vertex w of s as the basepoint, and let $\varphi : G \rightarrow V(T)$ denote the G -equivariant map such that $\varphi(e) = w$. Then Z equals Z_s . Let v denote the second vertex of s . As v does not lie in the G -orbit of w , it follows that $\varphi^{-1}(v)$ is empty. If k denotes the element of G which acts on T by a reflection fixing v , then clearly $kZ_s = Z_s^*$ so that $kZ = Z^*$. Thus Z is invertible as claimed.

Here is the way to resolve this problem. If some isolated X_i is equivalent to an invertible almost invariant set Y_i , we need to make slight adjustments at each stage of our construction. The above example shows that we may not be able to find almost invariant sets Z_i equivalent to X_i , such that the Z_i 's are in good enough position and not invertible. Thus we will not be able to construct our algebraic regular neighbourhood of the X_i 's as described on pages 46–50. Instead of replacing each X_i by such a non-invertible set Z_i , we replace each X_i by an equivalent almost invariant set Y_i so that the Y_i 's are in good enough position. This can be done by Lemma 3.13. Then we simply construct the graph of groups $\Theta(Y_1, \dots, Y_n)$. If no Y_i is isolated and

invertible, then this is $\Gamma(X_1, \dots, X_n)$. If some Y_i is isolated and invertible, we subdivide the corresponding edge of $\Theta(Y_1, \dots, Y_n)$, as discussed in Examples 3.10 and 3.11 on pages 44 and 45, and take $\Gamma(X_1, \dots, X_n)$ to be the resulting graph of groups.

This involves corresponding changes in the statements and proofs at several points in chapters 3, 4, 5 and 6:

Pages 50-51: the statement of Summary 3.16.

Pages 51-52: the proof of Lemma 3.17.

Pages 53-54: the discussion of the construction of an algebraic regular neighbourhood of an infinite family.

Page 62: the proof of part 2) of Lemma 4.10.

Page 67: Lemma 5.1 and its proof are correct, but the comment just before Lemma 5.1 that "it suffices to prove that the V_0 -vertices of $\Gamma(\{X_\lambda\}_{\lambda \in \Lambda} : G)$ enclose the given X_λ 's in the case when the X_λ 's are in good position and isolated X_λ 's are not invertible" is not correct. Thus Lemma 5.1 should be stated and proved without this last assumption.

Page 75: the proof of Lemma 5.10.

Page 78: the proof of Theorem 5.16.

Pages 80-83: the discussion of the construction of an algebraic regular neighbourhood of a family of almost invariant sets over groups which need not be finitely generated.

Page 89: the proof of Theorem 6.6.

Splittings over non-finitely generated groups

The second problem is in our treatment of splittings over non-finitely generated groups. Before discussing the details, we need some background discussion.

Let G be a finitely generated group. A HNN extension σ of G is said to be *ascending* if $G = A*_C$ where at least one of the two injections of C into A is an isomorphism. Note that if both injections are isomorphisms, then A is normal in G with infinite cyclic quotient. The difficulty arises when one considers ascending HNN extensions $G = A*_C$ for which A is not finitely generated. Such extensions are not at all unusual, as all that is needed is a surjection from G to \mathbb{Z} whose kernel A is not finitely generated. Whether or not A is finitely generated, ascending HNN extensions have the property that they can only be compatible with ascending HNN extensions. (Recall that two splittings σ and τ of a group G are compatible if G is the fundamental group of a graph of groups with two edges such that the associated edge splittings are σ and τ .) The precise result is the following.

Lemma 1 *Let σ be an ascending HNN extension of a group G . If σ is compatible with a splitting τ of G , then τ is also an ascending HNN extension.*

Proof. If σ is compatible with τ , then G is the fundamental group of a graph Γ of groups such that Γ has two edges s and t and the associated edge splittings are σ and τ respectively. As σ is HNN, it follows that Γ has at most two vertices. We claim that Γ has two vertices each of valence 2, so that Γ is a circle. For otherwise, the edge s would be a loop, and the subgraph given by t would carry the vertex group A of σ . But this is impossible as one of the inclusions of the edge group of s into A is an isomorphism. Now we know that Γ is a circle, it is easy to see that τ is also an ascending HNN extension. ■

Note that compatible ascending HNN extensions need not be conjugate. To construct examples, let K be a subgroup of a group H such that K itself has a subgroup L isomorphic to H . Let G be the fundamental group of a graph of groups with underlying graph a circle, with two vertices labeled by H and K and two edges labeled by K and L , so that the inclusion of the edge group L into the vertex group H is an isomorphism and the other inclusions are clear. If K and L are not isomorphic, the two edge splittings cannot be conjugate. A simple such example can be found with H (and hence L) free of rank 2, and K free of rank 3.

The problem in our arguments is a technical one which relates to the question of when two splittings are compatible. In particular, we are grateful to Vincent Guirardel for showing us an example which demonstrates that Theorem 5.16 is incorrect. His example consists of two splittings which have intersection number zero but are not compatible. Both splittings are ascending HNN extensions over non-finitely generated groups. However such splittings are the only source of problems, and we can correct our development of the theory of algebraic regular neighbourhoods fairly easily. This requires many changes to the paper including a modification of the definition of an algebraic regular neighbourhood (Definition 6.1).

Here is Guirardel's example. Let G denote the free group on two generators a and b , and let $f : G \rightarrow \mathbb{Z}$ be given by $f(a) = 0$, and $f(b) = 1$. Let K denote the kernel of f , so that K is freely generated by the elements $b^k ab^{-k}$, $k \in \mathbb{Z}$. Let τ denote the splitting of G as $K *_K$, where both injections of the edge group into the vertex group are isomorphisms. Let H denote the subgroup of G generated by the elements $b^k ab^{-k}$, $k \geq 0$, and let σ denote the splitting of G as $H *_H$, in which one inclusion is the identity and the other is conjugation by b . We will show that the splittings σ and τ have intersection number zero and cannot be compatible.

Let S and T denote the G -trees corresponding to σ and τ respectively. Thus T is a copy of the real line, with a vertex at each integer point. Think of T as the x -axis in the plane with S above it. We can describe S pictorially by saying that each vertex of S has integer x -coordinate, no edge of S is vertical, and at each vertex v of S , there is exactly one edge incident to v from the right. Now the projection of the plane onto the x -axis induces a G -equivariant map $p : S \rightarrow T$ which induces the identity on G . Let s be an edge of S and let t denote the edge $p(s)$ of T , and orient s and t to point to the left. Choose a base vertex $*$ for S and let $p(*)$ be the base vertex of T . Thus we have $\varphi : G \rightarrow V(S)$ given by $\varphi(g) = g(*)$, for all $g \in G$. If we remove the interior of s from S , we are left with two subtrees of S . The one which contains the terminal vertex of s is denoted by Y_s , and we let Z_s denote $\varphi^{-1}(Y_s)$. Similarly removing the interior of t from T yields two half lines in T , and the one which contains the terminal vertex of t is denoted by Y_t , and we let Z_t denote $\varphi^{-1}(p^{-1}(Y_t))$. The sets Z_s and Z_t are almost invariant subsets of G over H and K respectively which are associated to the splittings σ and τ of G . Clearly $Z_s \subset Z_t$. As gZ_t is equivalent to Z_t for every g in G , we have $Z_s < gZ_t$, for every g in G . Thus σ and τ have intersection number zero. Now consider the set E of all translates of Z_s and Z_t and their

complements in G . If σ and τ were compatible, E would correspond to the edges of a G -tree. In turn this would imply that there must some translate of Z_t (or of Z_t^*) which is $< Z_s$. As $Z_s < gZ_t$, for every g in G , this is clearly impossible.

It is interesting to see how our construction of an algebraic regular neighbourhood fails in this case. As no two elements of E cross, each element of E is isolated in E and so forms a CCC by itself. It follows immediately that the CCC's of E form a pretree. The problem is that this pretree is not discrete. For if b acts on T by translating one unit to the left, then we have the inclusions $\dots \subset b^2Z_t \subset bZ_t \subset Z_t$ and $Z_s < b^kZ_t$ for each $k \geq 0$. Thus there are infinitely many CCC's of E between Z_s and Z_t .

The error which causes all the problems occurs in the proof of Proposition 5.7. On page 72, lines -13 to -12, we assert that "there must be an element g of G such that $gX \subset U$." This is not correct in general. In order to appreciate the problem, we need to discuss the proof of Proposition 5.7. Recall that we have a family $\{X_\lambda\}$ of almost invariant subsets of G in good position such that the regular neighbourhoods of this family can be constructed as in chapter 3. Recall also that E denotes the collection of all translates of the X_λ 's and X_λ^* 's, and that X is a nontrivial H -almost invariant subset of G which crosses no element of E . Proposition 5.7 asserts that X is enclosed by a V_1 -vertex of Γ so long as either H is finitely generated or X is associated to a splitting of G . Our proof starts by showing that X is sandwiched between two elements of E , i.e. there are elements U and V of E such that $U < X < V$. If this condition holds then the remainder of our proof is correct. Further our proof that X must be sandwiched between two elements of E is correct, so long as H is finitely generated. In the case when H is not finitely generated, so that X is associated to a splitting σ of G , our proof is also correct so long as σ is not an ascending HNN extension. This was the case which we discussed incorrectly on page 72, line -12. For convenience in what follows we will say that a splitting of a group which is an ascending HNN extension over a non-finitely generated group is *special*.

Given a collection E of almost invariant subsets of G , we will say that an almost invariant subset X of G is *sandwiched by E* if there are elements U and V of E such that $U \leq X \leq V$. We can now correct the statement of part 2) of Proposition 5.7 by simply adding the hypothesis that X be sandwiched by E . Our published proof of Proposition 5.7 shows that this is automatic except possibly when X is associated to a special splitting of

G. This weakens Proposition 5.7, but almost all of our applications will still follow from this weakened version. Note that the proof of this weakened form of Proposition 5.7 is substantially shorter.

Here is a list of consequential changes:

Lemmas 5.10 and 5.15 need an extra sandwiching assumption. In the last paragraph of the proof of Lemma 5.10, on page 75, our arguments use the assumptions that A is sandwiched between two elements of E and that any element of E is sandwiched between two translates of A and A^* . The first assumption follows from the hypotheses of the lemma, but the second does not and needs to be added to the list of hypotheses. Of course, the second assumption is correct unless some edge splitting of Γ is special, and in this case, we know that Γ must be a circle. The same point arises on page 77 in the proof of Lemma 5.15.

The existence part of Theorem 5.16 is correct so long as we exclude special splittings. Otherwise the two splittings σ and τ described above form a counterexample. But the uniqueness part of Theorem 5.16 is correct, so that Theorem 5.17 is still correct.

Lemma 5.19 is correct.

Lemma 5.21 is also correct. We note that the assumption in this lemma and several later results that the regular neighbourhood has been constructed as in chapter 3 avoids many of the above worries about sandwiching. In particular, it implies that if P denotes the pretree of CCC's of E , the set of all translates of all the X_λ 's and X_λ^* 's, then P is discrete.

On pages 80-83, in the discussion of algebraic regular neighbourhoods of almost invariant sets over possibly non-finitely generated groups, we need to add a sandwiching assumption in order to prove existence for finite families. For example, we could assume that each X_i is sandwiched by $E(X_j)$, for each i and j .

Proposition 5.23 also needs an additional sandwiching assumption.

In part 2) of the definition of an algebraic regular neighbourhood (Definition 6.1), we need to add the assumption that the almost invariant set associated to the splitting σ is sandwiched by E .

Our main existence result (Theorem 6.6) is fine if the H_i 's are all finitely generated. In general, we need to add a sandwiching assumption. For example, we could assume that each X_i is sandwiched by $E(X_j)$, for each i and j .

Our main uniqueness result (Theorem 6.7) is correct as stated. The proof

uses Lemma 5.10 crucially, but the sandwiching assumption which needs to be added to the statement of Lemma 5.10 holds automatically in the present context.

The change in the definition of an algebraic regular neighbourhood means that we need to change the statements of many later theorems in which we describe the properties of the regular neighbourhoods we construct. For example, part (5) of Theorem 9.4 needs the addition of a sandwiching assumption. Note that we also used Proposition 5.7 to prove part (9) of Theorem 9.4, but this needs no change as the splittings considered are over finitely generated subgroups of G . The same comments apply to all the similar later results.

Having discussed the two errors in the construction of an algebraic regular neighbourhood, we now continue this list of errata in order of occurrence.

Page 28: There is an error in the second sentence of Example 2.34. In this example, G denotes the free group of rank 2. The sentence in question asserts that if C is a subgroup of G which is not finitely generated, then $e(G, C) = \infty$. This is incorrect. For example, if C is the kernel of the abelianisation map $G \rightarrow \mathbb{Z}^2$, then $e(G, C) = e(\mathbb{Z}^2) = 1$. However this does not invalidate Example 2.34. It is easy to see that there are many subgroups C of G which are not finitely generated for which $e(G, C) = \infty$. For a specific example, let H be a subgroup of G of index 2, so that H is free of rank 3, and let C denote the kernel of some surjection from H to the free group F_2 of rank 2. Then $e(G, C) = e(H, C) = e(F_2) = \infty$.

Note that the argument in Example 2.34 does not need $e(G, C) = \infty$. All that is needed is that $e(G, C) > 1$ in order that there be a nontrivial C -almost invariant subset of G .

Page 103: Theorem 7.11 is incorrect as stated, and this requires some minor modifications in the proofs of the applications of this result.

The wording of the statement needs to be modified. Let Γ_1 and Γ_2 be graphs of groups decompositions of a group G . We will say that Γ_2 is a *proper refinement* of Γ_1 if it is obtained from Γ_1 by splitting at a vertex so that the induced splitting of the vertex group is nontrivial. A sequence of proper refinements will also be called a proper refinement. Our proof of Theorem 7.11 assumes that Γ_{k+1} is a proper refinement of Γ_k . This assumption is implicit in the second sentence of the proof, where we assert that we have only to bound the length of chains of splittings of G over descending subgroups.

All of the above comments apply also to Theorem 7.13.

The first paragraph of the proof of Theorem 8.2 uses Theorem 7.11, and this paragraph needs changing as follows. Lines 1-6 of this paragraph are fine. But the next sentence is incorrect. It should assert that if G possesses a splitting σ' over a two-ended subgroup C' commensurable with H which has intersection number zero with the edge splittings of \mathcal{G} , then this splitting is enclosed by some V_0 -vertex v of \mathcal{G} , and determines a trivial splitting of the vertex group $G(v)$. This does not imply that σ' is conjugate to one of the edge splittings of \mathcal{G} . This requires changing the proof of Theorem 8.2 on lines 17-18 of page 111. Our choice of \mathcal{G} does not imply that σ must be conjugate to one of the edge splittings of \mathcal{G} , as claimed, but it does imply

the result of the next sentence which is all we need.

At the end of the proof of Proposition 8.4, we quote Theorem 7.11 to show that one cannot have an unbounded chain σ_i of compatible splittings of G over strictly descending subgroups, and this does follow from the corrected version of Theorem 7.11. The use of Theorem 7.11 in the proof of Theorem 9.2 is correct as Γ_{i+1} is a proper refinement of Γ_i for each i .

Theorems 7.11 and 7.13 are used in several other places. The key point to note is that the only problem which might occur is the existence of an unbounded chain σ_i of compatible splittings of G over strictly ascending subgroups H_i . If this occurs and all the H_i 's are *VPC* of the same length, then they are all commensurable, and hence each H_i has large commensuriser. In particular, when constructing an algebraic regular neighbourhood, the splittings σ_i must all be enclosed by a single V_0 -vertex of large commensuriser type.

Page 132: Theorem 10.8 is false. This clarifies why it is so important for our theory to consider *all* almost invariant subsets not just those which correspond to splittings. The generalisations of Theorem 10.8 in Theorems 12.6, 13.15 and 14.11 are also false. We will describe a counterexample.

Let K denote the Baumslag-Solitar group $BS(1, 2) = \langle a, t : tat^{-1} = a^2 \rangle$. Thus K has a natural expression as a HNN extension with infinite cyclic vertex group generated by a . It follows that K is finitely presented and torsion free. The map from K to the integers \mathbb{Z} given by killing a has kernel C which is isomorphic to the additive group of the dyadic rationals $\mathbb{Z}[\frac{1}{2}]$. If we choose the isomorphism so that a corresponds to 1, then $t^{-1}at$ corresponds to $\frac{1}{2}$ and in general $t^{-n}at^n$ corresponds to $\frac{1}{2^n}$. We let A denote the cyclic subgroup of C generated by a and let A_n denote the cyclic subgroup of C generated by $t^{-n}at^n$. Thus $A = A_0$ and the union of all the A_n 's equals C . It is easy to see that K admits no free product splitting. As K is torsion free, it follows that K admits no splitting over any finite subgroup and so is one-ended. Next we let G denote the group $K \times \mathbb{Z}$. Then G admits no splitting over any *VPCk* subgroup, for $k \leq 1$. Let H be an isomorphic copy of G , let D , B and B_n denote the subgroups of H which correspond to C , A and A_n respectively, and define \overline{G} to be $G *_A H$, the double of G along A . Note that \overline{G} is also finitely presented, and it is easy to see that \overline{G} is also one-ended. Thus the regular neighbourhood $\Gamma_1(\overline{G})$ exists. Note that \overline{G} commensurises A so that A has large commensuriser in \overline{G} .

Now \overline{G} admits splittings σ_i over A_i and τ_i over B_i which can be described

in the following simple way. We define σ_i by writing

$$\overline{G} = G *_A H = (G *_A A_i) *_A H = G *_A (A_i *_A H),$$

and define τ_i by writing

$$\overline{G} = G *_B H = G *_B (B_i *_B H) = (G *_B B_i) *_B H.$$

Note that $\sigma_0 = \tau_0$. It is easy to see that all these splittings are compatible. In fact the definitions already show that σ_i and σ_0 are compatible and that τ_i and τ_0 are compatible. The lemma below states that these are the only splittings of \overline{G} over a two-ended subgroup, up to conjugacy. It follows that the family \mathcal{S}_1 of all splittings of \overline{G} over two-ended subgroups contains an infinite collection of isolated splittings of \overline{G} , which shows that \mathcal{S}_1 cannot have a regular neighbourhood. Thus Theorems 10.8, 12.6, 13.15 and 14.11 are all false.

Lemma 2 *Any splitting of \overline{G} over a two-ended subgroup is conjugate to some σ_i or τ_j .*

Proof. Suppose that \overline{G} has a splitting σ over a two-ended subgroup. Let T be the corresponding \overline{G} -tree, so that $\overline{G} \backslash T$ has a single edge. Recall that G admits no splitting over any $VPCk$ subgroup, for $k \leq 1$. It follows that the subgroups G and H of \overline{G} must each fix a vertex of T . These vertices must be distinct as otherwise \overline{G} itself would fix a vertex of T . Also note that G and H can each fix only a single vertex of T , as otherwise G or H would fix some edge of T and hence itself be $VPCk$, for $k \leq 1$, which is not the case. Let v and w denote the vertices fixed by G and H respectively, and let λ denote the edge path in T which joins them. As G and H together generate \overline{G} , it follows that σ is an amalgamated free product. Further, by considering canonical forms of elements, it is easy to see that λ must consist of a single edge. Thus $\overline{G} = P *_R Q$, where P , Q and R denote the stabilisers of v , w and λ respectively. Note that this splitting of \overline{G} is conjugate to σ . As $G \subset P$ and $H \subset Q$, it follows that $G \cap H = A \subset R$. As R and A are each two-ended, it follows that R contains A with finite index. In particular we have the well known fact that R must be conjugate into G or into H . Now consider the subgroup GR of P generated by G and R . If R is contained in G , this subgroup equals G . Otherwise the fact that R lies in some conjugate of G or H implies that $GR = G *_A R$. Similarly the subgroup HR of Q generated by

H and R is equal either to H or to $R *_A H$. Now the inclusions $GR \subset P$ and $HR \subset Q$ induce a natural injection of $GR *_R HR$ into $\overline{G} = P *_R Q$. As G and H together generate \overline{G} , this injection must also be surjective and hence an isomorphism. It follows that $GR = P$ and $HR = Q$. If R lies in G and in H , the fact that R contains A implies that $R = A$. In this case $GR = G$ and $HR = H$, and the splitting σ is conjugate to σ_0 . If R lies in G , the facts that G is isomorphic to $K \times \mathbb{Z}$ and R contains A with finite index imply that R must equal A_n , for some $n \geq 0$. In this case $GR = G$ and $HR = A_n *_A H$, and the splitting σ is conjugate to σ_n . Similarly if R lies in H , then R must equal B_n , for some $n \geq 0$, and in this case the splitting σ is conjugate to τ_n . This leaves the case when R is not contained in G or in H . In this case we have an isomorphism between $G *_A H$ and $(G *_A R) *_R (R *_A H) = G *_A R *_A H$. But this is impossible as we know that R is conjugate into G or H . The result follows. ■

Page 144: The statement of Theorem 12.3 is not quite correct. There is a special case when Γ_n consists of a single V_0 -vertex. In this case there is a possibility which is not mentioned in the statement. Namely G may be $VPC(n+1)$. This possibility is contained in the paper by Dunwoody and Swenson but we omitted it in error when we applied their results.

The same omission occurs in the statements of Theorems 12.5, 12.6, 13.12, 13.13, 14.5, and 14.6.

Page 161: Example 14.1 is wrong. Recall that in chapter 14, we gave a construction of the regular neighbourhood $\Gamma_{1,2,\dots,n}$ in which we restrict attention to almost invariant sets over virtually abelian subgroups. The point of Example 14.1 was to show that this construction does not work if one considers almost invariant sets over virtually polycyclic subgroups. After discussing the error in Example 14.1, we will give a new example which demonstrates the phenomenon which Example 14.1 was supposed to demonstrate. Thus it is still correct to say that the construction of $\Gamma_{1,2,\dots,n}$ does not work if one considers almost invariant sets over virtually polycyclic subgroups.

Example 14.1 is supposed to be an example of a one-ended group G with incommensurable polycyclic subgroups H and K of length 3, and 2-canonical almost invariant sets X and Y over H and K respectively which cross weakly. The sets X and Y described in the example do cross weakly but they are not 2-canonical, as we will now show.

Recall that G is constructed by amalgamating H , K and a third group L along a certain infinite cyclic subgroup C . Thus G has subgroups $H *_C K$, $H *_C L$, $K *_C L$. Further G can be expressed as the amalgamated free product of the first two groups over H , and X is an H -almost invariant subset of G associated to this splitting. Similarly, the first and third groups give an amalgamated free product decomposition of G over K , and Y is associated to this splitting. Recall that Γ denotes the graph of groups structure for G which is a tree with four vertices carrying the subgroups C , H , K and L such that each edge group is C . We will label the vertices c , h , k and l correspondingly. Note that all three edges are incident to c .

The error in our argument occurs where we claim that if W is a nontrivial almost invariant subset of G over a two-ended subgroup A , then W is enclosed by the vertex l of Γ which carries L . The two edge splittings given by the edges ch and ck of Γ are clearly not enclosed by l , and so the associated C -almost invariant subsets of G are also not enclosed by l .

Now we will show that X and Y are not 2-canonical. In fact, they are not 1-canonical. Recall that X is associated to the splitting σ of G as $(H *_C K) *_H (H *_C L)$. We claim that this splitting is not compatible with the splitting τ of G over C as $H *_C (K *_C L)$. This means that X crosses the C -almost invariant subset of G associated to τ , so that X is not 1-canonical. A similar argument shows that Y is also not 1-canonical. To prove our claim suppose that σ and τ are compatible. This implies that G is the fundamental group of a graph Γ' of groups which has two edges such that the associated edge splittings are σ and τ . As neither splitting is HNN, Γ' must be homeomorphic to an interval with vertices P and R at the endpoints and one interior vertex Q . Choose notation so that σ is the edge splitting associated to PQ , and that τ is the edge splitting associated to QR . Thus the edge group associated to QR is C . Also the group G_P must be $H *_C K$ or $H *_C L$, and the group G_R must be H or $K *_C L$. If G_R is H , the fact that in either case G_P also contains H implies that H is contained in each edge group of Γ' . But this is impossible as C cannot contain H , as C is cyclic and H is not. It follows that G_R must be $K *_C L$. But G_P contains one of K or L , which implies that K or L is contained in C which is again impossible. This completes the proof of the claim.

Here is an example of a one-ended, finitely presented group G which has 2-canonical splittings σ_1 and σ_2 over $VPC3$ subgroups C_1 and C_2 respectively, such that σ_1 and σ_2 cross weakly and C_1 and C_2 are not commensurable.

This replaces the incorrect Example 14.1.

Let B denote the free abelian group of rank 2, let θ be a hyperbolic automorphism of B (i.e. θ has two real eigenvalues with absolute value not equal to 1), and let C be the extension of B by \mathbb{Z} determined by θ . Thus B is normal in C with quotient isomorphic to \mathbb{Z} . Fix an element α of C such that conjugation of B by α induces the automorphism θ , and let A denote the infinite cyclic subgroup of C generated by α . Thus A projects onto the infinite cyclic quotient of C by B . A crucial property of C , which follows from the hyperbolicity of θ , is that any *VPC2* subgroup of C must be a subgroup of B . Let nA denote the subgroup of A of index n . As A is a maximal cyclic subgroup of C , it follows that $Z_C(nA) = A$, where $Z_C(X)$ denotes the centraliser in C of X . Note that C is the fundamental group of a closed 3-manifold M which is a torus bundle over the circle.

For any integer i , let B_i denote a copy of B , let θ_i denote the automorphism of B_i which corresponds to θ , let C_i denote the corresponding copy of C , and let α_i denote the corresponding element of C_i . For any pair i, j of distinct integers, let C_{ij} denote the extension of $B_i \oplus B_j$ by \mathbb{Z} , given by the automorphism $\theta_i \oplus \theta_j$. Fix an element α_{ij} of C_{ij} such that conjugation of $B_i \oplus B_j$ by α_{ij} induces the automorphism $\theta_i \oplus \theta_j$, and let A_{ij} denote the infinite cyclic subgroup of C_{ij} generated by α_{ij} . Thus there is a natural inclusion of C_i into C_{ij} which sends α_i to α_{ij} , and hence sends A_i to A_{ij} .

We note the following facts about the C_{ij} 's. Each C_{ij} is one-ended and torsion free, and any *VPC2* subgroup of C_{ij} must be a subgroup of $B_i \oplus B_j$. It follows that each C_{ij} has no nontrivial almost invariant subsets over any *VPC* subgroup of length ≤ 2 . It is easy to see that $Z_{C_{ij}}(nA_{ij}) = A_{ij}$.

We construct a group $H = C_{13} *_{C_1} C_{01} *_{C_0} C_{02} *_{C_2} C_{24}$, where the inclusions of the edge groups into the vertex groups are the natural ones. Thus we have in H the equations $\alpha_{13} = \alpha_1 = \alpha_{01} = \alpha_0 = \alpha_{02} = \alpha_2 = \alpha_{24}$, and we abuse notation and denote this element by α and the cyclic subgroup of H it generates by A . The natural homomorphisms $C_{ij} \rightarrow \mathbb{Z}$ all fit together to yield a homomorphism $H \rightarrow \mathbb{Z}$ which maps A onto \mathbb{Z} . We also choose a group D which is the fundamental group of a closed hyperbolic 3-manifold and has $H_1(D) \cong \mathbb{Z}$. Then D is one-ended and torsion free. It has no nontrivial almost invariant subsets over any *VPC1* subgroup and contains no *VPC2* subgroups. Thus D has no nontrivial almost invariant subsets over any *VPC* subgroup of length ≤ 2 . We define $G = H *_A D$, where A is identified with a cyclic subgroup of D which maps onto $H_1(D)$. Thus the homomorphism $H \rightarrow \mathbb{Z}$ extends to one from G to \mathbb{Z} which maps A onto \mathbb{Z} . As H and D are

both torsion free, so is G . Note that as A is a maximal cyclic subgroup of D , it follows that $Z_D(A) = A$.

The group H is the fundamental group of a graph Δ of groups with underlying graph an interval divided into three edges. The edge groups are C_1 , C_0 and C_2 , and we denote the corresponding edges by e_1 , e_0 and e_2 respectively. The vertex groups are C_{13} , C_{01} , C_{02} and C_{24} , and we denote the corresponding vertices by v_{13} , v_{01} , v_{02} and v_{24} respectively. The group G is the fundamental group of a graph Γ of groups obtained from Δ by adding one edge e with associated group A . One end of e is at v_{13} and the other end has associated group D .

We take the splitting σ_1 of G to be the edge splitting of Γ associated to the edge e_1 . Thus σ_1 is a splitting of G of the form

$$G = \langle C_{13}, D \rangle *_{C_1} \langle C_{01}, C_{02}, C_{24} \rangle.$$

To describe the splitting σ_2 of G , we first slide the edge e so as to be attached to v_{24} instead of v_{13} . Now σ_2 is the edge splitting of this new graph of groups which is associated to the edge e_2 . Thus σ_2 is a splitting of G of the form

$$G = \langle C_{13}, C_{01}, C_{02} \rangle *_{C_2} \langle C_{24}, D \rangle.$$

It is easy to check that σ_1 and σ_2 are not compatible. If σ_1 and σ_2 cross strongly, then some conjugate of C_1 must intersect C_2 in a $VPC2$ subgroup. But our construction of G shows that conjugates of C_1 and C_2 intersect trivially or in a conjugate of A . It follows that σ_1 and σ_2 must cross weakly. Note that it also follows that C_1 and C_2 are not commensurable subgroups of G .

It remains to prove that the splittings σ_1 and σ_2 are 2-canonical. We will show that G has no nontrivial almost invariant subsets over any $VPC2$ subgroup, and that it has only one (up to equivalence, complementation and translation) nontrivial almost invariant subset over a $VPC1$ subgroup, which arises from the splitting τ of G over A associated to the edge e of Γ . Thus τ is the splitting of G as $G = H *_A D$. As each of σ_1 and σ_2 is clearly compatible with τ , it follows that they do not cross the associated almost invariant subset over A , and so are 2-canonical, as required.

First we prove the following technical result.

Lemma 3 *Let K be a subgroup of G such that, for any conjugate L of K in G ,*

1. the number $e(C_{ij}, L \cap C_{ij}) = 1$, for $ij = 13, 01, 02$ or 24 , and
2. the number $e(D, L \cap D) = 1$, and
3. $L \cap C_i$ has infinite index in C_i , for $i = 0, 1$ or 2 , and
4. $L \cap A$ has infinite index in A .

Then $e(G, K) = 1$.

Proof. Corresponding to the graph of groups decomposition Γ of G , there is a graph of spaces decomposition of a space X with fundamental group G . The covering space X_K of X with fundamental group K is a graph of spaces, where each vertex space has fundamental group equal to the intersection of K with a conjugate of some C_{ij} or of D , and each edge space has fundamental group equal to the intersection of K with a conjugate of C_0, C_1, C_2 or A . The hypotheses imply that each vertex space has 1 end and that each edge space is non-compact. It follows easily that X_K has 1 end, so that $e(G, K) = 1$ as required. ■

Now we summarise the main properties of G .

Lemma 4 *The group G has the following properties.*

1. G has 1 end.
2. If K is a $VPC1$ subgroup of G , then either $e(G, K) = 1$ or K is conjugate commensurable with A .
3. $Comm_G(A) = A$.
4. If K is a $VPC2$ subgroup of G , then $e(G, K) = 1$.

Proof. 1) Each C_{ij} is one-ended, as is D , and each of C_1, C_2 and A is infinite. Now we apply Lemma 3, with K equal to the trivial group, to see that $e(G) = 1$, as required.

2) Let K be a $VPC1$ subgroup of G . Thus the intersection of K with any subgroup of G is $VPCk$, for some $k \leq 1$. Now we apply Lemma 3. It follows that conditions 1)-3) of that lemma are satisfied. Hence $e(G, K) = 1$ unless condition 4) fails. This would mean that there is a conjugate L of K such that $L \cap A$ has finite index in A . As K and A are both $VPC1$, it follows that $e(G, K) = 1$ unless K is conjugate commensurable with A .

3) Let g denote an element of $Comm_G(A)$. As there is a homomorphism of G to \mathbb{Z} which maps A onto \mathbb{Z} , it follows that g must lie in $Z_G(kA)$, the centraliser of kA in G , for some non-zero k . Now recall that G is the fundamental group of the graph Γ of groups. As A is contained in each vertex group, and the centraliser in each vertex group of kA is equal to A , it follows that $Z_G(kA) = A$. Hence $Comm_G(A) = A$, as required.

4) Let K be a $VPC2$ subgroup of G . Thus the intersection of K with any subgroup of G is $VPCk$, for some $k \leq 2$. Now we apply Lemma 3. It follows that condition 1) and 3) of that lemma hold. As the intersection of any conjugate of K with D must be $VPCk$, with $k \leq 1$, it also follows that condition 2) holds. Hence $e(G, K) = 1$ unless condition 4) fails. This would mean that there is a conjugate L of K such that $L \cap A$ has finite index in A . As any $VPC2$ group has a subgroup of finite index isomorphic to $\mathbb{Z} \times \mathbb{Z}$, it follows that any $VPC1$ subgroup of K has large commensuriser in K . Thus this would imply that A has large commensuriser in G . Now part 3) of this lemma shows that this also is impossible. It follows that if K is a $VPC2$ subgroup of G , then $e(G, K) = 1$, as required. ■

Next we consider almost invariant subsets of G which are over a $VPC1$ subgroup.

Lemma 5 *G has only one (up to equivalence, complementation and translation) nontrivial almost invariant subset over a $VPC1$ subgroup, which arises from the splitting τ of G over A as $G = H *_A D$.*

Proof. Suppose that G has a nontrivial almost invariant subset over a $VPC1$ subgroup K , so that $e(G, K) > 1$. Recall from part 2) of Lemma 4 that if K is a $VPC1$ subgroup of G , then either $e(G, K) = 1$ or K is conjugate commensurable with A . As $Comm_G(A) = A$, by part 3) of Lemma 4, this implies that K is conjugate to a subgroup of A .

Now suppose that K is a subgroup of A and, as in the proof of Lemma 3, consider the cover X_K of X with fundamental group K . Part 3) of Lemma 4 tells us that $Comm_G(A) = A$, so that exactly one edge space of X_K is compact. Thus, as in the proof of Lemma 3, each of the two complementary components of this edge space has one end. It follows that $e(G, K) = 2$, for any subgroup K of finite index in A . Hence G has only one (up to equivalence, complementation and translation) nontrivial almost invariant subset over a $VPC1$ subgroup. As the splitting τ of G over A has such an almost invariant subset of G associated, the result follows. ■

JSJ DECOMPOSITIONS OF POINCARÉ DUALITY PAIRS
BY LAWRENCE REEVES, PETER SCOTT AND GADDE A. SWARUP

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