IDENTITIES FOR *SIN(X)*, MEDICAL IMAGING, AND SEVERAL COMPLEX VARIABLES

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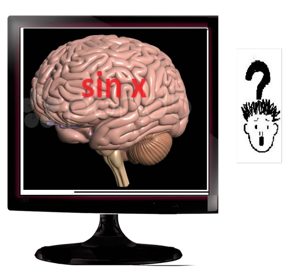
Dedicated to the memory of Leon Ehrenpreis and Sigurdur Helgason.

1. Introduction

More than three decades ago, the authors were working on mathematics of a medical imaging modality called Single Photon Emission Computed Tomography (SPECT), see e.g., [5, 6, 11]. More specifically, we were aiming at describing the range of the Radon type transform arising there [7, 8]. We have succeeded in doing so, but the necessity of the range conditions that we found, which is usually the "easy" part of the proof, suddenly produced an infinite sequence of very interesting algebro-differential identities for $\sin x$ (yes, our usual sine function!).They look like some kind of binomial formulas in non-commutative variables. Proving them directly, without using integral geometry (c'mon, it is just the $sin\, x$!) turned out to be not that easy. Analogous (although much easier to prove) identities also hold for the exponentials. Such identities are held in any commutative algebra with differentiation and cancelation. However, other natural extensions are not clear. Moreover, the formulas in question should be related to something deeper - group representation theory, special functions, D-moduli, whatever. Frustratingly, we have not been able to find such explanations and hope that a reader could succeed with this.



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2. The identities

For brevity, we avoid discussing the medical imaging part, where the formulas originated from, referring the interested reader to [1, 6, 15], with the proofs presented in [7–10].

Theorem 1. *For any odd natural number n, the following identity holds:*

$$\sum_{k=0}^{n}\{\begin{pmatrix} n \\ k \end{pmatrix}(\frac{d}{dx}-\sin x)\circ(\frac{d}{dx}-\sin x+i)\circ \cdots\circ(\frac{d}{dx}-\sin x+(k-1)i)\}(\sin x)^{n-k}\equiv 0. \tag{1}$$

Here i is the imaginary unit, $\begin{pmatrix} n \\ k \end{pmatrix}$ is the binomial coefficient "n choose k," the expressions in braces are considered as differential operators, and $\circ$ denotes their composition. For instance,

$$\left(\frac{d}{dx}-\sin x\right)u(x)=\frac{du}{dx}-\sin x\ u(x).$$

Similar identities also hold for the exponential e^x, but now for **all** natural values of n:

Theorem 2. *For any natural number n, the following identity holds:*

$$\sum_{k=0}^{n}\{\begin{pmatrix} n \\ k \end{pmatrix}(\tfrac{d}{dx}-e^x)\circ(\tfrac{d}{dx}-e^x+1)\circ\cdots\circ(\tfrac{d}{dx}-e^x+(k-1))\}(e^x)^{n-k}\equiv 0. \tag{2}$$

In the next section we also present an algebraic formulation of these identities (not only for $\sin x$, but even for some more "elementary" functions such as linear and exponential ones).

As we have already stated, the true meaning of these identities has been escaping and thus irritating us for all these years. We would be happy if someone could shed some more light onto their meaning and possible generalizations.

3. Algebraic formulation of the identities

Let us notice that e^x and $\sin x$ satisfy the following differential identities:

$$\begin{gathered}\tfrac{d}{dx}e^x=\lambda e^x,\ \text{where } \lambda=1,\\ (\tfrac{d}{dx})^2\sin x=\lambda^2\sin x,\ \text{where } \lambda=i.\end{gathered} \tag{3}$$

Hmm, comparing with differential identities of the previous section, one wonders whether their abstract versions could hold. And lo and behold, they do indeed:

Theorem 3. *let A be a commutative algebra (over a field K) with cancelation property and with a differentiation[1] D. Then*

(1) *Any solution $u \in A$ of the first-order equation $Du = \lambda u, \lambda \in K$ satisfies for any natural number n the following identity:*

$$\sum_{k=0}^{n} \left[\binom{n}{k} (D-u) \circ (D-u+\lambda) \circ ... \circ (D-u+(k-1)\lambda) \right] u^{n-k} = 0. \tag{4}$$

(2) *Any solution $u \in A$ of the second-order equation $D^2 u = \lambda^2 u$ satisfies for any* ***odd*** *natural number n the identity (4).*

Here are some useful algebraic hints (for the details see [9]). Each term of the sums in our identities is the composition of two non-commuting operators. Appropriate gouge transforms allow one to simplify these equations. For instance,

Lemma 4. *If $Du = \lambda u$, then*

$$u^{-m}(D-u)u^m = D - u + m\lambda, \tag{5}$$

where the use of negative powers (i.e., of the corresponding field) is justified by the assumed cancelation property of A.

This allows us to rewrite the expression in question as

$$\sum_{k=0}^{n} \left[\binom{n}{k} [(D-u) \circ u^{-1}]^k \right] u^n = 0. \tag{6}$$

We now deal with an operator binomial

$$\sum_{k=0}^{n} \binom{n}{k} A^k,$$

where

$$A = (D-u) \circ u^{-1}.$$

Thus, the usual binomial formula works and gives

$$[(D-u) \circ u^{-1} + 1]^n u^n = (D \circ u^{-1})^n u^n = 0. \tag{7}$$

A somewhat more sophisticated transform can be used in the case $D^2 u = \lambda^2 u$ (see details in [9]).

[1]I.e., $D(uv) = (Du)v + u(Dv)$.

4. Several complex variables - why?

It has been noticed before (see, e.g., [6–9, 11–14]) that several complex variables approach clarifies various formulas arising in SPECT. In particular, the inversion formulas are related to attaching analytic disks to totally real surfaces in $\mathbb{C}^2$. It was still a surprise when a relation was discovered between the identities studied here and the so-called separate analyticity theorems (Hartogs-Bernstein theorems, see [2] and references therein). For instance, the following amazing theorem is essentially equivalent to these identities [1, 16, 17]:

Theorem 5. *Let Ω be a disk in $\mathbb{R}^2$ and $f(x)$ be a continuous function in the exterior of Ω. Suppose that when restricted to any tangent line L to $\partial\Omega$, the function $f|_L$, as a function of one real variable, extends to an entire function on the complexification of L. Then f, as a function on $\mathbb{R}^2 \setminus \Omega$ extends to an entire function on $\mathbb{C}^2$.*

One wonders whether the circular shape of the "hole" Ω is relevant. And indeed, A. Tumanov found [20] a beautiful short proof based on attachment of analytic disks, which works for any strictly convex body Ω with a mild conditions on the smoothness of its boundary. More discussion of the relations of the range conditions with complex analysis can be found for instance in [4, 6–8, 16, 17].

5. Remarks and open questions

(1) It is clear that the identities discussed above cannot be accidental and must be related to special function theory and/or group representations. It would be interesting to discover such relations.

(2) The formulations of the theorems above lead to the following natural question: are there any analogs of these identities for solutions of the 3rd and higher orders equations:

$$D^k u = \lambda^k u \text{ for } k \geq 3.$$

A natural guess would be that similar identities hold, but for sparser arithmetic sub-sequence of natural numbers n (with the difference equal to k). It is hard to compute these expressions by hand even for small values of n. It is reasonable to use a symbolic computation system. This was done and shown in [19] that this conjecture fails for $k = 3$. So, are solutions of higher order differential equations $D^m u = \lambda^m u$ deprived of any such identities?

(3) The considerations of [10] show that the formulas we discuss can be derived from some commutator relations for the operators involved.

(4) Are there higher dimensional analogs of Theorem 5?

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