The dimer model

or how Mathematicians play dominoes

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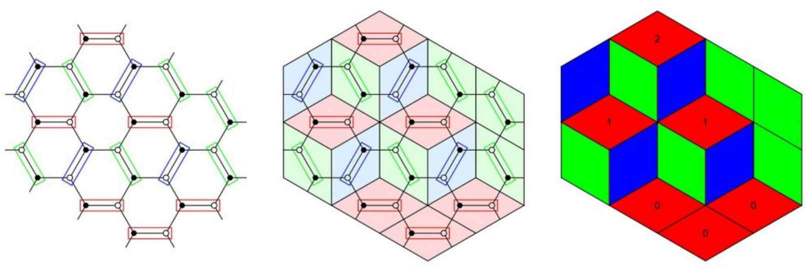

Abstract

Introductory text on the dimer model and its links to combinatorics, statistical physics, and geometry.

Note: *This article is a translation of the French orginal published in 2016. In the meantime, the study of the dimer model has seen several interesting developments, reflected in the final section, which was added at the time of translation in October 2025.*

Consider a chessboard. We want to cover the entire board with dominoes of size 1×2. For this we need 32 dominoes so that each domino is placed on exactly two squares of the 8×8 board, with no dominoes overlapping. A simple way to do this is to lay all dominoes in the same direction (horizontal or vertical). This is not very interesting. A more interesting, less regular tiling is shown in Figure 1 in the center. There are many other ways to tile an 8×8 chessboard with dominoes. To list all different tilings is quite a task since there are $12,988,816$ of them! There is more to say: for example, if we change the setup very slightly, namely remove two squares from two opposite corners, still wanting to tile the board with dominoes, now 31 of them, we quickly get stuck. This seemingly innocuous change turns out to be radical: we have to face the fact that the task is impossible. We leave it to the reader to find the simple reason while contemplating the coloring of the chipped board.

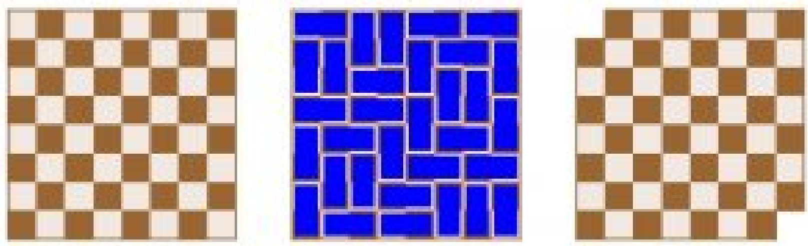

Figure 1: From left to right: a standard chessboard that allows 12,988,816 domino tilings; one such tiling; a chipped chessboard with no possible domino tilings

Let us look at the quintessence of domino tiling and try to extract its mathematical core. We notice that there are black and white squares that alternate (no two squares of the same color are adjacent). Then we remark that when we place a domino on the chessboard, we cover a white square and a black square, thus connecting them together. The standard way to mathematically encode this data uses graph theory. We replace the chessboard with a finite bipartite graph. A finite graph is a finite set of vertices connected by edges. A bipartite graph is a graph with vertices colored either black or white; moreover, in our setting every edge connects vertices of different colors. Thus black vertices of the graph correspond to black squares of the board and white vertices to white squares. Placing a domino is equivalent to choosing an edge in the graph. So instead of having to cover the chessboard with dominoes, the goal of the game now is to choose a set of (necessarily disjoint) edges that connect every black vertex to a (single) white vertex. This is called a *perfect matching*. An illustration on a small chessboard 4×4 is presented in Figure 2.

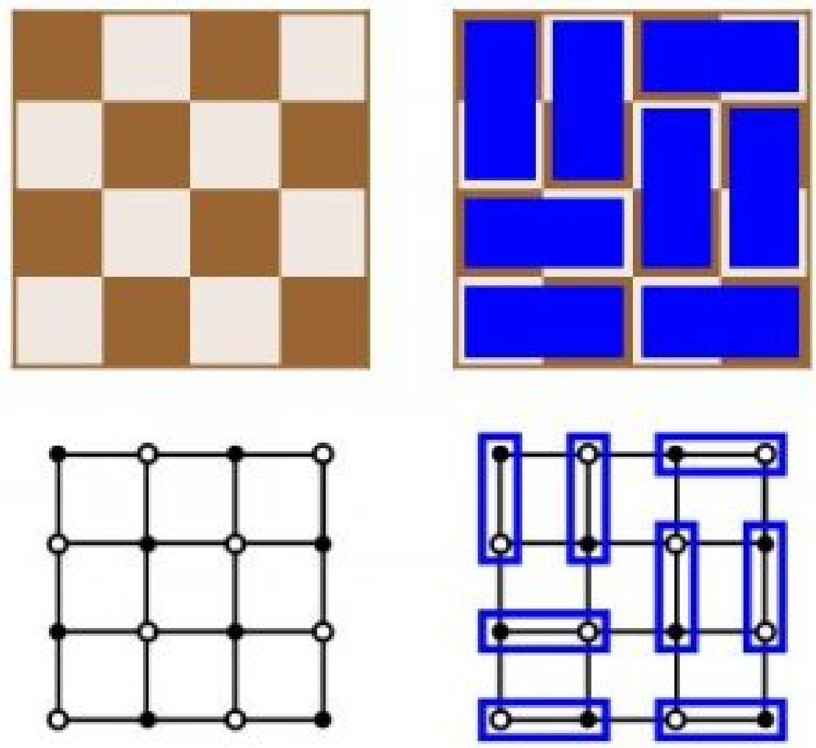

Figure 2: Above: a chessboard and one of its domino coverings. Below: the associated bipartite graph, and the corresponding perfect matching

Combinatorics and complexity

In the example of the chipped chessboard, it is very easy to demonstrate that there is no perfect matching since a necessary condition for the existence of a perfect matching is that there are as many black squares as white squares (a domino covers exactly two squares: one black, the other white).

This is not a sufficient condition; it depends on the shape of the chessboard, as shown by the example in Figure 3. [1]

In the general case, determining whether a bipartite graph has a perfect matching can be done using Hall's criterion: a bipartite graph has a perfect matching if and only if, for any subset of black vertices (resp. white vertices),

[1] Thurston [10] discusses more refined necessary conditions as well as sufficient conditions for the existence of tilings. These conditions are based on the notion of a height function, which we will introduce below.

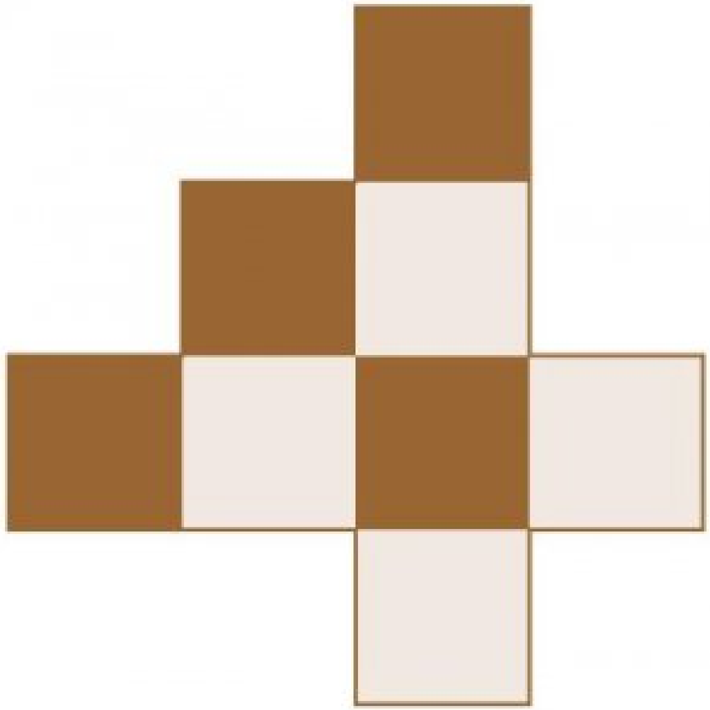

Figure 3: A portion of the starting chessboard containing as many black and white squares which cannot be covered by 4 dominoes. This illustrates the importance of the shape of the area to be covered.

the number of neighboring white vertices (resp. black vertices) is at least the size of the subset. This criterion can be verified using the example in Figure 3. A nice application of this criterion to a geometry problem has been described in an Image des Maths article by Frédéric Le Roux.

To go further

Finding a particular matching is a relatively straightforward problem since there is an algorithm for solving it (called the Hungarian algorithm) that runs in polynomial time in the number of vertices[2]*. This problem is in the complexity class* P.

[2]This algorithm is due to Kuhn in 1955 and is based on the earlier work of the Hungarian mathematicians Egerváry and Kőnig. Successive improvements to this algorithm are due to Munkres in 1957, and Hopcroft and Karp in 1972. Although it seems that a polynomial-time algorithm had already been proposed by the German mathematician Jacobi at the beginning

The question of determining the total number of different tilings (as in our initial example of a chessboard) is much more difficult. Its difficulty can be expressed using the language of complexity theory (these notions have been discussed by Pierre Pansu in an Image des maths article "Cutting graphs"). This problem is in the class $\#\mathsf{P}$ *(it is a famous theorem of Leslie Valiant obtained in 1979), which means that there is no algorithm allowing this number to be calculated exactly in polynomial time, assuming that* $\mathsf{P} \subsetneq \mathsf{NP}$[3]*. This problem is among the list of* $\#\mathsf{P}$*-complete problems. In particular, it means that to show that any counting problem is in the class* $\#\mathsf{P}$*, it suffices to show that it is at least as difficult as counting the number of perfect matchings of a graph. More details on the combinatorial and algorithmic aspects of matchings, in particular, their approximate counting, can be found in the seminal work of Lovász and Plummer [7].*

The fact that one can explicitly determine the number of perfect matchings in the case of a chessboard of size 8×8 is rather exceptional and is due to the fact that the graph is planar[4]. This important result is due to the Dutch physicist Pieter Kasteleyn [4], who in 1961 obtained an exact formula for the number of perfect matchings of any planar graph. The formula uses a basic notion of linear algebra, the determinant of a matrix, which can be calculated in polynomial time in the size of the matrix (by elementary matrix operations such as Gaussian row-reduction).

A bit of linear algebra

Consider a 3×3 *matrix* $M_0 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$*. The determinant of an* $n \times n$ *matrix* M *is*

$$detM = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{k=1}^{n} m_{k,\sigma(k)},$$

where S_n *denotes the permutation group of* n *elements and* $\epsilon(\sigma) = \pm 1$ *is the signature of the permutation* σ*. For the matrix* M_0 *we get*

$$detM_0 = aei - afh - bdi + bfg + cdh - ceg.$$

of the 19th century, as attested by a posthumous text in Latin [9].

[3] Deciding on the validity of this assertion remains a major open problem to this day, with a Clay Mathematics Institute major reward for its solution.

[4] A graph is said to be planar if it can be drawn in the plane without its edges crossing. The graph in Figure 2 is planar. An example of a non-planar graph is the complete graph on 5 vertices, that is, a pentagon with all its diagonals.

The determinant of an $n \times n$ matrix has $n!$ terms.

Consider a 3×3 matrix $M_0 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. *The determinant of an $n \times n$ matrix M is*

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$$detM_0 = aei - afh - bdi + bfg + cdh - ceg.$$

The determinant of an $n \times n$ matrix has $n!$ terms. The permanent is a number associated with a matrix which differs from the determinant by the absence of signs[5]*:*

$$permM = \sum_{\sigma \in S_n} \prod_{k=1}^{n} m_{k,\sigma(k)}.$$

For M_0 it is $permM_0 = aei + afh + bdi + bfg + cdh + ceg.$

Let us introduce the adjacency matrix A of a bipartite graph. Consider a graph with n black vertices, numbered from 1 to n and n white vertices, numbered from 1 to n. It gives rise to an $n \times n$ matrix with rows indexed by black vertices and columns by white vertices. A coefficient a_{ij} of this matrix is zero unless the vertices i and j are connected by an edge, in which case $a_{ij} = 1$. The number of perfect matchings of a bipartite graph is

$$N = \sum_{\sigma \text{ permutation of } \{1,\dots,n\}} \prod_{k=1}^{n} a_{k,\sigma(k)}.$$

This is exactly the permanent of the matrix A!

For example, for the complete bipartite graph with $2n$ vertices (each of the n black vertices is connected to each of the n white vertices), there are $|S_n| = n!$ possible perfect matchings.

The permanent of an arbitrary matrix is usually difficult to calculate, and there is no efficient algorithm to do so.

[5] More prosaically, the permanent of an $n \times n$ matrix is a sum of products of coefficients of the matrix: each product contains exactly one coefficient from each row and one coefficient from each column, and the sum covers all the ways of choosing n coefficients satisfying these conditions.

To go further

If the matrix has only 0 *and* 1 *entries, then the problem of calculating its permanent is* #P*-complete. In fact it is the same problem as counting the number of perfect matchings on a bipartite graph! Indeed a perfect matching corresponds to a product of matrix entries with no repetitions in neither row indices nor column indices. Thus the problem of counting perfect matchings is reformulated in the language of linear algebra.*

Kasteleyn's idea was to show that for planar graphs one can always construct a matrix K starting from the adjacency matrix A so that the determinant of K computes the number of perfect matchings:

$$\operatorname{perm} K = \det A,$$

This is achieved by introducing some signs in the entries of A (signs on the edges of the graph) so that the signs in the standard determinant formula cancel and the permanent of the original matrix effectively coincides with the determinant of the new one[6]. Thus counting the number of perfect matchings, a difficult problem involving a permanent, becomes a simpler problem of calculating a determinant. The matrix K is called *the Kasteleyn matrix*.

In our first example of an 8×8 chessboard, a modification of the Kasteleyn matrix was suggested first by Jerome Percus and then by Richard Kenyon: they took K to be a square adjacency matrix with the entries encoding the edges between black and white vertices by 1 if the vertices are connected by a vertical edge and by i (where $i^2 = -1$) if the connecting edge is horizontal.

Let us consider a toy example of a graph with 6 vertices first (see Figure 4). In this case the adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

[6] It suffices to choose the signs in such a way that their product along each face (a face being a connected component of the complement of the graph in the plane) is $(-1)^{k/2+1}$, where k is the number of edges bordering the face. We leave it to the reader as an exercise to verify that such a choice is indeed possible (hint: use a spanning tree) and that it leads to the expected result. More generally, the Kasteleyn method applies to all planar graphs, including those that are not bipartite. In the general case, the choice of signs is somewhat more subtle and is based on the notion of a Kasteleyn orientation. Furthermore, one must then consider the full adjacency matrix and take the square root of the determinant.

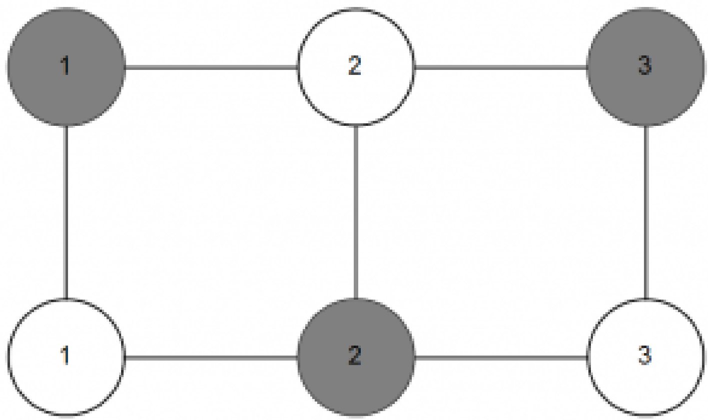

Figure 4: In this example the black and white vertices are numbered, which allows us to write down the adjacency matrix A and the Kasteleyn matrix K with rows indexed by black vertices and columns by white vertices.

and the Kasteleyn matrix

$$K = \begin{pmatrix} 1 & i & 0 \\ i & 1 & i \\ 0 & i & 1 \end{pmatrix}.$$

Hence $\det K = 3$ and so $\operatorname{perm} A = \det K = 3$ which is exactly the number of perfect matchings of this graph. For the 8×8 chessboard, one obtains

$$|\det K| = \prod_{k=1}^{8} \prod_{\ell=1}^{8} \sqrt{\left| 2\cos\left(\frac{\pi k}{9}\right) + 2i \sin\left(\frac{\pi \ell}{9}\right) \right|} = 12\,988\,816 = 2^4 \times 17^2 \times 53^2.$$

This gives the value mentioned earlier[7]. In general, the number N_n of perfect matchings on the $2n \times 2n$ chessboard is given by the sequence A004003 from the Encyclopedia of Sequences [8]. This sequence grows exponentially with n^2.

[7] It may seem surprising that a product of complicated numbers (with trigonometric functions) gives an integer: this is nevertheless the case since the determinant of a matrix with integer coefficients is an integer, even if its eigenvalues (obtained by diagonalization) are non-integer real numbers. A computer calculation yields 12, 988, 816. We refer the curious reader to the following article, which offers another demonstration of this formula, valid for rectangular $2n \times 2m$ chessboards, as a product of special values of Chebyshev polynomials of the second kind.

When n goes to infinity

$$N_n \to \exp\left[\frac{4}{\pi}Cn^2(1+o(1))\right],$$

where

$$C = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = \int_0^1 \frac{\arctan x}{x}\,dx = 0.915\ldots$$

is the Catalan constant — named after the Franco-Belgian mathematician Eugène Charles Catalan[8].

A hexagonal lattice is also quite interesting (see Figure 5). The dominoes in this case are lozenges. We can see the tiling as a plane projection of a surface of piled cubes. For a hexagonal portion of the hexagonal lattice (with integer side lengths a, b, c, a, b, c), the number of tilings is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

This result was obtained before Kasteleyn's work, and it is a particular case of the famous Percy MacMahon formula for the number of plane partitions. In the example of Figure 5 this formula gives 50 possible tilings.

In what follows all considered graphs are bipartite and planar.

Statistical Physics

In the 1960s, Kasteleyn, who also had a background in chemistry, was trying to understand a model of polymers in solution. Because of the difficulty of the general problem, he was forced to restrict himself to a particular case, that of a solution of dimers (polymers with only one bond). This is what is now called the dimer model. If Kasteleyn was interested in perfect matchings, it is in connection with this model [9]

[8] Another sequence, probably more familiar to the reader, is the Fibonacci sequence $\forall n \geq 2, u_n = u_{n-1} + u_{n-2}, u_0 = 0, u_1 = 1$, which grows exponentially like the powers of the golden ratio. The Fibonacci sequence can also be interpreted in terms of dominoes: the n-th term of this sequence counts the number of tilings of a rectangular $2 \times n$ chessboard. This classic exercise is easily solved without resorting to Kasteleyn's method.

[9] The importance of the dimer model also lies in the fact that the Ising model can be studied through it. This model was introduced in 1920 by the German physicist Wilhelm Lenz,

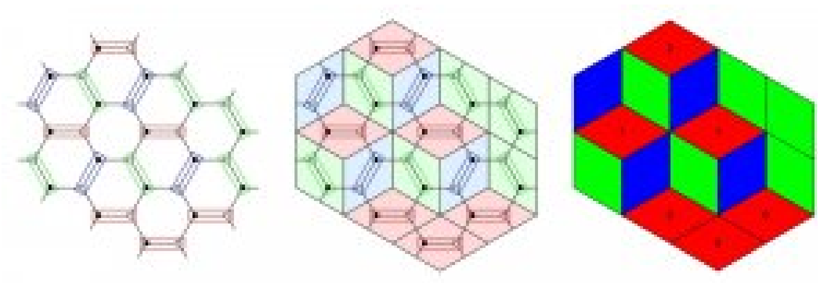

Figure 5: Center: tiling by lozenges of a hexagonal region with $a = b = c$ and $c = 3$ with 50 possible tilings. Left: Representation of the tiling by a perfect matching. Right: interpreting a hexagonal tiling as a pile of cubes; numbers indicate the height with respect to the horizontal reference plane for cubes of side length 1.

From a physical point of view, we can think of the adsorption of diatomic molecules on the surface of a crystal in solution as being modeled. In this case, a dimer is a diatomic molecule, and to describe the dimer model, we think of a dimer as an edge of a perfect matching of a very large finite graph that models the structure of the crystal. To understand this phenomenon, statistical physics leads us to study a random perfect matching. In the following, we will therefore discuss probability measures on the set of perfect matchings.

To go further

In statistical physics, any edge e is associated a positive weight $\nu(e) > 0$, representing the logarithm of some form of internal energy, and the total weight of a matching can be defined as the product of the weights on the edges of the matching:

$$\mathcal{Z}_\nu = \sum_{\omega} \prod_{e \in \omega} \nu(e)$$

For a finite graph, the weighted sum over all possible perfect matchings is called the partition function of the model. In the case where all weights are equal to 1, the partition function is nothing other than the number of perfect matchings. The probability

then the doctoral supervisor of the German physicist Ernst Ising, as a simplified model of magnetic interaction. The study of the Ising model in dimension 2 via the dimer model was further developed by Kasteleyn and, in parallel, by the English physicists Michael Fisher and Harold Temperley. This approach gave a detailed understanding of the Ising model in dimension 2. An introduction to this model has been published in this article from Images des maths.

measure under consideration associates to each matching a probability

$$\mathbb{P}(\omega) = \frac{\prod_{e\in\omega} \nu(e)}{\mathcal{Z}_\nu}.$$

From a physical and mathematical point of view, it is interesting to study the probability measures on the perfect matchings of an infinite graph, which we will assume to be periodic. We therefore imagine an infinite biperiodic planar bipartite graph like the square lattice. But be careful, this graph is weighted (by weights that we also assume to be periodic).

We are not interested in all possible measures. A first obvious constraint is that we impose translation invariance, which represents a form of isotropy of the model. Two physical principles determine the additional characteristics that probability measures of interest must verify. The first comes from the theory of the American physicist Josiah Willard Gibbs of systems in thermodynamic equilibrium, which predicts the structure that a probability measure must take: in short, the measure in a given region must be invariant under resampling in a disjoint region (the Gibbs measures of the Ising model were discussed here). The second comes from the ergodicity hypothesis formulated by the Austrian physicist Ludwig Boltzmann. In technical mathematical jargon, this is called an **ergodic Gibbs measure**.

Probability and algebraic geometry

In a remarkable work from 2003, Richard Kenyon, Andrei Okounkov and Scott Sheffield succeeded in classifying and describing precisely all the ergodic Gibbs measures of the dimer model on a biperiodic planar graph. Let us try to sketch the heuristic outlines of this major result.

If a fundamental domain of the graph[10] contains vertices of each color, there exists an explicit Laurent polynomial P in two variables of degree d, called the characteristic polynomial, which serves as an elementary building block to calculate all the interesting quantities of the model. For example, if the graph is the hexagonal lattice (with all weights equal to 1), the fundamental domain is represented in Figure 6, $d = 1$ and we have $P(z, w) = 1 + z + 1/w$.

The Newton polygon of P is the convex hull $\mathcal{N}(P)$ of integer points $(i, j) \in \mathbb{Z}^2$ such that $z^i w^j$ is a monomial in P; see Figure 6 for the Newton polygon

[10] A fundamental domain denotes a smallest portion of the graph which determines the infinite graph by translation in two directions.

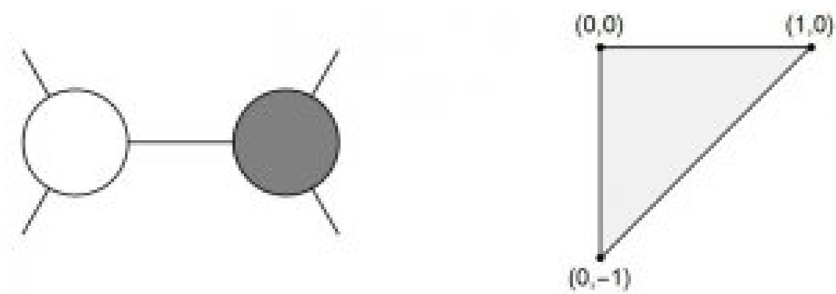

Figure 6: A fundamental domain of a hexagonal lattice whose characteristic polynomial is $P(z, w) = 1 + z + 1/w$ and the corresponding Newton polygon

corresponding to the hexagonal lattice. A theorem of Sheffield shows that the Newton polygon $\mathcal{N}(P)$ provides a parametrization of all ergodic Gibbs probability measures.

In order to understand how a point in the Newton polygon $\mathcal{N}(P)$ corresponds to a probability measure, a *height function* is introduced. The representation of a tiling via the height function goes back to John Conway, and was popularized by William Thurston [10].

The height function of a tiling is an integer-valued function on the internal faces of the graph (the bounded connected components of the complement of the graph in the plane: in the square lattice they are the unit squares; in the hexagonal lattice, the hexagons). There is some freedom in how one defines it. For the square lattice (of our chessboard), it is often defined as follows (up to an additive constant). Given its value h on one face, it is defined for the neighboring faces by moving along the sides of a domino: the height function decreases by 1 whenever one passes a black vertex on the left (and therefore a white vertex on the right) and increases by 1 when passing a white vertex on the left (and therefore a black vertex on the right).

To define it everywhere, one starts by extending the function to faces adjacent to the same domino, and then gradually to all faces (this procedure is well defined since the net change in height going around a domino is 0).

This definition might seem a bit mysterious, but we will see later that it gives rise to some nice limits. It is important to understand that the height function, or more precisely the difference of two height functions, has as its level lines paths formed by double dominoes as in Figure 7, this provides a more direct definition valid for any planar bipartite graph as we now explain.

Modulo a multiplicative factor, we can define this function as follows (see

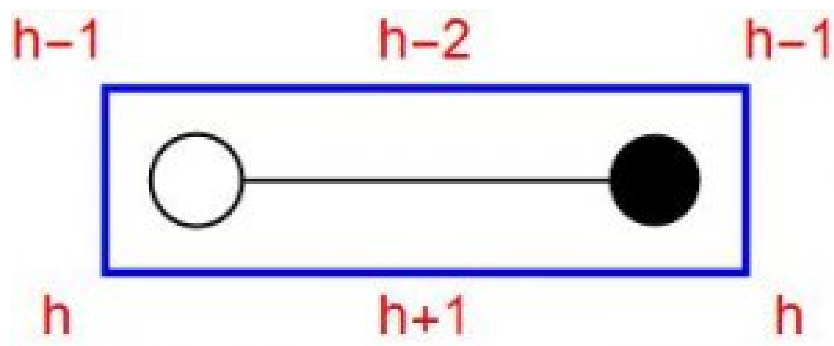

Figure 7): it associates an integer to any interior face of the graph. We are given two bags of dominoes: some are blue, others red. Construct a first blue tiling. On top, place a red tiling. We remove the pairs of dominoes that overlap exactly. Now let's see what remains. These are simple loops of even length. Moreover, they are oriented if we agree that a cycle inherits the black-white orientation from the red dominoes' tiling. By fixing the value of the function on one face, we define it for the other faces by adding 1 or -1 each time we cross a cycle oriented clockwise or counterclockwise respectively. Given the blue configuration, this method associates a function to any red tiling. The height function is not unique and to determine it one must specify a reference choice — the choice of the blue tiling (this can be expressed in terms of boundary conditions).

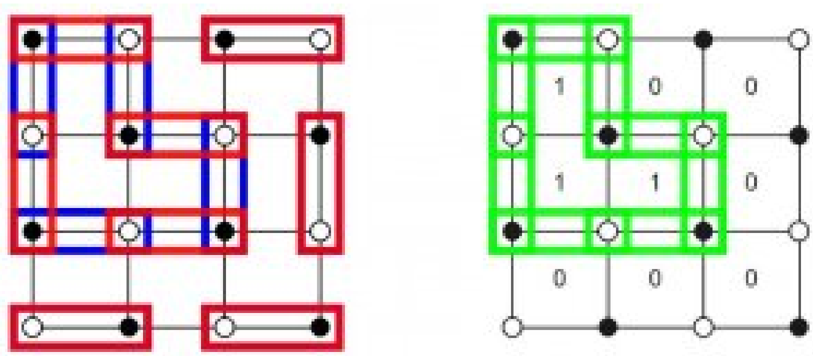

Figure 7: On the left: two superposed domino tilings; on the right: the associated loops and their height function.

It turns out that in the case of a hexagonal lattice, the graph of the height function is, up to a scaling factor, the surface whose tiling is a projection when seen as a pile of cubes in Figure 5).

If one fixes a bounded domain in the plane whose boundary is analytic and covers it by a large finite portion of the infinite graph with a very fine lattice such that there exist dimer coverings, one can look at the sequence of random tilings. One then obtains (after rescaling) a *limit surface* for the corresponding height function (this is a kind of law of large numbers) which is a minimal surface determined by its surface tension; this is a theorem of Henry Cohn, Richard Kenyon and James Propp. The slope of this surface reflects the local randomness of the tiling. In particular, near the boundary there is no randomness: the configuration is frozen and we speak of a *limit shape*. See an example in Figure 8 [11] Sheffield proved that the set of possible gradients (slopes in the two

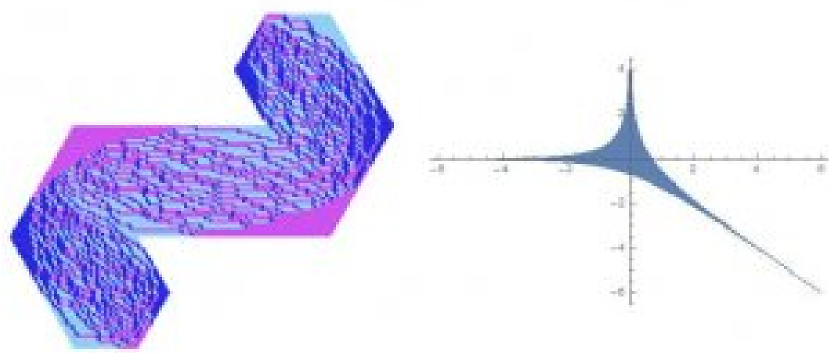

Figure 8: Left: a limit shape in a domain approximated by the hexagonal lattice (sampled by Richard Kenyon); Right: the amoeba of the polynomial $P(z, w) = 1 + z + 1/w$ associated with the hexagonal lattice.

coordinate directions) is $\mathcal{N}(P)$, but the link between a slope (s, t) and the corresponding probability measure is very subtle [12] — yet it is possible to understand this link heuristically, as we will see now. As can be seen from Figure 8[13], some

[11] The circular limit shape obtained in the case of a particular shape, called the Aztec diamond (a certain portion of the square lattice), has been described in in an Images des Maths article by Élise Janvresse and Thierry de la Rue.

[12] For specialists: the free energy of the probability measure associated with (s, t) is given by the Legendre dual evaluated at (s, t) of the Ronkin function of P.

[13] it is possible to sample the dimer model using the Coupling from the past algorithm of Propp and Wilson.

parts are more deterministic and others are more random. There is a classification of this randomness that mimics the usual physical classification: gaseous, liquid, or solid (frozen) state. To describe this phase diagram elegantly, Kenyon, Okounkov, and Sheffield used a subtle change of variable using another set associated with the polynomial P, namely, its *amoeba*.

To go further: Amoebas

The amoeba $\mathscr{A}(P)$[14] *of a polynomial* P *is defined as the logarithmic image of the modulus of elements of the set* $\{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\}$, *called the spectral curve of* P. *In other words,*

$$\mathscr{A}(P) = \{(\log|z|, \log|w|) \mid P(z,w) = 0,\ (z,w) \in \mathbb{C}^2\}.$$

When the amoeba has the same area as the Newton polygon, one says the spectral curve is Harnack. This is related to algebraic properties of the curve (Mikhalkin, Rullgård). One of the results of Kenyon, Okounkov and Sheffield was to prove that the spectral curve of a dimer model is a Harnack curve. Moreover, Kenyon and Okounkov have demonstrated the spectacular result that any Harnack curve is the spectral curve of such a dimer model. See the overview texts by Kenyon [5] and Felder [2] for more detail.

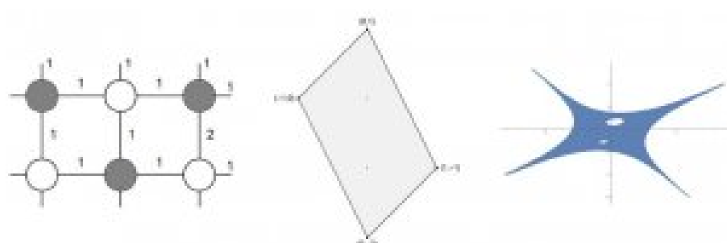

Figure 9: A fundamental domain of a weighted infinite square graph whose characteristic polynomial is $P(z,w) = 9 - 2w + 1/w^2 - 7/w + 1/z + z/w$; the corresponding Newton polygon; the corresponding amoeba. The amoeba represents the phase diagram: the white zone outside corresponds to frozen measures, the blue zone to liquid measures, and the two white zones inside to gaseous measures.

[14] This notion was introduced in 1994 by I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky in their influential book from 1994.

Another example providing gaseous measures has been studied on an Aztec diamond domain. In Figure 10, one clearly sees the three different phases from the center outward: gaseous, liquid, solid.

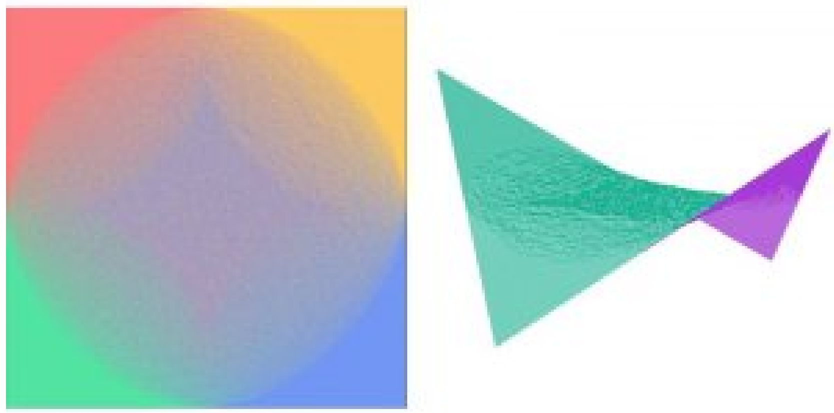

Figure 10: A dimer model on a weighted square graph in a bounded domain shaped as an Aztec diamond (rotated 45° for convenience). Left: the dominoes are of 4 colors depending on orientation and parity. Corners: frozen; circular region: liquid; center: gaseous. Right: the height function (by Vincent Beffara).

The fundamental domain of a weighted graph is given in Figure 11 along with the Newton polygon and the associated amoeba.

For more details please refer to Kenyon's lectures [6] as well as [3].

Conformal geometry and random fractals

The limit of the height function when the graph edges tend to zero (the scaling limit) was discussed earlier. We can understand this convergence result as a law of large numbers. This limit is very sensitive to the shape of the domain but also to the type of periodic graph that approaches it.

But what about its fluctuations (in short: the central limit theorem)? And what about the cycles associated with the overlapping of the blue and red dominoes? It turns out that there are underlying universal random objects that are independent of the microscopic structure of the tiling. These objects somehow forget all the complexity of the tiling and retain only its random fluctuations.

Under good assumptions (see the work of Richard Kenyon, Béatrice de Tilière, and more recently Leonid Petrov), the fluctuations of the height function in a

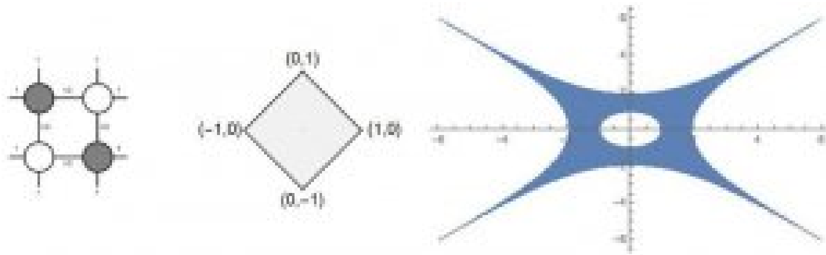

Figure 11: Fundamental domain of the underlying weighted infinite biperiodic square graph. Its characteristic polynomial is $P(z,w) = -(5/2) - 1/(2w) - w/2 - 1/(2z) - z/2$; its Newton polygon and amoeba are shown. The "hole" in the center of the amoeba corresponds to the gaseous phase.

liquid region of the phase diagram converge towards the Gaussian free field (see Figure 12), a distribution (in the sense of Laurent Schwartz) which is random [15].

It is conjectured[16] that the loops obtained from the double-domino model converge to the conformal set of closed curves with parameter $k = 4$ (see Figure 13). This set of loops is part of a one-parameter family of fractal objects introduced by Scott Sheffield and Wendelin Werner in 2010.

These two random fractal objects share two fundamental properties: the Markov property (what happens in two distant disjoint parts is independent of each other) and conformal invariance. It is this second notion that makes the link with complex geometry. It means that if we transform the domain into another by a map that preserves the angles locally (such a map is said to be conformal), then the law of these two random objects is unchanged. A popular discussion of this can be found here and there.

For the past 20 years, the probabilistic study of conformal geometry (or conformal field theory in physics) has been expanding rapidly. See this and that

[15] Even though the random field shown in Figure 12 already looks very irregular, what we see here is actually a discrete regularization by discrete approximation of the real object — the object itself is not a function, and so we cannot draw its graph.

[16] Works by Richard Kenyon and Julien Dubédat support this conjecture.

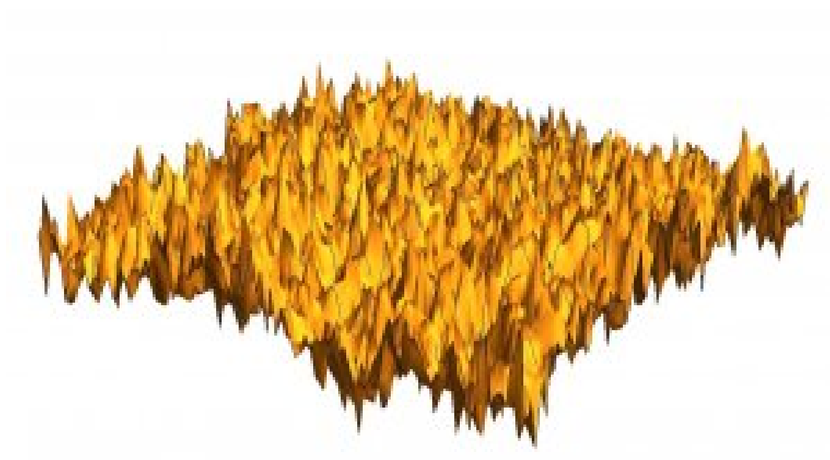

Figure 12: A Gaussian free field

article. Significant progress has been made since then, notably with the recent advent of so-called imaginary geometry, notably thanks to the work of Oded Schramm, Wendelin Werner, Greg Lawler, Scott Sheffield, Jason Miller, Julien Dubédat, Bertrand Duplantier and many others. See also the related articles here and in [1]. Two of the most important objects in this theory are the conformal set of loops with parameter $k = 4$ and the Gaussian free field.

These fractal objects possess fascinating properties. In particular, one can make sense of the following assertion: the loops of the conformal set of loops (Figure 13) are the level curves of the Gaussian free field (Figure 12). From the point of view of their discrete analogues (i.e., height functions), this is the definition, but proving it for fractal objects is another matter!

Open problems

It seems reasonable to think that the domino game still has many surprises in store for us. In conclusion, we mention three areas of active research related to dominoes.

In algorithmic and combinatorial optimization, many applied problems require efficient algorithms to find a perfect matching (the so-called assignment problem) for a huge graph with certain properties (for example, if it is a sparse graph, i.e., one with few edges relative to the number of vertices, or, conversely,

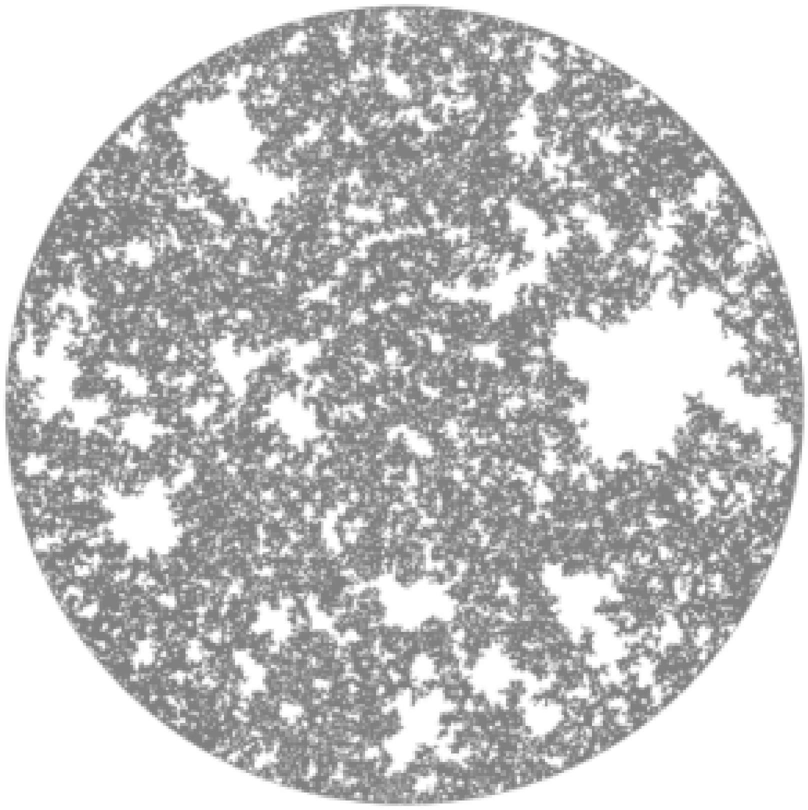

Figure 13: Conformal set of closed curves with the parameter $k = 4$ (example by David Wilson).

if the graph is dense), and it is important to optimize them. It is also important to calculate an approximate number of possibilities.

In physics, it is interesting to consider polymer models that are closer to reality. However, Kasteleyn's method no longer works in this case. Some models allow interactions between dominoes or consider graphs that are no longer planar. Breakthroughs in the first direction have recently been made by a group of Italian researchers (Alessandro Giuliani, Vieri Mastropietro, Fabio Lucio Toninelli) using ideas from the physical theory known as renormalization flow.

From a mathematical point of view (in combinatorics and geometry), it is tempting to change the initial rules and replace the domino with other elementary tiles, such as triominoes (size 1×3), other elongated rectangles, or other polyominoes. In this case, the techniques mentioned above no longer apply directly, and new ones must be invented.

State of the art in 2025

In the decade since the original write-up of this invitation to the dimer model, the research community has produced new results, insights, and developments in the study of this model. A comprehensive survey would go far beyond the scope of this article, but let us highlight a few advances closely related to the themes discussed above.

Boutillier, Cimasoni and de Tilière have extended the description of the spectral data associated with a dimer model on an infinite (so-called minimal) bipartite planar graph, following the steps taken by Goncharov-Kenyon and Fock, based on the initial Kenyon-Okounkov-Sheffield description mentioned above. They managed to describe completely the set of spectral curves appearing, Harnack curves of higher arbitrary genus, thus establishing a beautiful correspondence between a probabilistic model and the classical theory of compact Riemann surfaces.

Lozenge tilings of planar domains with holes have been successfully studied by Gorin, Petrov, and others. In joint work with Borot and Guionnet, they established connections with discrete beta ensembles and random matrix theory, thus contributing to the now-vast field known as integrable probability.

The connection between the level lines of the dimer height function and the conformal loop ensemble with parameter 4 has been further substantiated in works by Basok–Chelkak and Lis–Rey–Ryan.

New and impressive results by Catherine Wolfram, in collaboration with Nishant Chandgotia and Scott Sheffield, have established large deviation principles for three-dimensional dimers extending the analysis beyond the integrable planar case. A recent preprint by Caroline J. Klivans and Nicolau C. Saldanha Domino tilings beyond 2D surveys what is currently known about dimers in higher dimensions.

In another line of research, Kenyon, Prause, Wolfram, and others introduced and studied N-fold dimer models, a generalization of the double-dimer model. They obtained intriguing large-N limits and found exactly solvable solutions even on non-planar graphs, despite the general non-solvability of the dimer model in such settings. Kenyon and collaborators have also explored quantum deformations of these N-fold dimer models.

Recent progress on the study of trimer tilings has also been presented by

James Propp.

Interested reader may consult a list of recent papers by a France-based group of researchers on this topic and view talks from a recent conference dedicated to the subject – and honoring one of its central figures, Richard Kenyon.

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References

[1] Vincent Beffara , *Raconte-moi... le processus SLE*, Gaz. Math. (144), 59—63, 2015.

[2] Giovanni Felder, *The work of Andrei Okounkov*, International Congress of Mathematicians. Vol. I, 55—64, Eur. Math. Soc., Zurich, 2007. Link

[3] Patrick Ferrari and Béatrice de Tilière, *Dimer models and random tilings*, Panoramas et Synthèses, SMF, 2015.

[4] Frank den Hollander, *Pieter Willem Kasteleyn: October 12, 1924 — January 16, 1996*, Journal of Statistical Physics, 85(5):801—806, 1996. Link

[5] Richard Kenyon *Les travaux d'Andrei Okounkov sur le modèle des dimères*, Gaz. Math., (112):18—22, 2007.

[6] Richard Kenyon, *Lectures on dimers*, In Statistical mechanics, volume 16 of IAS/Park City Math. Ser., 191—230. Amer. Math. Soc., Providence, RI, 2009. Link

[7] László Lovász and Michael D. Plummer, *Matching theory*, AMS Chelsea Publishing, Providence, RI, 2009.

[8] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences.*

[9] Yann Olivier, *Jacobi's Bound.*

[10] William P. Thurston, *Conway's tiling groups*, Amer. Math. Monthly, 97(8):757—773, 1990. Link

English translation by Olga Kravchenko with an addition by the author.