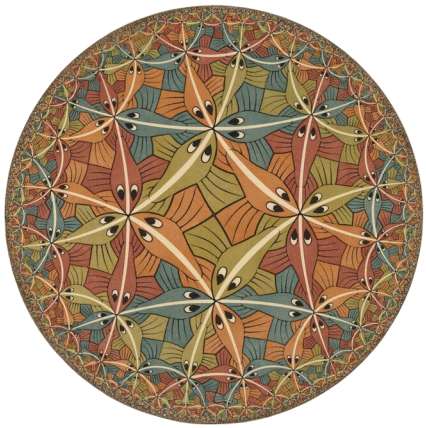
A Bit of Geometric Group Theory

by Gilbert Levitt



Maurits Cornelis Escher, *Circle Limit III*, Print, 1959

Discrete groups appear in every area of mathematics — and even in Escher's art. Even if they are defined algebraically, we often understand them better by their action on geometric objects. More and more often, they are viewed as geometric entities in their own right. Their properties are especially striking when the curvature is negative.

1. A Few Examples of Groups

Groups we will consider will be generally non-commutative. We will write the group operation multiplicatively, with the neutral element in a group G denoted $\mathbf{1}_G$ or simply $\mathbf{1}$. Groups that will interest us most are finitely generated — that is, they can be generated by a finite number of elements. Let's take a look at a few examples.

- The *free Abelian group* $\mathbb{Z}^2$, also written as $\mathbb{Z} \times \mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z}$, is the set of pairs of integers (m, n), with addition defined by $(m, n) + (m', n') = (m + m', n + n')$. To write it multiplicatively, let $a := (1, 0)$, $b := (0, 1)$, and view $\mathbb{Z}^2$ as the set of elements $a^m b^n$, equipped with the multiplication rule $(a^m b^n)(a^{m'} b^{n'}) = a^{m+m'} b^{n+n'}$. The neutral element $a^0 b^0$ is denoted $\mathbf{1}$, and the inverse of $a^m b^n$ is $a^{-m} b^{-n}$.
- Let us consider the group $\mathrm{Aff}(\mathbb{R})$ acting on the real line $\mathbb{R}$ by homotheties and translations — that is, transformations of the form $x \mapsto a \cdot x + b$ with $a, b \in \mathbb{R}$ and $a \neq 0$, the product being given by composition: $(f \circ g)(x) = f(g(x))$.

  This is a "continuous" group (a Lie group), but we can consider finitely generated subgroups, for example, the group G_1 generated by $t : x \mapsto x + 1$ and $h : x \mapsto 2x$. One can deduce that G_1 is the set of transformations f_{mnp}

of the form[a] $f_{mnp}(x) = 2^m x + n/2^p$, with $m, n, p \in \mathbb{Z}$. Indeed the identities

$$(f_{mnp} \circ f_{m'n'p'})(x) = 2^{m+m'} x + \frac{n' \cdot 2^{m+p} + n \cdot 2^{p'}}{2^{p+p'}} = f_{m''n''p''}(x)$$

with $m'' = m + m'$, $n'' = n' \cdot 2^{m+p} + n \cdot 2^{p'}$, $p'' = p + p'$, and $f_{mnp}^{-1} = f_{m'n'p'}$ where $m' = -m$, $n' = -n$, and $p' = m + p$, show that the set $\{f_{mnp} : m, n, p \in \mathbb{Z}\}$ is a subgroup. It contains h and t, and it is the smallest such subgroup because $f_{mnp} = h^{-p} t^n h^{m+p}$ is contained in every subgroup that contains h and t.

- The group $\mathrm{GL}(n, \mathbb{R})$ of invertible (with determinant $\neq 0$) $n \times n$ matrices with real coefficients is also a Lie group.

  The matrices with integer entries do not form a subgroup, because the determinant appears in the denominator when computing the inverse of a matrix. However, $\mathrm{SL}(n, \mathbb{Z})$, the set of matrices with integer entries and determinant 1, is a subgroup.

  We will consider the group $G_2 \subset \mathrm{SL}(n, \mathbb{Z})$, generated by

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

2. Free Groups

In a group G, the subgroup generated by $g_1, \ldots, g_k$ is the set of all elements of G that can be written as a reduced word $g_{i_1}^{n_i} \ldots g_{i_p}^{n_p}$, where the n_j are nonzero integers and $i_j \neq i_{j+1}$. For example, a^2, $b^{-1}c$, and $c^{-3}a^3b^2acb^{-5}$ are reduced words in a, b, c. Care must be taken not to forget the empty word, denoted 1, which represents the identity element 1_G. The length $|W|$ of a word W is the total number of letters, taking exponents into account, for example $|c^{-3}a^3b^2acb^{-5}| = 15$.

We say that elements $g_1, \ldots, g_k$ of G are *independent*[b] (or form a *free family*) if two different reduced words always represent two different elements of G, or equivalently, if there is no nontrivial relation $g_{i_1}^{n_1} \cdots g_{i_p}^{n_p} = 1$. For example, the family $\{g\}$, consisting of the single element $g \in G$, is free if and only if there is no nontrivial relation $g^n = 1$, that is, if g has infinite order.

In the examples above, the families $a, b \subset \mathbb{Z}^2$ and $h, t \subset G_1$ are not free, because of the relations $ab = ba$ and $hth^{-1} = t^2$.

We will, however, show — using the so-called ping-pong technique — that the pair of matrices A and B is a free family in $\mathrm{SL}(n, \mathbb{Z})$.

To this end, let us make $\mathrm{SL}(n, \mathbb{Z})$ act on $P = \mathbb{R} \cup \infty$ (the real projective line) by associating to the $\mathrm{SL}(n, \mathbb{Z})$-matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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[a] Such a transformation does not determine the triple m, n, p uniquely, e.g. if $n' = 2n$ and $p' = p+1$ then $f_{mnp} = f_{mn'p'}$. (Ed.)

[b] Often, in English literature, *independence* of the collection $\{g_1, \ldots, g_k\}$ of elements of a group, means another, in general strictly stronger, property — namely, that no element is equal to a reduced word in other elements of the collection; thus, *free family* seems to be a more appropriate term and it is used below. (Ed.)

the *homography*[c]

$$h_M : x \mapsto \frac{ax+b}{cx+d}$$

with the usual conventions, in particular $h_M(-d/c) = \infty$ and $h_M(\infty) = a/c$ if $c \neq 0$. The definition is constructed so that $h_{MN} = h_M h_N$ for all pairs of $\mathrm{SL}(n,\mathbb{Z})$-matrices M and N.

Let $P_A = (-1,1)$, and let P_B be the complement of $[-1,1]$ in P. We have $h_A(x) = x+2$, and therefore $h_A^n(P_A) \subset P_B$ for all $n \neq 0$. Similarly, $h_B(x) = x/(2x+1)$ and $h_B^n(P_B) \subset P_A$ for $n \neq 0$.[d] Let's now play ping-pong with P_A and P_B.

To show that the pair A and B is a free family, consider a nontrivial reduced word, for example $W = B^2AB^{-3}A^5$. Apply $h_W = h_B^2 h_A h_B^{-3} h_A^5$ to P_A. The element h_A^5 sends it into P_B, the element h_B^{-3} sends it back into P_A, and so on, and finally $h_W(P_A)$ is contained in P_A, but not equal to it. This prevents h_W from being the identity, and therefore W from being equal to 1 in $\mathrm{SL}(2,\mathbb{Z})$.

This reasoning applies to any word W beginning with a power of B and ending with a power of A. The other cases are treated similarly: if W begins and ends with a power of A, we have $h_W \neq \mathrm{id}$ because $h_W(P_A) \subset P_B$; if W ends with a power of B, we apply h_W to P_B.

Since A and B form a free family, every element of G_2 can be written uniquely as a reduced word in A and B. At this point we can forget that A and B are matrices and regard G_2 as the set $F(A,B)$ of reduced words in two abstract symbols A and B. Multiplication consists of concatenation and reduction; for example, $(B^2AB^{-3}A^5)(A^{-5}BA^4) = B^2AB^{-2}A^4$, and the inverse of $B^2AB^{-3}A^5$ is $A^{-5}B^3A^{-1}B^{-2}$.

We say that G_2 is the free group of rank 2, often denoted F_2. Similarly, we define F_n, the free group of rank n, for $n > 2$.

Many groups contain subgroups which are free groups. For example, one can show that two randomly chosen rotations of the sphere generate a free group, as do the transformations $x \mapsto x+1$ and $x \mapsto x^3$ on $\mathbb{R}$.

The group F_2 contains arbitrarily large free families: it is easy to see, that the infinite family $\{A^nBA^{-n}\}_{n\in\mathbb{N}}$ is free, because the B's do not cancel when these elements are multiplied. The free group of rank 2 therefore contains free groups of any rank, and even free groups that are not finitely generated. The Nielsen–Schreier theorem guarantees that every subgroup of a free group is free, that is, it is generated by a free family.

3. Tits Alternative

We have already noted that G_1 is not free,[e] since its generators satisfy $hth^{-1} = t^2$. To find other relations, observe that in $\mathrm{Aff}(\mathbb{R})$, and therefore in G_1, every

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[c] *Homography* is a synonym for *projective transformation.* (Ed.)

[d] Note also, that the inclusions $h_A^n(P_A) \subset P_B$ and $h_B^n(P_B) \subset P_A$ are strict. (Ed.)

[e] Strictly speaking, it has only been shown that $\{h,t\}$ is not a free family of generators. This doesn't imply that G_1 is not a free group. However, looking at the relation, one immediately concludes that the abelianization of G_1 is generated by the single element h and therefore is of rank one, while every noncommutative free group must have abelianization of higher rank, $\mathrm{Ab}(F_n) \cong \mathbb{Z}^n$. (Ed.)

commutator $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ is a translation, and that two translations commute. Therefore any two commutators commute, $[g_1, g_2][g_3, g_4] = [g_3, g_4][g_1, g_2]$ for all $g_1, g_2, g_3, g_4 \in G_1$. This "universal" relation expresses that G_1 is *metabelian*, or equivalently, *solvable* of class 2.

More generally, we say that G is solvable of class $\leq p$ if the subgroup generated by all commutators $[g_1, g_2]$ is solvable of class $\leq p-1$, that is, if any $2p$ elements of G satisfy a certain identity built from iterated commutators. Solvable groups are those that can be obtained by successive extensions from commutative groups. The impossibility of solving algebraic equations of degree 5 by radicals is due to the non-solvability of the symmetric group S_5. This is the subject of the *Galois theory*.

It is easy to verify that, for any field K, the subgroup of $\mathrm{GL}(n, K)$ consisting of invertible upper triangular matrices is solvable (of class n). The famous Tits alternative (1972) states that if a finitely generated group G is linear — that is, isomorphic to a subgroup of some $\mathrm{GL}(n, K)$ — then either

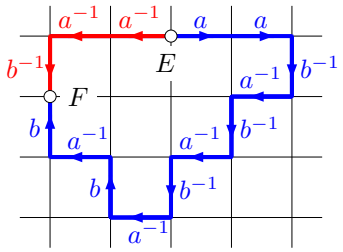
- G contains a subgroup isomorphic to F_2, or
- a subgroup of finite index in G is solvable.

In other words, either G contains arbitrarily large free families, or (up to finite index) the elements of G satisfy a universal relation. The Tits alternative has been extended to other classes of groups. For instance, Bestvina, Feighn, and Handel have recently proved it for subgroups of the group $\mathrm{Out}(F_n)$ of automorphisms of a finitely generated free group, modulo conjugations.

4. Relations and Presentations

If a group G is not free, different reduced words may represent the same element; we say that such words are equivalent in G. The *word problem* is the problem of determining, by an algorithm, whether two given words represent the same element. In fact, it suffices to determine which words are trivial, that is, which represent the neutral element $\mathbf{1}_G$.

This is easy in $\mathbb{Z}^2$, which is commutative. For instance, it is immediately clear to us that $a^{100}b^{100}a^{-100}b^{-100} = \mathbf{1}$. But a machine that could only apply mechanically the basic relation $ab = ba$ (and, to be generous, the relations $a^{\pm 1}b^{\pm 1} = b^{\pm 1}a^{\pm 1}$) would find it tedious to show this equality: it would, in fact, have to move each of the one hundred a's past each b — that is, about 10 000 operations for a word of length 400. In general, the number of operations required to show that a word of length n in $\mathbb{Z}^2$ is trivial, in the worst case, is of the order of n^2 for large n. We say that $\mathbb{Z}^2$ has a quadratic isoperimetric inequality.



The words $a^2b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}b$ and $a^{-2}b^{-1}$ represent the same element of $\mathbb{Z}^2$: the corresponding paths have the same endpoint F.

The geometric interpretation is as follows (see figure). Tile the plane with a grid whose horizontal edges oriented to the right are labeled a, and whose vertical edges oriented upward are labeled b. Fix a vertex E of this graph as the origin. A word in a and b can then be represented as a path starting from E; for example, $a^2b^{-1}a^{-1}b^{-1}$ moves two units to the right, one down, one to the left, and one down again.

We note that two words are equivalent if and only if their associated paths have the same endpoint; for example, $a^2b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}ba^{-1}b$ is equivalent to $a^{-2}b^{-1}$. In particular, the vertices of the graph correspond to the elements of $\mathbb{Z}^2$, and a word is trivial if and only if the associated path is a loop (it closes back at E). Thus, $a^{100}b^{100}a^{-100}b^{-100}$ represents the boundary of a square of side 100. Applying the relation $a^{\pm1}b^{\pm1} = b^{\pm1}a^{\pm1}$ amounts to changing the loop to bypass one cell of the grid, and 10 000 is simply the area of the square.

The exponent 2 obtained above is thus the one that expresses the area of a square as a function of its side. We can see the analogy with the classical isoperimetric inequality, which bounds the area enclosed by a simple plane curve by the square of its length (divided by 4π, though that detail is not important here).

Returning to algebra, we now explain how to solve the word problem in G_1 using only the relation $hth^{-1}t^{-2} = 1$. Thanks to the equations $ht^{\pm1} = t^{\pm2}h$ and $t^{\pm1}h^{-1} = h^{-1}t^{\pm2}$, one can, in any word, move all positive powers of h to the right of the word and all negative powers to the left. In other words, any word W is equivalent in G_1 to a word of the form $h^{-m}t^nh^p$ with $m, p \geq 0$. Such a word represents the transformation $x \mapsto 2^{p-m}x + 2^{-m}n$, which is the identity if and only if $n = 0$ and $p = m$, that is, if the word is empty. Therefore, $W = 1$ in G_1 if and only if the word $h^{-m}t^nh^p$ associated to W is the empty word: the word problem is solved.

This reasoning actually shows that all relations satisfied by h and t can be formally deduced from the relation $hth^{-1}t^{-2} = 1$. We say that G_1 is presented by the generators h and t subject to the relation $hth^{-1}t^{-2} = 1$.

In general, we say that $G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$, where the r_j are words in the g_i, is a *presentation* of G if G is generated by elements g_i satisfying the relations $r_j = 1$, and if every relation among the g_i can be formally deduced from the relations $r_j = 1$ (more precisely, G is isomorphic to the quotient of the free group $F(g_1, \ldots, g_k)$ by the subgroup generated by all products of conjugates of the r_j and their inverses).

Fix an integer m, and now ask a machine to prove the relation $[h^mth^{-m}, t] = 1$ from $hth^{-1}t^{-2} = 1$. This is easy for us, since we can see that $h^mth^{-m} = t^{2^m}$. But for the machine, the number of operations will be on the order of 2^m, that is, an exponential function of the length of $[h^mth^{-m}, t]$, which is equal to $4m + 4$. Since the words $[h^mth^{-m}, t]$ are representatives of the general case, G_1 satisfies an exponential isoperimetric inequality.

5. The Dehn Function

Given a finite presentation $G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$, we define the *Dehn function* $\varphi(n)$, whose growth determines the isoperimetric inequality satisfied by G. A replacement such as $a^{\pm1}b^{\pm1} \mapsto b^{\pm1}a^{\pm1}$ or $ht^{\pm1} \mapsto t^{\pm2}h$ amounts to multiplying the word by a conjugate of some $(r_j)^{\pm1}$, and a word W is trivial in G if and only if, in the free group $F(g_1, \ldots, g_k)$, it can be written[f] as $W = \prod_{m=1}^{s} u_m(r_{j_m})^{\pm1}u_m^{-1}$ for some elements $u_m \in F(g_1, \ldots, g_k)$. For each trivial word W, we consider the smallest possible s, and define $\varphi(n)$ to be the maximum of these s for all trivial words of length $\leq n$.

[f] Note that the right-hand side in the expression for W need not be a reduced word. (Ed.)

The Dehn function depends on the presentation, but the manner of its growth[g] (quadratic, exponential, etc.) depends only on G. We have already seen that φ is quadratic for $\mathbb{Z}^2$ and exponential for G_1; here is an example of a linear φ.

Let G_3 be the group with presentation $\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle$ (the fundamental group of the closed orientable surface of genus 2). Given a word W in a, b, c, d, we can shorten it if it contains more than half of the relation (or its inverse), up to cyclic permutation. For instance, we may replace $aba^{-1}b^{-1}c$ by dcd^{-1}, or $d^{-1}c^{-1}bab^{-1}$ by $c^{-1}d^{-1}a$, or $dc^{-1}d^{-1}ab$ by $c^{-1}ba$, and so on. We then reduce the resulting word (if possible) and repeat the process as long as possible.

Dehn showed (around 1910) that in G_3, this procedure — called Dehn's algorithm — solves the word problem: W represents 1 if, and more importantly only if, the above algorithm terminates with the empty word. Since the length of the word decreases at each step, the number of operations is bounded by the length of the word; thus G_3 satisfies a linear isoperimetric inequality.

Groups in which the word problem can be solved by Dehn's algorithm (shortening the word whenever it contains more than half of a relation) have a Dehn function that is at most linear. Conversely, one can show that a group with at most linear Dehn function admits a presentation for which Dehn's algorithm applies. These groups are precisely the hyperbolic groups defined by Gromov around 1985; we will discuss their geometric aspects below.

If G is not hyperbolic, its Dehn function is at least quadratic. On the other hand, there is no "gap" beyond the exponent 2: N. Brady and M. Bridson have recently shown[h] that the set of exponents α for which there exists a group whose Dehn function is equivalent to n^α is dense in $[2, +\infty)$. Note that the set of isomorphism classes of finitely presented groups is countable, and therefore so is the set of these α.

Knowing the Dehn function of a finitely presented group explicitly allows one to solve the word problem algorithmically in that group: to determine whether a word W of length n is trivial, it suffices to compare it with all expressions $\prod_{m=1}^{s} u_m (r_{j_m})^{\pm 1} u_m^{-1}$ with $s \leq \varphi(n)$, of which there are only finitely many (the lengths of the words u_m can be bounded a priori). Conversely, an algorithm that solves the word problem makes it possible to compute φ.

It is known that there exist finitely presented groups in which the word problem cannot be solved algorithmically, because the Dehn function is non-recursive: it grows faster than any recursive function, thus no algorithm can compute it. Therefore, in complete generality, nothing can be said about a group given by generators and relations — not even whether the group is trivial or not. However, in most cases, any algebraic or geometric information about G, even minimal, allows one to analyze it.

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[g] The growth rate is captured by the following preorder (transitive, reflexive, but not antisymmetric relation) on the set of nondecreasing functions defined on $\mathbb{N}$: function φ is dominated by ψ if there is $C \in \mathbb{N}$ such that $\varphi(n) \leq C \cdot \psi(C \cdot n + C) + C$ for all $n \in \mathbb{N}$. (Ed.)

[h] N. Brady and M. R. Bridson. *There is only one gap in the isoperimetric spectrum.* Geometric and Functional Analysis 10, no. 5 (2000), pp. 1053-1070. (Ed.)

6. The Hyperbolic Plane $\mathbb{H}^2$

Let us look at the group G_3 from a geometric point of view (as Dehn did). Let us try to construct a graph as we did for $\mathbb{Z}^2$. The graph[i] to consider is no longer of degree 4, but of degree 8: from each vertex emerge 4 edges labeled a, b, c, d, and 4 edges arrive there. The cells of the grid are octagons, corresponding to the relation $aba^{-1}b^{-1}cdc^{-1}d^{-1}$.

We can try to draw this graph, but we quickly run out of space to fit 8 edges at each vertex: the Euclidean plane cannot be tiled by regular octagons. This graph must actually be drawn not in the Euclidean plane, but in the *hyperbolic plane* $\mathbb{H}^2$.

Imagine a circular swimming pool (more mathematically, the open unit disk D in the plane) filled with a viscous fluid that becomes denser as one approaches the edge: the viscosity coefficient is proportional to $\frac{1}{1-r^2}$, where r is the Euclidean distance from the center of the disk. The (hyperbolic!) distance between two points x, y in D is defined as the time it would take a swimmer to go from x to y.

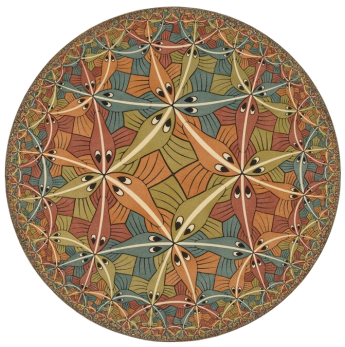
There always exists a *shortest path* from x to y (called a *geodesic*), but it does not appear straight to us: it bends toward the center of the disk to allow a faster route, just as an airplane climbs in altitude to reduce air resistance. The geodesics are in fact arcs of circles perpendicular to the boundary of D, as well as the diameters (note that the boundary of D is "at infinity"; it cannot be reached in finite time).

Like the Euclidean plane, the hyperbolic plane is a homogeneous metric space: any point can be sent to any other by an isometry; in particular, the center of D plays no special role; any Möbius transformation of the complex plane that maps D onto itself induces an isometry.

But hyperbolic plane has negative curvature, whereas the Euclidean plane has zero curvature, and the sphere has positive curvature.

To study G_3, we tile $\mathbb{H}^2$ with regular octagons whose sides are geodesic segments of equal length and whose angles are $2\pi/8$, in such a way that eight octagons meet at each vertex. The graph associated with such a tiling is precisely the grid we were seeking for G_3.

The Dehn function of G_3 is therefore linear, since $\mathbb{H}^2$ satisfies a linear isoperimetric inequality: the area enclosed by a curve can be bounded by a linear function of its length. For example, a disk of radius R has area $2\pi \sinh R$, which is comparable to its perimeter $2\pi(\cosh R - 1)$. (It's better to do jigsaw puzzles in the hyperbolic plane: once you've placed the border, you have already set a non-negligible proportion of the pieces.)



M. C. Escher used tilings of $\mathbb{H}^2$; for example, *Circle Limit III* evokes a tiling by regular triangles and quadrilaterals, separated by white geodesic lines.

Elementary geometry in $\mathbb{H}^2$ holds other surprises. The Euclidean parallel postulate does not hold, and the sum of the angles of a triangle is not equal to π: it is equal to π minus the area of the triangle; in particular, the area of a triangle is at most π.

[i] Here we mean directed graph, that is, with oriented edges. (Ed.)

Another fundamental property of $\mathbb{H}^2$ is the thinness of triangles: there exists a constant δ (equal to $\log(2-\sqrt{2}+1)$) such that every point on one side of a geodesic triangle lies at distance at most δ from some point on one of the other two sides. This property is called δ-*hyperbolicity*, or simply hyperbolicity.

7. Hyperbolic Groups and Quasi-Isometries

We have seen that the group $\mathbb{Z}^2$ resembles the Euclidean plane, while G_3 resembles the hyperbolic plane. Following M. Gromov, we formalize this idea by viewing a group G, equipped with a finite generating set S, as a metric space: the distance between two elements g and h of G is the minimal length of a word (written with the elements of S) representing $g^{-1}h$.

This discrete space is more easily visualized as the set of vertices of the *Cayley graph* of G: we place an edge between two vertices g, h if h is obtained from g by right-multiplication by an element of S, and we declare that each edge is a segment of length 1. The distance between two points is then the length of a shortest path connecting them (such a path is again called a geodesic).

The Cayley graph[j] of the free group G_2 is a tree (it has no loops). That of $\mathbb{Z}^2$ is the square grid used earlier, with the so-called Manhattan (or taxicab) distance (note that in general there may be several geodesics between two given points). This distance is not the Euclidean one, which corresponds to "as the crow flies", but it is comparable: the ratio of the two distances lies between two strictly positive constants (here 1 and $\sqrt{2}$). Similarly, the Cayley graph of G_3 is formed by the geodesics bounding the octagons in the tiling of $\mathbb{H}^2$ mentioned above, with a distance comparable to the hyperbolic distance.

A group G is called *hyperbolic* if there exists a constant δ such that its Cayley graph is δ-hyperbolic, meaning that the triangles in the Cayley graph, like those in $\mathbb{H}^2$, are thin. Thus G_2 and G_3 are hyperbolic, while $\mathbb{Z}^2$ is not (and neither is G_1).

In our examples, we have always chosen the most natural, simplest generating set. But a finitely generated group has infinitely many generating sets, and hence infinitely many distinct Cayley graphs, so it is not immediately clear that all of them are hyperbolic if one of them is.

In fact, all these graphs resemble one another — just as the Cayley graph of $\mathbb{Z}^2$ resembles the Euclidean plane and that of G_3 resembles $\mathbb{H}^2$. This resemblance is to be understood in the sense of quasi-isometry.

Two metric spaces X, Y are *quasi-isometric* if there exists a map $f : X \to Y$ and a constant $\lambda > 1$ such that f does not distort distances too much for sufficiently distant points (the ratio between $d_Y(f(x), f(x'))$ and $d_X(x, x')$ lies between $\frac{1}{\lambda}$ and λ whenever $d_X(x, x') > \lambda$), and f is almost surjective (every ball of radius λ in Y contains a point of the image).

Every bounded space is quasi-isometric to a point; quasi-isometry is meant to capture asymptotic properties of spaces — that is, properties "at infinity." Two geodesic spaces that are quasi-isometric are simultaneously hyperbolic or not, which justifies the definition of a hyperbolic group given above.

Thus, to every finitely generated group one can associate a well-defined metric space, up to quasi-isometry. One can therefore speak of groups that are quasi-isometric to each other. Many algebraic or geometric properties of groups are

[j] with respect to a free generating set. (Ed.)

invariant under quasi-isometry: for example, being finite, finitely presented, containing a commutative subgroup of finite index, being hyperbolic, or having a Dehn function of a given growth type. In general, one may attempt to classify groups up to quasi-isometry.

Let us mention a recent rigidity result due to Farb and Mosher.[k] For an integer $n \geq 2$, let H_n be the subgroup of $\mathrm{Aff}(\mathbb{R})$ generated by $x \mapsto x+1$ and $x \mapsto nx$ (so H_2 is the G_1 studied above). If n is a power of m, then H_n is a subgroup of finite index in H_m, hence quasi-isometric to it. Conversely, Farb and Mosher have shown that H_n and H_m are quasi-isometric if and only if n and m are powers of the same integer, and (essentially) that any finitely generated group quasi-isometric to some H_n contains a finite-index subgroup isomorphic to H_{n^p}.

8. Literature

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