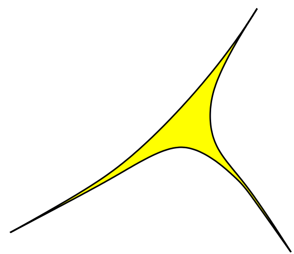
Triangles after Euclid, Gauss and Gromov

A small fragment of Misha Gromov's work

by Étienne Ghys



Euclid

For centuries, geometry was Euclid's geometry — the one we learn in school, with its right, isosceles and equilateral triangles and its theorems of Pythagoras and Thales; the geometry of "the world in which we live." Euclid established its foundations in the third century BCE in the book — a landmark for mathematicians, titled "The Elements".[1,a] For more than twenty centuries, this book stood at the heart of mathematics, so definitive did it seem.

Yet one of Euclid's axioms — a statement he asked readers to accept at the beginning of his book without proof — left a lingering unease: "Through a point chosen outside a given line, it is possible to draw exactly one line parallel to the given one." Generations of mathematicians tried to deduce it from the other, seemingly more natural axioms. Many believed they had succeeded; many were mistaken.

Non-Euclidean Geometry

At the beginning of the nineteenth century at least three mathematicians had the same idea almost simultaneously: there must exist a non-Euclidean geometry. These were Bolyai, Lobachevsky, and Gauss. All three broke with a long-held dogma — they proposed an entirely new conception of space. It was a genuine mathematical revolution, comparable to the Copernican revolution of the fifteenth century.

Of course, it would take very lengthy historical explanations to do this justice, but that is not the purpose of this article. One of the central questions concerns existence: what does it mean for another geometry to exist besides Euclidean? Several geometries? (I note in passing that my spell-checker refuses to put the word "geometry" in the plural!) After all, we live in only one space, not two... Isn't the purpose of geometry to describe the relationships between distances of points in *our* world?

Indeed, this was one of the first times mathematicians needed the courage to assert that one can (and should) work with spaces that may be entirely imaginary, with no obvious connection to practical problems. The links between mathematics

[1]Euclid, *The Elements*

[a]English translation available for download, for example, at `https://archive.org/details/euclid\_heath_2nd_ed/1_euclid_heath_2nd_ed/` (Ed.)

and physics — which, in principle, deals with what happens in our real world — have always been complex.

In any case, at the beginning of the nineteenth century a few visionaries began studying a "new geometry" in which Euclid's parallel postulate does not hold. They proved theorems that differ from Euclid's. For example, right triangles still exist, but Pythagorean theorem is different... Let there be no mistake: this does not mean that the Pythagorean theorem we learn in school is false; it means that it is true in Euclidean space, and that a different theorem holds in non-Euclidean space — there is no contradiction here. This "new geometry" is sometimes called non-Euclidean, sometimes hyperbolic, and sometimes the Bolyai–Lobachevskian geometry...

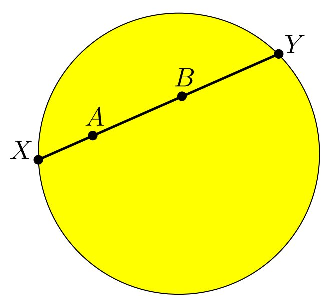
But all this still seemed somewhat suspicious and its very existence was called into question. Things gradually became clearer throughout the nineteenth century. As always, it took time and the efforts of many mathematicians, but eventually it was understood that if one accepts Euclidean geometry, one must also accept the other.

Here is one of the models of non-Euclidean geometry used, often called the Cayley model. Consider a disk in the Euclidean plane, and forget all the points that are not inside this disk — only the points within the disk matter to us. Take two points, A and B, connect them with a straight line, and mark the points X and Y, where this line intersects the boundary circle.

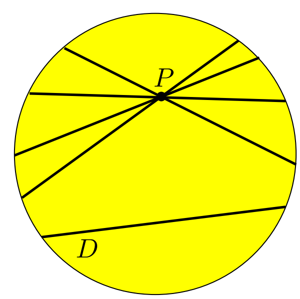
Now, define the non-Euclidean distance between the points A and B by the following magical formula (don't worry if you don't quite remember what a logarithm is — it isn't essential for what follows):

$$\text{distance}_{\text{noneucl}}(A, B) = \log\left(\frac{XB \cdot YA}{XA \cdot YB}\right)$$

where XB, YA etc., stand for the usual Euclidean distances between the corresponding points.



Two points, A and B, in the Cayley model.



Many lines through point P parallel to line D.

Well then, one can imagine some fictitious inhabitants (mathematicians, perhaps?) living inside the disk, who would know nothing of the world outside of it, nothing of Euclidean geometry. They would teach their children that between any two points there exists a certain distance — which, of course, they would not call "non-Euclidean" since they would have no knowledge of Euclid.

These inhabitants would have a Ministry of Education, which would write school curricula containing theorems in complete contradiction with those of our Euclidean world![2] Moreover, it would seem obvious to them that through a point taken outside a line, one can draw an infinite number of parallels. Look at the figure below: the four lines passing through point P appear parallel to line D, don't they? And if you thought they intersect D, I would reply that they meet outside the disk — and the outside of the disk simply does not exist in their world. Thus, if we accept Euclidean geometry, we are compelled to accept non-Euclidean geometry at the same time.

You can learn a little more about this new geometry in the article "Une chambre hyperbolique".[b,c]

Metric Spaces

Starting from the beginning of the twentieth century, the concept of a metric space appeared. We say that a set E is a metric space (note the *indefinite* article, implying that there are many such spaces — obvious today, but a heresy not so long ago) if for each pair of elements x, y in E (which we naturally call *points* of space E), there is a number called the distance between x and y. This number must be non-negative (the least one could ask of a distance), but it must also satisfy a few additional properties (sometimes called axioms, though that word has lost some of its force over time... Should it be "axiom" or "definition"? — not so clear anymore).

- $\mathrm{d}_E(x,y) = \mathrm{d}_E(y,x)$ for all points x, y. (Feel free to object: sometimes it's harder to go from one point to another than the other way around — for example, when climbing up a mountain path. But still, we'll stick with this assumption/axiom/definition.)
- $\mathrm{d}_E(x,y) = 0$ if and only if $x = y$. (Ah yes, mathematicians can't resist saying such incantations — "if and only if", which they love to abbreviate as *iff*) We agree: two points at zero distance are, in fact, one and the same point.
- $\mathrm{d}_E(x,z) \leq \mathrm{d}_E(x,y) + \mathrm{d}_E(y,z)$ for any three points x, y, z in E. This is *the triangle inequality.* It says that the path from x to z via y is longer than the direct path from x to z. Obvious? Perhaps, yet most mathematicians are amazed at how Misha Gromov managed to make such a seemingly trivial axiom yield profound insights.

How About Examples?

Today, it's almost obvious to most mathematics students that there are many metric spaces. Barely two centuries ago only one was known — ours or at least the one Euclid had presented as ours.

So, here are a few examples nonetheless:

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[2] The ministers of education in our world, on the other hand, tend to write curricula without any theorems at all... See the post by Valerio Vassallo at `https://images-archive.math.cnrs.fr/Requiem-pour-la-Geometrie.html` (in French)

[b] Jos Leys, *Une chambre hyperbolique*, `https://images-archive.math.cnrs.fr/Une-chambre-hyperbolique.html`

[c] See also "Hyperbolic geometry: The first 150 years" by John Milnor, available at `https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-6/issue-1/Hyperbolic-geometry-The-first-150-years/bams/1183548588.full` (Ed.)

- **The Non-Euclidean Plane.** We've already encountered it: it's the interior of the disk, with a distance defined by that mysterious formula that seems to have come out of nowhere...
- **The Sphere.** After all, we live on the surface of Earth. Just as we imagined a nation unaware of the world outside a disk, most of us (myself included) know nothing beyond what lies on Earth. The distance between two points on Earth is the length of the shortest path connecting them, while staying on Earth, of course.

  Anyone who has looked at the routes taken by airplanes crossing the Atlantic will understand that the geometry on Earth is not quite the same as Euclidean geometry.
- **The Population of the Earth.** Let's think of all the men and women who are alive today or who have lived in the past. That will be our set E. We define the distance between two persons as the length of the shortest path connecting them up in the family tree. The distance between a parent and their child is one unit. In general, the distance between two people is the length of the shortest chain linking them through parent/child connections — father or mother $\rightarrow$ son or daughter.[d] For example, the distance between my nephew and me is 3, since we must go up to my father, then down to my sister, and finally down to her son.

  This gives us a metric space whose geometry one might very much like to understand.
- **The Internet.** Here, the set E consists of all the web pages on the planet. The distance between two pages is 1 if you can go from one to the other with a single click, it is equal to 2 if you must pass through an intermediate page, and so on. Note that in this case, the distance need not be symmetric: sometimes it's easy to go from one page to another, but the reverse isn't true.

  We could imagine many other communication networks as well — Facebook, for instance, where the distance is the length of the shortest chain of friends connecting two people.
- **French railway network.** Here, the set E is the French territory. One of its points is called Paris. We define the SNCF-distance[e] between two points A and B as follows: If the straight line (AB) does not pass through Paris, the SNCF-distance between A and B is the sum of their usual (Euclidean) distances to Paris. On the other hand, if (AB) does pass through Paris, the SNCF-distance is simply the usual Euclidean distance between A and B.

  In short, this models a world where all routes go through Paris — a playful geometric nod to the structure of the French railway network.

[d] In this example we measure distances only through common ancestors, not common children or further descendants. (Ed.)

[e] SNCF (**S**ociété **n**ationale des **c**hemins de **f**er français) is France railway operator. French railway network is indeed almost star-shaped centered at Paris, so that one is often forced to travel through Paris even if the straight line between the origin and destination doesn't even come close to the city. (Ed.)

There's no need to continue the list — it could be made endless. The geometry of metric spaces became a central theme in mathematics during the first half of the twentieth century.

Hyperbolic Spaces

Let us now turn to Misha Gromov, who was honored with the Abel Prize in 2009 (see the special feature[f] devoted to him on "Images des Mathématiques"). Like many mathematicians of his generation, he was fascinated by non-Euclidean geometry. But what struck him most were the qualitative properties of this remarkable geometry. He observed a kind of "stability", which I'll try to explain.

Gradually — probably beginning in the early 1970s — Gromov observed a number of properties satisfied by non-Euclidean geometry. He compared them, sought to understand which were essential and which were incidental, and ultimately arrived at a fundamental concept: that of *the hyperbolic metric space* (often specified as *in the sense of Gromov*, since mathematicians have an unfortunate tendency to label almost anything as hyperbolic).

The process was a long one. The text of his lecture[3] at the International Congress in Warsaw (1983)[4] already contained the major ideas, though still somewhat unpolished. In 1987, his book on hyperbolic groups[5] was published — a work that, nearly a quarter of a century later, can rightly be called marvelous. And yet, it was not easy to read. Some criticized it for a certain lack of detail, but that can be forgiven in a book intended as a *qualitative* analysis of geometry.

An aside: The Style

A brief digression is in order. Gromov's style is unique, inimitable, and far removed from mathematical convention. His writing is always difficult, even irritating at times, often lacking in precision, yet it always goes straight to the essence and possesses an incredible density. His proofs are often only sketches; sometimes they are "a little wrong," but they always contain wonderful ideas.

Perhaps the best thing is simply to quote here the opening lines of one of his articles: "Spaces and Questions," GAFA, Geometric and Functional Analysis, Special Volume (2000), 118–161.

> Our Euclidean intuition, probably, inherited from ancient primates, might have grown out of the first seeds of space in the motor control systems of early animals who were brought up to sea and then to land by the Cambrian explosion half a billion years ago. Primates' brain had been lingering for 30-40 million years. Suddenly, in a flash of one million years, it exploded into growth under relentless pressure of the sexual-social competition and sprouted a massive neocortex (70%

[3] M. Gromov, *Infinite groups as geometric objects*, Proc. Int. Congress Math. Warsaw 1983 1 (1984), 385-392.

[4] The Congress should have taken place in 1982, but was postponed for political reasons. Besides Gromov was unable to attend it.

[5] M. Gromov, *Hyperbolic groups*, in *Essays in Group Theory*, S. Gersten ed., MSRI Publications 8 (1987), 75-263 Springer.

[f] ???`https://images-archive.math.cnrs.fr/+-Mikhail-Gromov-geometre-+.html` (in French) (Ed.)

neurons in humans) with an inexplicable capability for language, sequential reasoning and generation of mathematical ideas. Then Man came and laid down the space on papyrus in a string of axioms, lemmas and theorems around 300 B.C. in Alexandria. Projected to words, brain's space began to evolve by dropping, modifying and generalizing its axioms. First fell the Parallel Postulate: Gauss, Schweikart, Lobachevski, Bolyai (who else?) came to the conclusion that there is a unique non-trivial one-parameter deformation of the metric on $\mathbb{R}^3$ keeping the space fully homogeneous.

A Definition

It's time to give some precise definitions. Gromov begins from the following remarkable observation:

In a triangle in non-Euclidean geometry, the radius of the inscribed circle cannot exceed a certain maximum value. Incredible — because this is, of course, completely false in Euclidean geometry: take a large equilateral triangle, then its inscribed circle will also be large. In non-Euclidean geometry, even if the sides of a triangle are enormous, there exists a point (the center of the inscribed circle) that remains close to each of the three sides.

Let's start with a definition that isn't actually essential but will allow us to speak about the sides of a triangle.

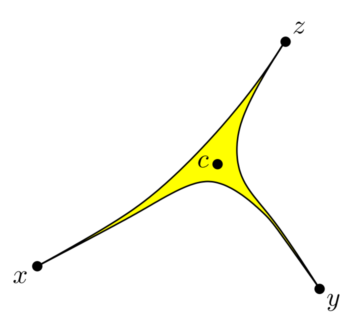
A metric space E is called geodesic if, any two points x and y can be connected by what we might naturally call a *segment* xy — that is, a curve $x(t)$ joining x and y, parametrized by a parameter t varying between 0 and the distance $\mathrm{d}_E(x, y)$, such that the distance between $x(t)$ and $x(s)$ equals $|s - t|$.

This simply means that x and y are connected by a segment of the correct length. Thus, whenever we have three points x, y, z, we can "draw" the three sides xy, yz, and zx.

Definition. *Let $\delta \geq 0$ be a real number (the Greek letter "delta" has become the traditional symbol in this context).*

A geodesic metric space E is said to be δ-hyperbolic if for every triangle formed by three points x, y, z there exists a point c that lies at a distance less than δ from each one of the three sides of the triangle xyz.

In other words, every triangle in E is "thin" — there is always a point close to all three sides.

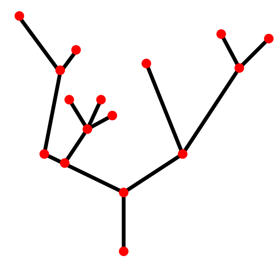


Two Examples and a (Very Small) Theorem of Gromov

Of course, the whole game would be pointless if the non-Euclidean plane of Bolyai–Lobachevsky–Gauss were not δ-hyperbolic (for some value of δ).[E: The logical flow is broken! Blue text added by me.] Indeed, it is possible to verify, that non-Euclidean plane does possess the δ-hyperbolicity property.

We now turn to another family of δ-hyperbolic (even 0-hyperbolic) spaces, called *trees*.

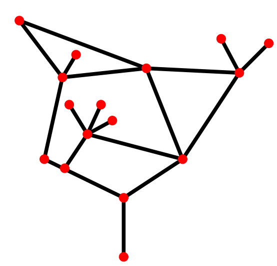
Here is a *tree*.



A tree.

It contains a certain number of vertices, some of which are connected by edges. The term tree is reserved for such a network provided it has no closed loops.

For example, the following is not a tree.

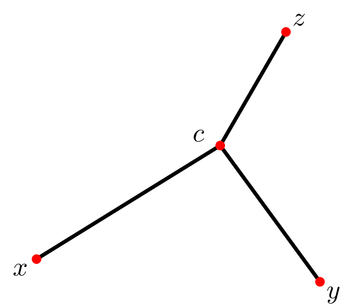


Not a tree.

For each edge of a tree, we can assign a non-negative length of our choosing. Then the tree can be viewed as a geodesic metric space: the distance between two points is simply the length of the shortest path connecting them. Such a structure is called a *metric tree*.

What do triangles look like in a tree? The figure below shows that in a tree, all triangles share a very distinctive property: there is always a point common to all three sides! In a sense, one could say that the inscribed circle has radius zero.

In any case, it should be clear that trees are 0-hyperbolic metric spaces. Instead of “triangle”, we more appropriately speak of a tripod.



The following theorem is certainly not among the most difficult ones proved by Gromov — far from it! Its proof takes only a few lines in one of his many books, and I'm sure he himself attaches only moderate importance to it.

Nevertheless, I present it here because it is quite typical of Gromov's "qualitative" approach to geometry and also because, who knows, perhaps the readers might even be able to prove it on their own?

There are many kinds of theorems. Some have statements that seem fairly "ordinary" but their difficulty lies in the proof, which might not be easy. Others, by contrast, are quite easy to prove, and the main challenge for their author was to find the right statement — the one that captures what truly matters.

Theorem. *Let (E, d_E) be a δ-hyperbolic geodesic metric space and let $(x_1, \ldots, x_n)$ be a collection of n points in E. Then it is possible to construct a metric tree A, with its own distance function d_A, and n corresponding points $(y_1, \ldots, y_n)$ in A, such that the distances in A and in E between the corresponding points are almost the same.*

More precisely, this means that for every pair (i, j), the distances satisfy:

$$|\,\mathrm{d}_E(x_i, x_j) - \mathrm{d}_A(y_i, y_j)| \leq 100\delta \cdot \log(n).$$

In other words, any finite collection of points in a δ-hyperbolic space can be approximated by points in a tree, up to a distortion bounded by $100\delta \cdot \log(n)$.

A Few Remarks

The above tree-approximation theorem is not particularly difficult, but it provides a genuine intuition for hyperbolic geometry. If you want to understand non-Euclidean geometry, look at trees, and you will at least gain a qualitative understanding.

Why "qualitative"? Because we study distances between points while allowing for a certain "uncertainty" or "error" — on the order of 100δ. The goal is not to obtain precise measurements, but rather to grasp the order of magnitude of things.

Gromov himself calls it *coarse geometry*, that is the geometry of rough shapes.

Also note the 100δ in the statement. The theorem is probably still true if we replaced 100 by 99, but it's not Gromov's style to chase after the best possible constants.

In his book, one finds here and there truly whimsical constants (though rarely incorrect ones) of the kind $10\,000\delta\ldots$ In some of his other articles, Gromov did not hesitate to use constants whose numerical values are absolutely enormous.

What's the Point?

All this would have only moderate interest if the following two conditions were not met: First of all, we have an excellent understanding of δ-hyperbolic spaces. Gromov's book, several of his later papers, and other works have revealed the full richness of these spaces. One can take as much pleasure in "doing geometry" within these spaces as in our familiar "old Euclidean space." There were proven many general theorems about hyperbolic spaces.

Secondly and above all, δ-hyperbolic spaces abound, and that is what makes them so interesting. We've already seen examples such as trees (for instance, genealogical trees) and the non-Euclidean plane. In a certain sense, one might even think that "almost all spaces" are δ-hyperbolic.

Examples arise from number theory[??? en provenance de l'arithmétique], but also from group theory. As early as 1990, Gromov even saw in this framework a possible interpretation of certain growth processes in biology. A certain class of groups introduced in the 1960s, known as "small cancellation groups," suddenly became much clearer thanks to this geometric perspective — geometry coming to the rescue of algebra, so to speak.

In this way, Misha Gromov identified a fundamental concept: that of the hyperbolic group, whose meaning is above all geometric. But of course, combining groups and geometry is hardly a new idea in mathematics.

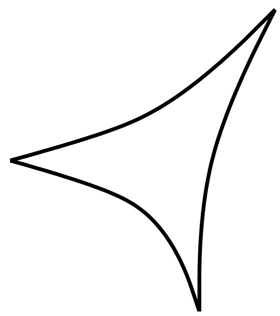
Much more recently, Gromov even introduced probabilistic arguments into his geometric approach. He showed that if one picks a group at random, there is a high probability that it will be hyperbolic.

On July 12, 1831, Gauss wrote a letter[g] to his friend Schumacher about non-Euclidean geometry. Here is an English translation of an excerpt from that letter.

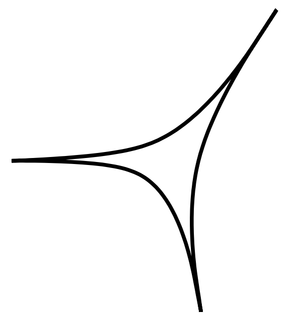
> ...
> In non-Euclidean geometry there are no similar figures without equality. For example, the angles of an equilateral triangle are not only dependent on R,[h] but also vary according to the size of the sides; and when the side increases without limit, the angles can become as small as one wishes. It is therefore inherently contradictory to try to draw such a triangle by means of a scaled-down one — one can, in essence, only designate it.

[g] Carl Friedrich Gauss, *Werke, Band VIII*, `https://link.springer.com/book/10.1007/978-3-642-92474-3` or `https://archive.org/details/briefwechselzwi03schugoog/page/n10/mode/2up` (Ed.)

[h] By letter R Gauss denotes the parameter of deformation of the geometry. Nowadays it is called (Gauss) curvature. (Ed.)

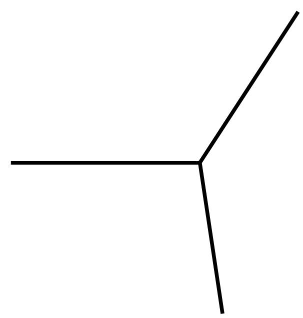


Small non-Euclidean triangle.[i]



The designation of a large non-Euclidean triangle.[i]

The designation of the infinite triangle in this sense would ultimately be



In Euclidean geometry there is nothing absolutely large, but in non-Euclidean geometry there is — and this is precisely its essential character. Those who do not admit this thereby assume ipso facto the entirety of Euclidean geometry; but, as I have said, in my conviction this is mere self-deception.

. . .

Incredible! Gauss explains that triangles in non-Euclidean geometry look quite different from Euclidean triangles. He even says that, "in the limit" they become triangles like those found in trees.

So, did Gauss already have a "qualitative" approach to geometry — a hundred and fifty years before Gromov? Of course, Gauss did not develop a true qualitative theory of geometry as Gromov later did.

I don't think Gromov ever saw this letter from Gauss, but I am convinced that he would be proud to know that his path crossed the one of Gauss.