From Ramsey's Theorem to the Erdős–Hajnal Conjecture[1,a]

by Patrice Ossona de Mendez

Part I: Order is inevitable

Ordnung muss sein! (Order is inevitable.) This saying, attributed to Theodore Motzkin, aptly summarizes Ramsey theory, which aims to solve problems of the form: "From what size on can a structure no longer avoid having a certain property?"[2]

A first example

A classic recreational problem is the "friends problem":

> When six people meet, we are sure that either three of them know one another, or three of them do not know one another.

The proof of this fact is a bit tedious, because one has to consider different cases. Nevertheless, this annoying complication will open the way to a more general and more interesting point of view. Let us look at it more closely:

So we have six people. Let us call Dominique[3] one of the people in the group. If Dominique knows at least three people in the group, two cases can occur:

[1]This article grew out of a conversation in Prague, over a glass of wine, with Shalom Eliahou. I am grateful to him for his encouragement, his insightful suggestions, and his valuable comments on the first draft of this article. I would also like to thank Bedaride Nicolas, Pierre Lescanne, Marielle Simon and Cidrolin for their kind feedback, wise advice, and careful proofreading. Finally, I wish to express my gratitude to Carole Gaboriau and Maï Sauvageot, indispensable editorial assistants (and tireless shell hunters) for their precious support.

[2]The logo was obtained from a photograph taken from the archives of the J. M. Keynes Papers (JMK B/4, p. 283) at King's College, Cambridge. The photographer and copyright holder of the original photograph are unknown; see Alexander Soifer, The Mathematical Colouring Book, Springer, 2008, p. 283.

[3]Note that this is an epicene name, as gender is of no relevance here.

[a]English translation by Rostislav Matveev of the article: Patrice Ossona de Mendez, *Du Théorème de Ramsey à la Conjecture d'Erdős–Hajnal (1)&(2)*, Images des Mathématiques (2017); https://doi.org/10.60868/tg9t-gq67 and https://doi.org/10.60868/q1m6-2n71. (Ed.)

- either two of them know each other (and together with Dominique, that makes at least three people who know one another);
- or they do not know each other (and that makes three people who do not know one another).

We see that in both cases we reach the desired conclusion. So we only have to consider the case where Dominique knows at most two people in the group. Again, two cases can occur among the people whom Dominique does not know:

- either two of them do not know each other (and together with Dominique, that makes three people who do not know one another);
- or they all know one another (and that makes at least three people who know one another).

In this case as well, we reach the desired conclusion. □

There is a graph grafted on

Let us now see how we can translate this result into the language of graph theory. A graph is a collection of vertices connected (or not) to one another by edges. Each edge has two incident vertices and represents a relation between two entities, the entities being the vertices. For example, the vertices can be people, and the edges can indicate who knows whom. Two vertices connected by an edge are said to be adjacent. We will also use the following two terms:

A set of vertices is an *independent set* if it does not contain two adjacent vertices. In our example, a set of people who do not know one another is an independent set.

A set of vertices is a *clique* if all vertices in the set are pairwise adjacent. In our example, a set of people who all know one another forms a clique.

Let us return to our example. We have six people (so six vertices), and the edges connect the vertices corresponding to people who know one another: an edge connects the vertex Dominique and the vertex Claude if Dominique and Claude know one another.[4]

In this language, the result we proved above can be stated as follows:

> In any graph with six vertices, there exist three vertices forming an independent set, or three vertices forming a clique.

In red and blue, Ramsey and his clique

We could also have constructed a graph differently, by connecting two people with a red edge if they know each other, and with a blue edge if they do not know each other. Thus, any two vertices are always connected by an edge, either red or blue. Our result can then be stated as follows:

> In any *complete graph*[b] on six vertices whose edges are colored red or blue, there exists a triangle (a 3-vertex clique) all of whose edges have the same color (all red or all blue).

This last formulation is a special case of Ramsey's Theorem.

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[4]Note that here we are considering symmetric relations, which would not have been the case if we had considered the property "Dominique knows Claude," which does not necessarily imply that, symmetrically, Claude knows Dominique.

[b]A graph is *complete* is every pair of distinct vertices is connected by an edge. (Ed.)

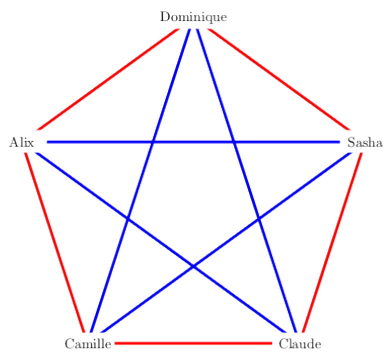
Theorem (Ramsey's Theorem)**.** *For any finite set of colors (red, green, blue, violet, ...) and any sequence of integers* $n_{red}, n_{green}, n_{blue}, n_{violet}, \ldots$, *there exists an integer* N *with the following property:*

For every complete[c] *graph on* N *vertices whose edges are colored with the colors red, green, blue, violet,... there exists either*

- *a clique of size* n_{red} *all of whose edges are red, or*
- *a clique of size* n_{green} *all of whose edges are green, or*
- *a clique of size* n_{blue} *all of whose edges are blue, or*
- *a clique of size* n_{violet} *all of whose edges are violet,*

etc. □

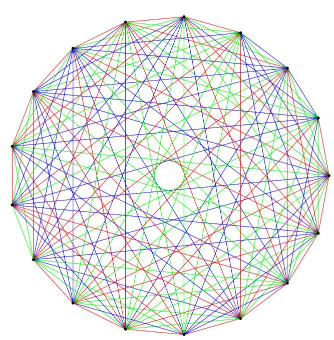
Our example above corresponds to the fact that $N = 6$ has the property stated by the theorem in the case of two colors, red and blue, and for the values $n_{\text{red}} = 3$ and $n_{\text{blue}} = 3$. By contrast, $N = 5$ does not have this property:



This group of five people is such that, in every subgroup of three people, one can find two people who know each other and two people who do not know each other. In other words, in a 5-cycle (corresponding to the red edges), one finds neither a clique of size 3 nor an independent set of size 3.

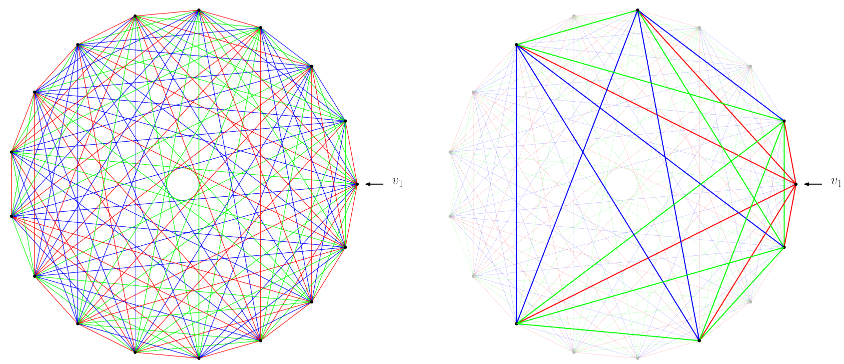
Big and colorful

Let us look at a slightly larger example and see how, in a complete graph on 17 vertices whose edges are colored red, green, and blue, we can always find a *monochromatic triangle*, that is, a triangle all of whose edges have the same color. Here is an example of a colored complete graph:



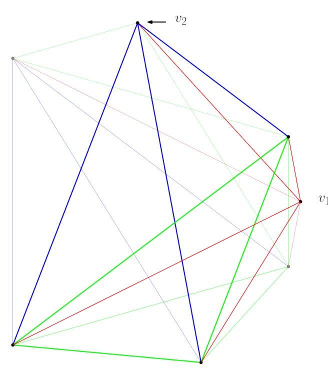
[c]A graph is called *complete* if any pair of vertices are connected by an edge. (Ed.)

Let us arbitrarily choose a vertex v_1. Consider the most prevalent color among the edges incident to this vertex (here, the red color, which appears on 6 edges), and focus on the complete subgraph whose vertices are v_1 and its neighbors connected to it by red edges:



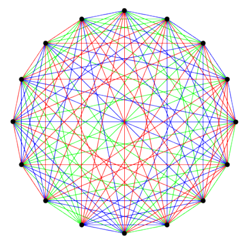
If two of the selected neighbors of v_1 were connected by a red edge, we would have found a red monochromatic triangle. Here, this is not the case.

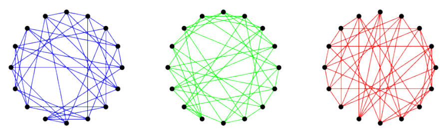
So we arbitrarily choose one of the selected neighbors of v_1 and call it v_2. At this vertex, we again consider the majority color (here blue, appearing on three edges), and we now select the vertices that are not only adjacent to v_1 by a red edge, but also adjacent to v_2 by a blue edge.



If two of the selected neighbors were connected by a blue edge, we would have found a blue monochromatic triangle. Since this is not the case, it means that the three neighbors of v_2 are all connected by green edges, so we have finally found a green monochromatic triangle. To find this triangle, it was necessary that v_2 have three neighbors in the majority color, hence at least 5 neighbors (after the selection among the neighbors of v_1). Therefore v_1 needed to have at least 6 neighbors in the majority color, hence have degree at least 16 — that is, the initial complete graph had to have size at least 17.

By contrast, the edges of a complete graph on 16 vertices can be colored red, green, and blue without creating any monochromatic triangle:





With just two colors it is already not so simple

Let us return to the two-color problem:

> For an integer t, what is the smallest integer n such that in every complete graph on n vertices whose edges are colored red and blue, we are guaranteed to find a monochromatic clique of size t?

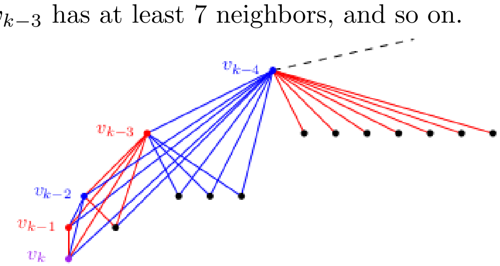
For example, we have already shown above that every complete graph on 6 vertices whose edges are colored with two colors contains a monochromatic triangle. In other words, for $t = 3$ it is enough to choose $n \geq 6$.

In the general case, we can apply the same proof method as the one we followed above.

Consider a complete graph on n vertices whose edges are colored blue and red. Let v_1 be any vertex, and let c_1 be the majority color among the edges incident to v_1. Select the neighbors of v_1 that are connected to v_1 by an edge of color c_1, and among these vertices choose an arbitrary v_2. Let c_2 be the majority color among the edges incident to v_2 that connect v_2 to vertices that are connected to v_1 by an edge of color c_1. Select the vertices that are both connected to v_1 by an edge of color c_1 and connected to v_2 by an edge of color c_2. Among these vertices we choose an arbitrary v_3, etc.

In this process, we select a sequence $v_1, v_2, v_3, \ldots, v_k$ of vertices, until v_k has no neighbors left. Moreover, we color vertex v_i with the color c_i.

Counting backwards, we see that v_{k-1} has at least 1 neighbor (v_k), v_{k-2} has at least 3 neighbors, v_{k-3} has at least 7 neighbors, and so on.



Another way to count is the following: the complete subgraph formed by v_{k-1} and the set of vertices connected to v_1 by an edge of color c_1, to v_2 by an edge of color c_2, …, to v_{k-1} by an edge of color c_{k-1} has size 2 (it contains only v_{k-1} and v_k). The complete subgraph formed by v_{k-2} and the set of vertices connected to v_1 by an edge of color c_1, to v_2 by an edge of color c_2, …, to v_{k-2} by an edge of color c_{k-2} is twice as large, of size 4, and so on.

To be sure to find a monochromatic clique of size t, it is enough that $k \geq 2t-1$, because then the vertices among $v_1, \ldots, v_{2t-1}$ of the majority color (which appears at least t times) form a monochromatic clique. So it is sufficient that the size n of the initial complete graph whose edges are colored blue and red be at least

$$n \geq \underbrace{2 \times 2 \times \cdots \times 2}_{2t-1} = 2^{2t-1}$$

It is rather surprising that this result is, up to a small margin, optimal. Indeed, if one colors at random[5] the edges of a complete graph of size 2^t in red and blue, there is a very high probability[6] that the size of the largest monochromatic clique is of the order of magnitude of t.[7]

In summary, Ramsey's theorem states that a certain amount of order is unavoidable, and the structures where this order is "minimal" are, quite intuitively,[8] the random structures.

Instead of looking at an edge-coloring of a complete graph with two colors, one can study an arbitrary graph G on a set of n vertices. Indeed, if we consider the complete graph with edges present in G to be blue and the edges absent from G to be red, Ramsey's theorem can be interpreted in the following form:

Theorem. *Let t be a natural number, and let n be a natural number greater than 2^{2t-1}. Then any graph G on n vertices necessarily contains a clique of size t or an independent set of size t (or both).*

We see that the number n grows very quickly with t. For example, to be sure of finding a clique or an independent set of size $t = 100$, we need to consider a graph of size

$$n \geq 2^{199} = 803469022129495137770981046170581301261101496891396417650688$$

A natural question then arises: under what conditions can we reduce this astronomical number?

The Erdős–Hajnal conjecture

We have seen that if the number n of vertices in a graph is large enough, then the graph contains a large clique or a large independent set.

What happens if one of the two possibilities is forbidden? More precisely, what if we consider a graph that contains no triangle, that is, no clique of size 3? Does the number n of vertices of G still need to grow as fast as 2^{2t-1} in order to guarantee the existence of an independent set of size t? We will see that in fact, instead of

[5] Independently, with probability 1/2 for each color.

[6] More precisely, with a probability that tends to 1 as t tends to infinity, that is, "asymptotically almost surely."

[7] This result can be derived from Markov's inequality. In the 1970s, Matula proved that the likely values of the size of the largest monochromatic clique are essentially concentrated on two values close to $2t$. For example, if $t = 10$ (that is, if we consider a complete graph of size $2^{10} = 1024$), there is about an 80% chance that the largest monochromatic clique has size exactly 15.

[8] In the sense that "at random" and "without order" are two notions that are often regarded, pragmatically, as equivalent.

requiring $n \geq 2^{2t-1}$, it is enough to have just $n \geq t^2$. For example, for $t = 100$, the necessary size is merely $n \geq 10000$.

Why is this so?

Consider a triangle-free graph G with $n \geq t^2$ vertices. There are two possibilities:

- Graph G contains a vertex v with at least t neighbors. Then the set of neighbors of v forms an independent set of size t: indeed, there cannot be an edge joining two neighbors of v, because otherwise those two vertices together with v would form a triangle.
- Every vertex of G has at most $t-1$ neighbors. We will color the vertices using the t colors: we take the vertices one by one. Each vertex receives one of the colors that is not already used by one of its neighbors. Since each vertex has at most $t-1$ neighbors, there is always an available color. In the end, we have colored the vertices of the graph and, by construction, two adjacent vertices never share the same color. Now choose the most frequently used color. Since the graph has n vertices, at least n/t vertices have this majority color. These vertices form an independent set. Indeed, two vertices of the same color are never adjacent. Since $n \geq t^2$, this set has size at least t.

In effect we have proved the following proposition:

Proposition. *If a triangle-free graph has at least t^2 vertices, then it contains an independent set of size t.*

We have seen that when we forbid the presence of a triangle (that is, a clique of size 3), we can find a large independent set. What happens if we forbid a larger clique? A priori, we expect the size of the graph required to guarantee the presence of an independent set of size t to grow progressively from t^2 up to 2^{2t-1}, depending on the size of the forbidden clique.

In what follows we will see that the size of the graph without cliques of size k required to guarantee the existence of an independent set of size t grows like t^{k-1}:

Proposition. *Every graph on at least t^{k-1} vertices and with no clique of size $k \geq 3$ contains an independent set of size t.*

Proof sketch by induction on k: It has already been shown that if we forbid all triangles (that is, all cliques of size 3), then every graph with t^2 vertices contains an independent set of size t. This will be the base of the induction. Now we show how to make one induction step: $(k = 3) \rightarrow (k = 4)$.

Consider a graph with no cliques of size 4 and with $n \geq t^3$ vertices.

If, on the one hand, G contains a vertex v with at least t^2 neighbors, then the subgraph induced[d] by the neighbors of v contains no triangle, because any triangle in the neighborhood of v would form, together with v, a clique of size 4. Moreover, the graph induced by the neighbors of v has at least t^2 vertices (since v has at least t^2 neighbors). By the previous result, we can conclude that the neighborhood of v contains an independent set of size t.

If, on the other hand, all vertices of G have at most $t^2 - 1$ neighbors, then, as we saw above, we can color the vertices of G using t^2 colors so that no two

[d] An induced subgraph is the one that contains all the possible edges of the ambient graph, given its set of vertices. (Ed.)

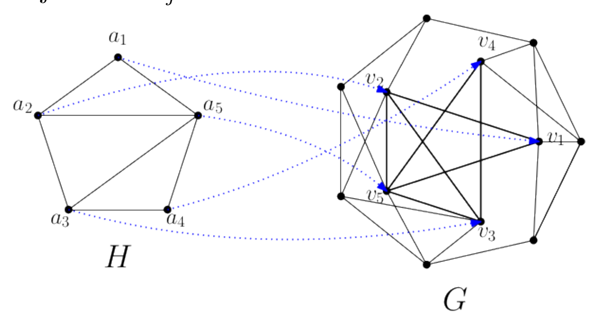
adjacent vertices share the same color. The most frequent color class then forms an independent set of size n/t^2. Hence G contains an independent set of size at least t, since $n \geq t^3$. The general induction step proceeds along the same lines. □

These results show that forbidding a clique of fixed size guarantees that the number of vertices needed to guarantee the presence of an independent set of size t is bounded by a power of t, with exponent depending on the size of the forbidden clique.

By passing to the complementary graph (obtained by swapping edges and non-edges), we see that the number of vertices needed to guarantee a clique of size t in a graph with no independent set of size k is at most t^{k-1} — a power of t whose exponent depends only on the size k of the forbidden independent set.

It is natural to ask whether such a phenomenon can be generalized. This is the point of the conjecture posed by Erdős and Hajnal in 1989. Before stating it, let us take a moment to introduce a few definitions.

Let G and H be two graphs. Let $a_1, \ldots, a_h$ be all vertices of H. A *copy* of H in G is a set of distinct vertices $v_1, \ldots, v_h$ of G such that v_i is adjacent to v_j in G if and only if a_i is adjacent to a_j in H:[e]



In particular, if H is a graph with t vertices and no edges, then a copy of H in G is an independent set of size t in G. Similarly, if H is a complete graph with t vertices, then a copy of H in G is a clique of size t.

Paul Erdős and András Hajnal proposed the following conjecture:

Conjecture (Erdős–Hajnal conjecture)**.** *For every graph F, there exists an integer $c(F)$ with the following property: any sufficiently large*[9,f] *graph G that contains no copy of F, and that has more than $t^{c(F)}$ vertices, contains either an independent set or a clique of size t.*

We can restate the Erdős–Hajnal conjecture in a dual form, emphasizing how it is related to a notion of "unavoidable order".

Conjecture (Erdős–Hajnal conjecture (second formulation))**.** *For every integer h, there exists an integer $K(h)$ with the following property: if a graph G contains neither an independent set nor a clique of size $t > h$, and has at least $t^{K(h)}$ vertices, then G contains a copy of every graph on h vertices.*

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[9] "Sufficiently large" means "larger than some size $N(F)$ depending only on F."

[e] In other words, a copy of H in G is an induced subgraph of G isomorphic to H. (Ed.)

[f] Formally speaking the "sufficiently large" quantifier is not necessary in the statement, since we can always increase the value of $c(F)$ to be bigger then $\log_2 N(F)$. However, one is interested in the minimal value of $c(F)$ for which the property in the conjecture is satisfied for large graphs. (Ed.)

Part II: The power of the forbidden

In the previous part, we saw that a certain amount of order is unavoidable, and, in particular, that every graph contains a large independent set or a large clique. The minimal size of this unavoidable independent set or clique is achieved by random graphs. What happens if we impose a strong local constraint? This is precisely the subject of the Erdős–Hajnal conjecture...

As we continue our journey from Ramsey's theorem to the Erdős-Hajnal conjecture, and before we dive deeper into mathematical "technicalities", let's take the time to contemplate the landscape. At this pivotal moment, the view invites a bit of mathematical daydreaming about order and chance, and offers a possibility to sharpen our intuition for this ever-surprising world.

> (...) the faculty which teaches us to see is intuition. Without it, the geometrician would be like a writer well up in grammar but destitute of ideas.
>
> *Henri Poincaré*[10,g]

A bit of random philosophy

In the first part, we saw that every sufficiently large graph contains an independent set or a clique of non-negligible size. Recall that an independent set is a set of vertices that none are adjacent, whereas a clique is a set of vertices that are all pairwise adjacent.

Despite this unavoidable order, we should be wary of drawing any hasty conclusions.

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[10] Henri Poincaré, *Science et Méthode,* Livre II, §II, p.9.

[g] See also: Henri Poincaré, *Science and Method,* p. 130; https://henripoincarepapers.univ-nantes.fr/chp/hp-pdf/hp1914sm.pdf (Ed.)

> The human understanding, from its peculiar nature, easily supposes a greater degree of order and equality in things than it really finds.
>
> *Francis Bacon*[h]

Nevertheless, to be sure of finding an independent set or a clique of size t, we must consider a graph with at least 2^t vertices. In particular, a random graph[11] on 2^t vertices very probably[12] contains neither an independent set nor a clique of size larger than 2^t.

Although this property holds for a large graph chosen at random, it is very difficult to construct such graphs explicitly, that is, graphs with many vertices (at least 2^t) and containing neither a clique nor an independent set significantly larger than the unavoidable minimum size t.

What role does randomness play? How can we characterize the chaos and combinatorial madness that randomness gives rise to? Once again, let us allow ourselves to borrow a philosophical quotation to illustrate our point.

> In fact, chaos is characterized less by the absence of determinations than by the infiinite speed with which they take shape and vanish. This is not a movement from one determination to the other but, on the contrary, the impossibility of a connection between them, since one does not appear without the other having already disappeared, and one appears as disappearance when the other disappears as outline.
>
> *Gilles Deleuze, Félix Guattari*[13]

In other words, when we build a graph at random, independent sets and cliques emerge only to destroy one another, and neither manages to grow significantly.

What happens if we move away from randomness? What if, through constraints, we impose some form of structure? This is the idea behind the Erdős–Hajnal conjecture:

> From constraint comes structure, and from structure comes order.

By forbidding the presence of a given subgraph, we force a global structure, and this global structure entails the appearance of independent sets or cliques that are significantly larger than what one would find in a random graph. More precisely, in 1989 Paul Erdős and András Hajnal proposed the following conjecture.

Conjecture (Erdős–Hajnal conjecture). *For every graph F, there exists an integer $c(F)$ with the following property:*

Any sufficiently large graph G that contains no induced copy of F, and that has more than $t^{c(F)}$ vertices, contains an independent set or a clique of size t.

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[11] Here, "random" means either that each edge is present, independently of the others, with probability 1/2, or that the graph is chosen uniformly at random among all graphs on 2^t vertices (the Erdős–Rényi model).

[12] That is, with probability converging to one, as t tends to infinity.

[13] What Is Philosophy? p. 42, (translation by Hugh Tomlinson and Graham Burchell)

[h] Francis Bacon, *Novum Organum* (1620), Book I, Aphorism 45. (Ed.)

We say that a graph F has the Erdős–Hajnal property if there exists an integer $c(F)$ such that every sufficiently large graph G that contains no induced copy of F, and that has more than $t^{c(F)}$ vertices, contains an independent set or a clique of size t. We then define $c(F)$ to be the smallest integer with this property.

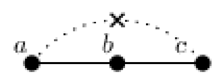
In the first part, we proved that complete graphs do have the Erdős–Hajnal property. In fact, if F is a complete graph of size k, then we have $c(F) \leq k-1$.

A textbook case: paths of length 3 and cographs

A short path to warm up...

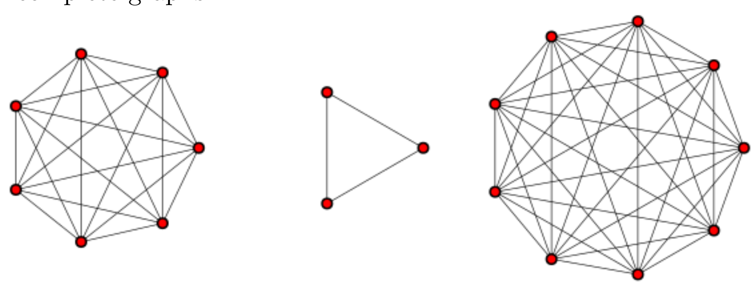
What about the path P_3 of length 2 (the "3" in P_3 refers to the number of vertices in the path, not its length)?

A graph contains an induced copy of the path P_3 of length 2 if it contains three vertices a, b and c such that a is adjacent to b and b is adjacent to c, but a is not adjacent to c.[i]



An induced copy of P_3 on the vertices a, b, c.

Graphs with no induced copies of P_3 have a very simple structure: they are disjoint unions of complete graphs!



A (disconnected) graph with no induced copies of a path of length 2.
In this graph, each connected component is a clique.

Proposition. *The path P_3 of length 2 has the Erdős–Hajnal property with $c(P_3) = 2$.*

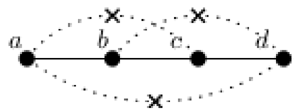
Proof: Let G be a graph with no induced copy of the path P_3 of length 2. Then G is a disjoint union of cliques. Suppose that G has at least t^2 vertices.

If G contains no clique of size t, then it must contain at least t cliques. By choosing one vertex from each clique, we obtain an independent set of size at least t. □

Let's join the dance: from a *pas de deux* to a *pas de trois*

A far less simple — however very instructive — example is the path of length 3. A graph contains an induced copy of the path P_4 of length 3 if it contains four vertices a,b,c and d such that a is adjacent to b, b to c, and c to d, and there are no other adjacencies; that is, if one can find in the graph the following configuration:

[i]Thus, absence of copies of P_3 means that any induced connected subgraph on 3 vertices is a triangle. (Ed.)

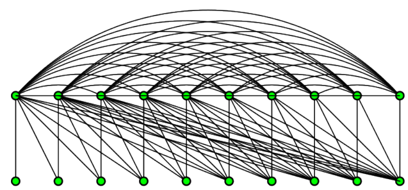


An induced copy of P_4 on the vertices a, b, c, d.

This example will show us how a local constraint can give rise to a global structure, and how that global structure forces the presence of a large clique or a large independent set.

From constraint comes structure.

Definition. *Graphs that contain no induced copy of the path P_4 of length 3 are called cographs.*

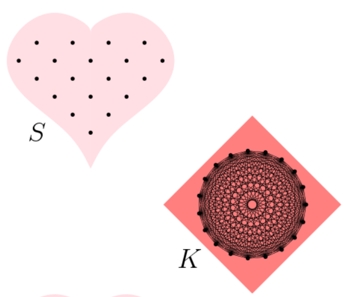


An example of a cograph.

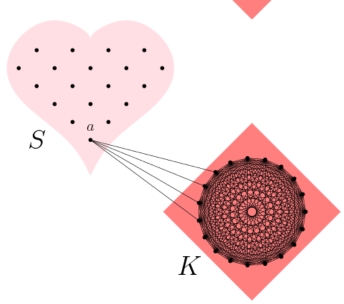
Cographs have a very special global structure, which, in particular, links *maximal independent sets* (that is, independent sets that are not strictly contained in any larger independent set) and *maximal cliques* (that is, cliques that are not strictly contained in any larger clique):

Proposition. *In a cograph, every maximal independent set intersects every maximal clique.*

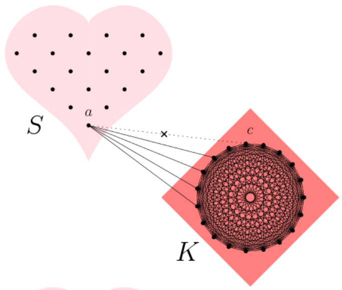
Proof: We prove this by contradiction, with the help of figures.



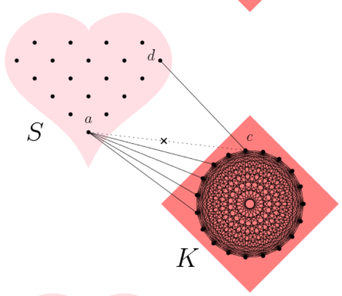
Suppose there exist a maximal independent set S and a maximal clique K that are disjoint.



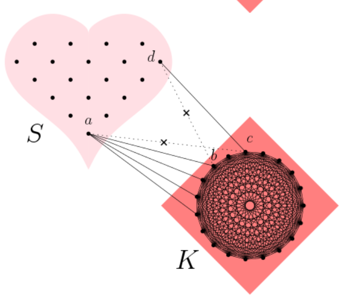
There is at least one vertex of S that has a neighbor in K. Otherwise, we could add one of the vertices of K to S and obtain a larger independent set (since no vertex of S is adjacent to the added vertex). Let a be a vertex of S with a maximum number of neighbors in K.



The clique K contains a vertex c that is not adjacent to a. Otherwise, a would be adjacent to every vertex of K, and by adding a to K we would obtain a larger clique.



The set S contains a vertex d that is adjacent to c. Otherwise, c could be added to S to form a larger independent set.
The vertex d is distinct from a (since c is adjacent to d but not to a) and is not adjacent to a (because a and d both belong to the independent set S).



The vertex a has at least one neighbor b in K that is not adjacent to d. Otherwise, d would be adjacent to all of a's neighbors in K, contradicting the choice of a as a vertex of S with the maximum number of neighbors in K.
Thus b is distinct from c (to which it is adjacent).

The vertices a, b, c, d therefore form an induced copy of the path of length 3, contradicting our initial assumption.

Hence, by contradiction, we have shown that in a cograph — that is, in a graph containing no induced copy of P_4 — every maximal independent set intersects every maximal clique. □

Proposition. *Let S be a maximal independent set of a cograph G, and let K be a maximum*[j] *clique of G.*

Let G be the graph obtained from G by deleting all vertices of S (together with all edges incident to them), and let K' be the clique of G' obtained by removing from K the unique vertex that lies in both K and S.

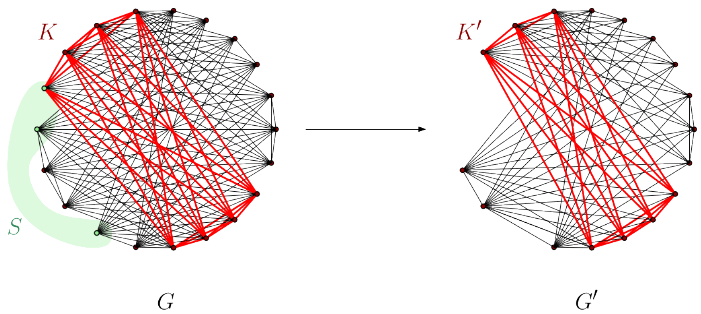
Then K' is a maximum clique of the cograph G'.

Proof: Let c be the number of vertices in K. Since every maximal clique intersects every maximal independent set, there exists a vertex v that belongs to both K and S.

Note that no other vertex can lie in both K and S: otherwise, that vertex would have to be adjacent to v (since both are in the clique K) and non-adjacent to v (since both are in the independent set S), which is impossible.

[j] Note the distiction, a *maximum* clique is the largest in size among all cliques of G, whereas a *maximal* clique is the one that can not be extended to a bigger one. One implies the other (which way?), but not the other way around. (Ed.)

If we delete all vertices of S, we in particular delete the vertex v from K, and we obtain the clique $K' := K \setminus \{v\}$ of size $c-1$.



Deleting a maximal independent set from a cograph.

Assume for contradiction that K' is not a maximum clique of G'. Let K' be a maximum clique of G', so $|K'| \geq c$. But K'' is also a clique of G, and in fact a maximum clique of G (since it has size c). Therefore K'' intersects S, contradicting the assumption that K'' is a clique of G'. □

And from structure comes order

We will see that this property will allow us to prove that P_4 has the Erdős–Hajnal property. More precisely:

Proposition. *The path P_4 of length 3 has the Erdős–Hajnal property with $c(P_4) \leq 2$.*

Proof: We will in fact prove a slightly stronger statement: if a cograph on n vertices (that is, a graph on n vertices with no induced copies of P_4) has neither an independent set nor a clique of size t, then $n \leq (t-1)^2$.

Indeed, suppose that our graph G contains no clique of size t, and let K be a maximum clique (so $|K| = k \leq t-1$). Choose a maximal independent set S (which therefore contains a vertex v of K) and delete it. We obtain a new graph G', which is still a cograph and $K \setminus \{v\}$ is a maximum clique of G'.

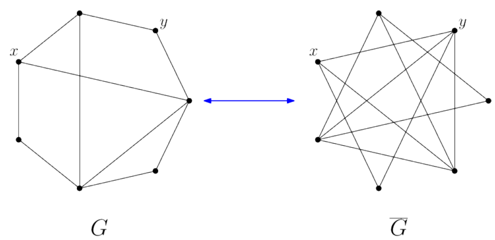
Repeat the same operation until the graph is empty (i.e., has no vertices left). Since at each step we delete exactly one vertex of K, we have deleted k independent sets in total. If $n > (t-1)^2 \geq k(t-1)$, then one of the independent sets we deleted had size at least t. Because deleting vertices does not change adjacency relations within that set, the original graph already contained an independent set of size t. □

Back to the general problem, or how to put old wine in a new bottle

We have succeeded in resolving one case of the Erdős–Hajnal conjecture. Studying the cases one by one would be a Herculean task. That is why, instead of considering all graphs F one by one, we will look for which constructions make it possible to obtain new graphs with the Erdős–Hajnal property from known examples.

Additional examples

The first construction consists in taking the complement of a graph. The complementary graph of a graph G is the graph, denoted $\overline{G}$, which has the same vertex set as G, and in which edges and non-edges are swapped. In other words, two vertices x and y are adjacent in $\overline{G}$ if and only if they are not adjacent in G:



Complimentary graph.

Note that the complement of the complement of a graph is the graph itself, that is $\overline{\overline{G}} = G$.

Next we will prove that the Erdős–Hajnal property is preserved under taking complements, as stated in the following proposition.

Proposition. *A graph F has the Erdős–Hajnal property if and only if its complement $\overline{F}$ has the Erdős–Hajnal property.*

Moreover, we then have

$$c(\overline{F}) = c(F)$$

Proof:[k] It suffices to show that if F has the Erdős–Hajnal property, then the same holds for $\overline{F}$, with exponent $c(\overline{F}) = c(F)$.

One checks immediately that G contains no induced copies of $\overline{F}$ if and only if the graph $\overline{G}$ contains no induced copies of F.

Let G be a sufficiently large graph containing no induced copies of $\overline{F}$. Then $\overline{G}$ (which is the same size as G) contains no induced copies of F, and therefore (since we assume that F has the Erdős–Hajnal property) if the size of $\overline{G}$ (i.e., the size of G) is at least $t^{c(F)}$, then $\overline{G}$ contains a clique or a stable set of size t. Now, a clique in $\overline{G}$ is a stable set in G, and a stable set in $\overline{G}$ is a clique in G.

Thus G contains a clique or a stable set of size t, which means that $\overline{F}$ has the Erdős–Hajnal property and that $c(\overline{F}) \leq c(F)$. Applying operation of taking complement a second time, we obtain

$$c(F) = c(\overline{\overline{F}}) \leq c(\overline{F})$$

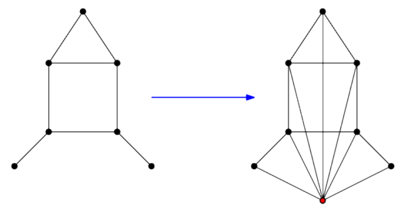
□

Since cliques have the Erdős–Hajnal property, the same is true of edgeless graphs. More precisely, we see that any graph on $n \geq t^{k-1}$ vertices with no stable set of size k contains a clique of size t.

Something universal

Adding a *universal vertex* to a graph F consists in creating a new graph F', obtained from F by adding a new vertex and connecting it by edges to all the vertices of F:

[k] One observes that both the assumption and the conclusion of the Erdős–Hajnal Conjecture are invariant under taking complements simultaneously of the graphs F and G, so the proposition follows immediately. (Ed.)



Adding a universal vertex.

Proposition. *If a graph F has the Erdős–Hajnal property and if the graph F' is obtained from F by adding a universal vertex, then F has the Erdős–Hajnal property. Moreover, we then have*

$$c(F') \leq c(F) + 1$$

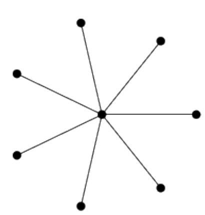
Proof: Assume that the graph F has the Erdős–Hajnal property, and let G be a sufficiently large graph with no induced copies of F'. Let t be an integer.

If G has a vertex v with at least $t^{c(F)}$ neighbors, then the graph G' induced by the neighbors of v (together with the edges between them) contains no induced copies of F, since any induced copy of F inside the neighborhood would form, together with v, an induced copy of F' in G. Therefore the graph G' has at least $t^{c(F)}$ vertices (because v has at least $t^{c(F)}$ neighbors). Since F has the Erdős–Hajnal property, we deduce that G' contains a clique or a stable set of size t, and therefore G does as well.

Otherwise, every vertex of G has at most $t^{c(F)} - 1$ neighbors. As we saw in Part I, we can then color the vertices of G with $n/t^{c(F)}$ colors, in such a way that two vertices of the same color are never adjacent. Taking the largest color class, we obtain a stable set of size at least $n/t^{c(F)}$. If $n \geq t^{c(F)}$, the graph G therefore contains a stable set of size t. □

Let us note that this proposition allows us to recover the fact that the clique K_k of size k has the Erdős–Hajnal property with $c(K_k) \leq k-1$: indeed, the clique K_2 of size 2 (formed by two vertices joined by an edge) clearly has the Erdős–Hajnal property with $c(k_2) = 1$, since any edgeless graph of size t^1 contains a stable set of size t. Since the clique K_k of size k is obtained by adding a universal vertex to the clique k_{k-1} of size $k-1$, we therefore obtain, by induction on k, the inequality $c(K_k) \leq k-1$.

A less immediate example is that of the star S_k, which is the graph obtained by adding a universal vertex to an edgeless graph on k vertices.



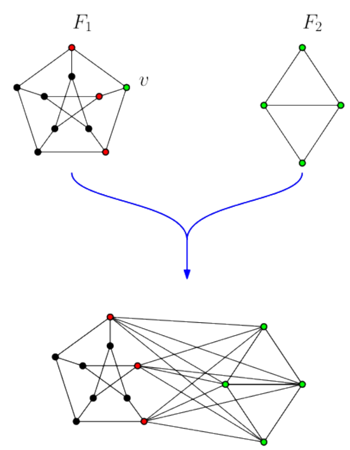
The star s_7.

We therefore can deduce from the previous proposition that the star S_k has, for every integer k, the Erdős–Hajnal property, and moreover that $c(S_k) \leq k$.

Substitutions

In the early 2000s, Noga Alon, János Pach, and József Solymosi proved that the Erdős–Hajnal property is preserved under an operation called *substitution*.

Let F_1 and F_2 be two graphs, and let v be a vertex of F_1. Substituting the vertex v in F_1 by the graph F_2 consists in taking the disjoint union of the graphs F_1 and F_2, deleting the vertex v from F_1, and joining all neighbors of v in F_1 to every vertex of F_2:



The substitution of vertex v in F_1
by the graph F_2.

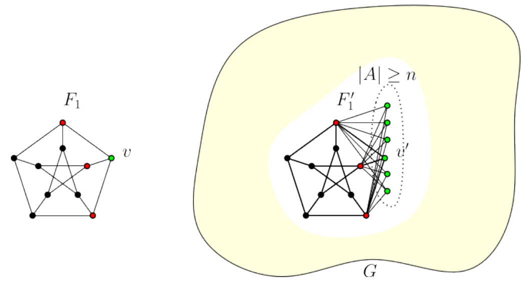
Proposition (Alon, Pach, Solymosi)**.** *If F_1 and F_2 have the Erdős–Hajnal property and v is a vertex of F_1, then the substitution of v in F_1 by F_2 has the Erdős–Hajnal property.*

This a difficult proposition. Nevertheless, we will outline the arguments and give the intuition behind the proof.

Sketch of the proof: Let F be the graph obtained by substituting vertex v in F_1 by the graph F_2.

Let t be an integer, and let k be a sufficiently large integer. Let $n = t^{\max\{c(F_1),c(F_2)\}}$. Consider a graph G with at least $N = n^k$ vertices that contains no induced copy of the graph F. Suppose (for a contradiction) that G contains no cliques or stable sets of size t. Since F_1 and F_2 have the Erdős–Hajnal property, if we take any subset of vertices A of size n, then the graph $G[A]$ induced by A (i.e., the vertices in A together with the edges between them) contains both an induced copy of F_1 and an induced copy of F_2.

A counting argument then shows (assuming k is large enough) that there exists an induced copy F_1' in G of F_1 with the following property: if we denote by v' the vertex in F' corresponding to v, then there exist at least n "alternative" vertices of G that allow one to extend $F_1' \setminus \{v\}$ (i.e., the induced copy F_1' of F_1 with the vertex v' removed) into an induced copy of F_1:



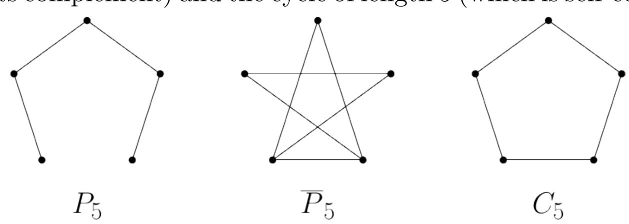
Replacing v by one of n alternative vertices.

The set A of "alternative" vertices that allow one to extend $F_1' \setminus \{v'\}$ into an induced copy of F_1 then induces a graph $G[A]$ that has more than n vertices and contains neither a stable set nor a clique of size t. Since F_2 has the Erdős–Hajnal property, we deduce that $G[A]$ contains an induced copy of F_2 which, together with $F_1' \setminus \{v'\}$, forms an induced copy of F. The graph G therefore contains an induced copy of F, contradicting our initial assumption. We have thus proved (or rather, sketched the proof) that the graph F has the Erdős–Hajnal property. □

What's next?

A few other examples of graphs having the Erdős–Hajnal property are known, such as all graphs on at most 4 vertices.

For graphs on 5 vertices, two graphs remain resistant: the path of length 4 (or, equivalently, its complement) and the cycle of length 5 (which is self-complimentary):



Difficult cases of Erdős–Hajnal conjecture.

And what about the others, you might ask? Even if the conjecture is not resolved in the general case, it is "almost" resolved:

Theorem (Loebl, Reed, Scott, Thomason, Thomassé)**.** *For every graph F, there exists an integer $c(F)$ with the following property:*

Almost all[14] *sufficiently large graphs G that contain no induced copies of F and have more than $t^{c(F)}$ vertices contain a stable set or a clique of size t.*

In other words, the examples of graphs G that may fail the conjecture, *are not* the random graphs with no induced copies of F.

Which goes to show…

> We can form a clear notion of order, but not of disorder.
>
> *Jacques-Henri Bernardin de Saint-Pierre*[15]

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[14] "Almost all graphs G" means that the proportion of graphs G on n vertices having this property tends to 1 as n tends to infinity.

[15] Jacques-Henri Bernardin de Saint-Pierre, *Paul et Virginie.*